Partial-order reduction

State-space explosion

We have learnt that there is a practical solution to LTL model checking in $\mathcal{O}(|\mathcal{K}| \cdot 2^{|\phi|})$.

In practice: ϕ small, but $\mathcal{K}$ very large, often exponentially larger than an underlying high-level description. Reasons: data, concurrency, ...

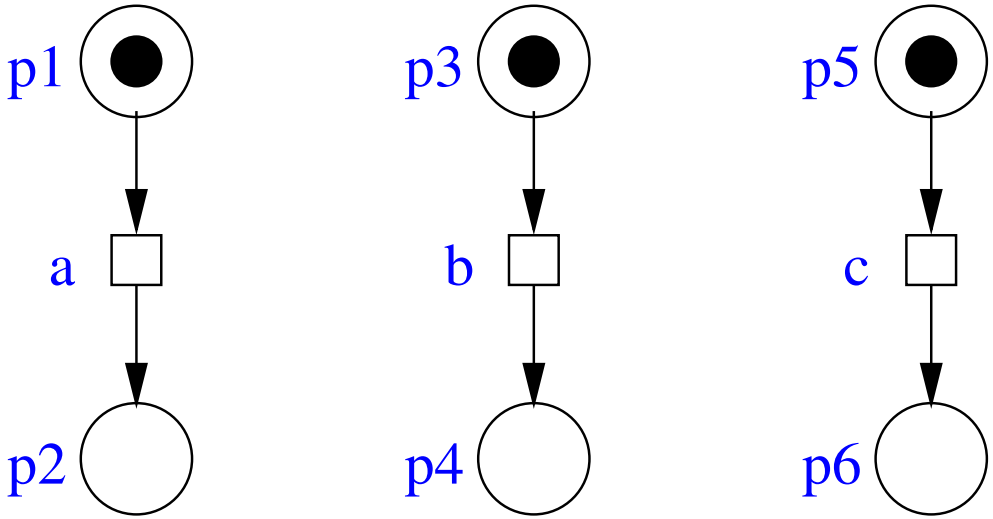
We shall explore a method that tackles state-space explosion for concurrent systems and LTL.

Input: a high-level description of a system (e.g., a description or programming language) and an LTL formula ϕ

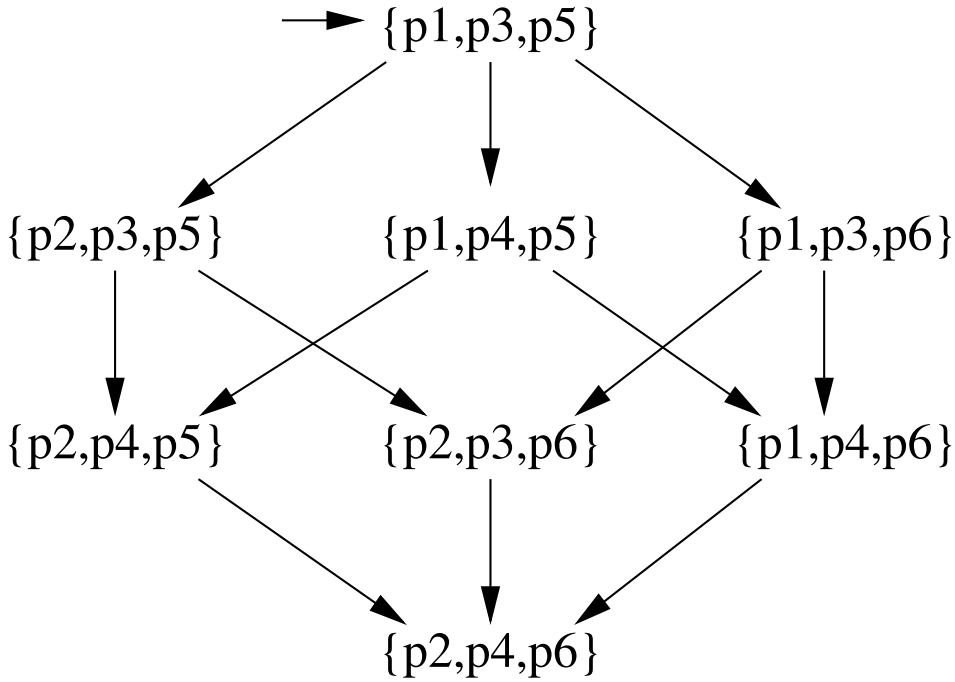
Goal: Determine whether ϕ holds without actually constructing the entire Kripke structure in the first place.

Example

Consider three processes in parallel, each of which can make one step.

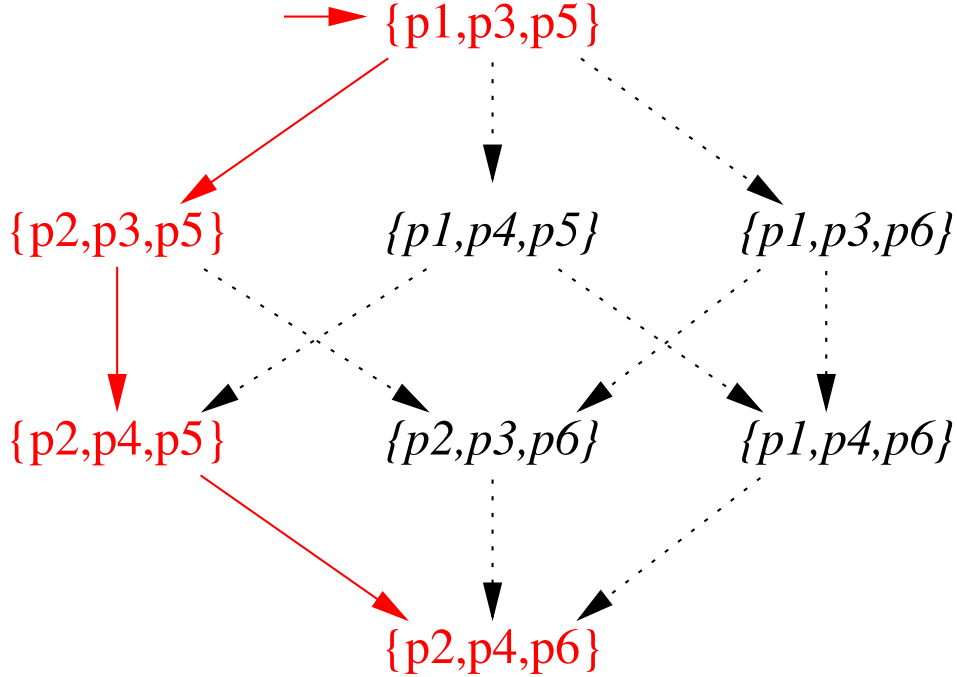


The reachability graph has got $8 = 2^3$ states and $6 = 3!$ possible paths.



For n components we have 2^n states and $n!$ paths.

All paths lead to $\{p_2, p_4, p_6\}$.



Idea: reduce size by considering only one path

Only possible if nothing "interesting" happens in those paths!

On-the-fly Model Checking

Important: It would be pointless to construct $\mathcal{K}$ first and then reduce its size.

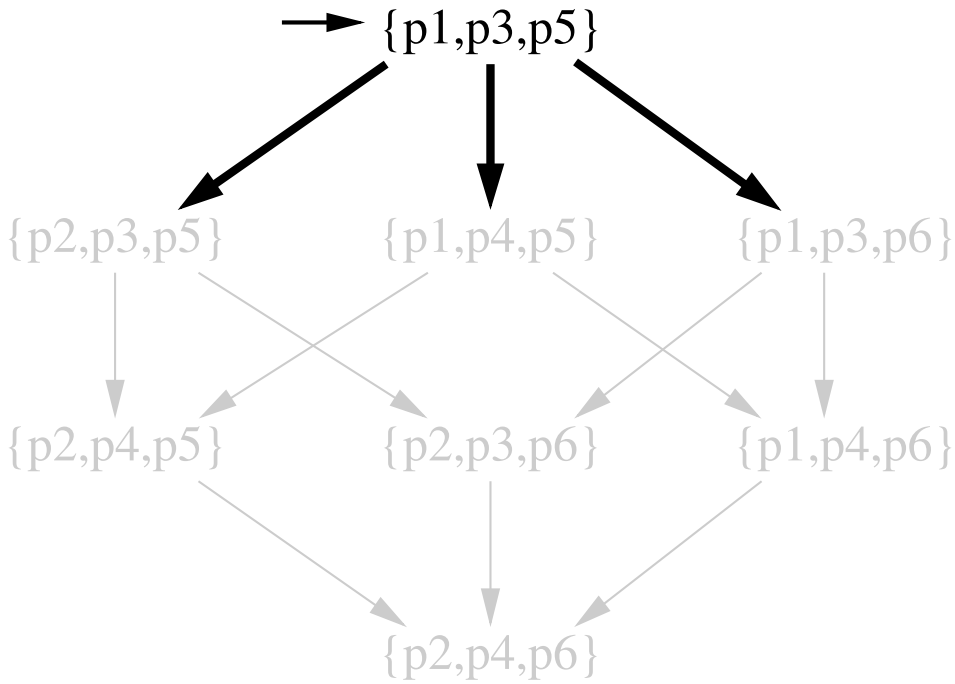
(does not save space, we can analyze $\mathcal{K}$ during construction anyway)

Thus: The reduction must be done "on-the-fly", i.e. while $\mathcal{K}$ is being constructed (from a compact description such as a Promela model) and during analysis.

$\Rightarrow$ combination with depth-first search

Reduction and DFS

We must decide which paths to explore *at this moment*.



I.e. before having constructed (or “seen”) the rest!

$\rightarrow$ only possible with additional information!

Partial-order techniques

Partial-order techniques aim to reduce state-space explosion due to concurrency.

One tries to exploit *independences* between transitions, e.g.

Assignments of variables that do not depend upon each other:

$$\mathtt{x} := \mathtt{z} + 5 \quad \| \quad \mathtt{y} := \mathtt{w} + \mathtt{z}$$

Send and receive on FIFO channels that are neither empty nor full.

Idea: avoid exploring all interleavings of independent transitions

correctness depends on whether the property of interest does not distinguish between different such interleavings

may reduce the state space by an exponential factor

Labelled Kripke structures

We extend our model with actions:

$$\mathcal{K} = \langle S, A, \rightarrow, r, AP, \nu \rangle$$

S, r, AP, ν as before, A is a set of actions, and $\rightarrow \subseteq S \times A \times S$.

We assume forthwith that transitions are deterministic, i.e. for each $s \in S$ and $a \in A$ there is at most one $s' \in S$ such that $\langle s, a, s' \rangle \in \rightarrow$.

$en(s) := \{ a \mid \exists s' : \langle s, a, s' \rangle \in \rightarrow \}$ are called the enabled actions in s.

Notation for (labelled) Kripke structures

S^* denotes the *finite*, S^ω the *infinite* sequences (words) over S.

We write $s \xrightarrow{a} t$ if $\langle s, a, t\rangle \in \rightarrow$, or simply $s \rightarrow t$ if a does not matter.

$p = s_0 \cdots s_n$ is a path of length n that reads $w = a_0 \cdots a_{n-1}$
if $s_i \xrightarrow{a_i} s_{i+1}$ for all $0 \leq i < n$.

$s \rightarrow^* t$ if there is a path from s to t.

$s \rightarrow^+ t$ if there is a path from s to t with at least one transition.

$\rho = s_0 s_1 \cdots$ is an infinite path reading $\sigma = a_0 a_1 \cdots$
if $s_i \xrightarrow{a_i} s_{i+1}$ for all $i \geq 0$.

Notation for (labelled) Kripke structures

If $\mathcal{K}$ is deterministic, then $\sigma \in A^\omega$ has at most one infinite path from a given state s, which we denote $s_{\mathcal{K}}(\sigma)$ (if it exists). If $r_{\mathcal{K}}(\sigma)$ is well-defined, we say $\sigma \in \mathcal{L}(\mathcal{K})$.

The set of finite prefixes of words in $\mathcal{L}(\mathcal{K})$ is denoted $\mathcal{L}^*(\mathcal{K})$.

Let $\sigma = a_0 a_1 \cdots \in A^*$ and $i \geq 0$.

$\sigma(i) = a_i$ denotes the i-th element of σ;

$\sigma^i = a_i a_{i+1} \cdots$ the suffix of σ obtained by deleting i letters at the start;

$\sigma_{|i} = a_0 \cdots a_{i-1}$ the prefix of σ of length i. (Note: $\sigma = \sigma_{|i}\, \sigma^i$)

For $\rho \in S^\omega$, we write $\nu(\rho)$ for $\nu(\rho(0))\nu(\rho(1)) \cdots$
By abuse of notation, for $\sigma \in \mathcal{L}(\mathcal{K})$, we write $\nu(\sigma)$ for $\nu(r_{\mathcal{K}}(\sigma))$.

Independence

$I \subseteq A \times A$ is called independence relation for $\mathcal{K}$ if:

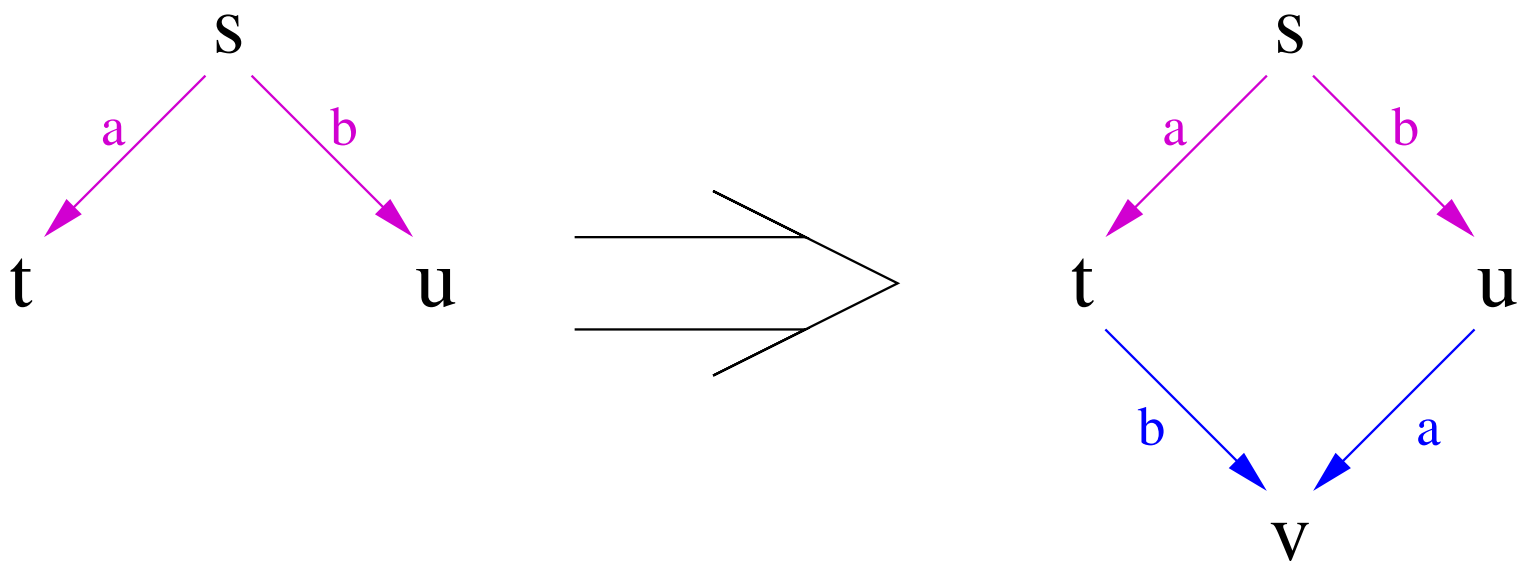
for all $a \in A$ we have $\langle a, a \rangle \notin I$ (irreflexivity);

for all $\langle a, b \rangle \in I$ we have $\langle b, a \rangle \in I$ (symmetry);

for all $\langle a, b \rangle \in I$ and all $s \in S$ we have:

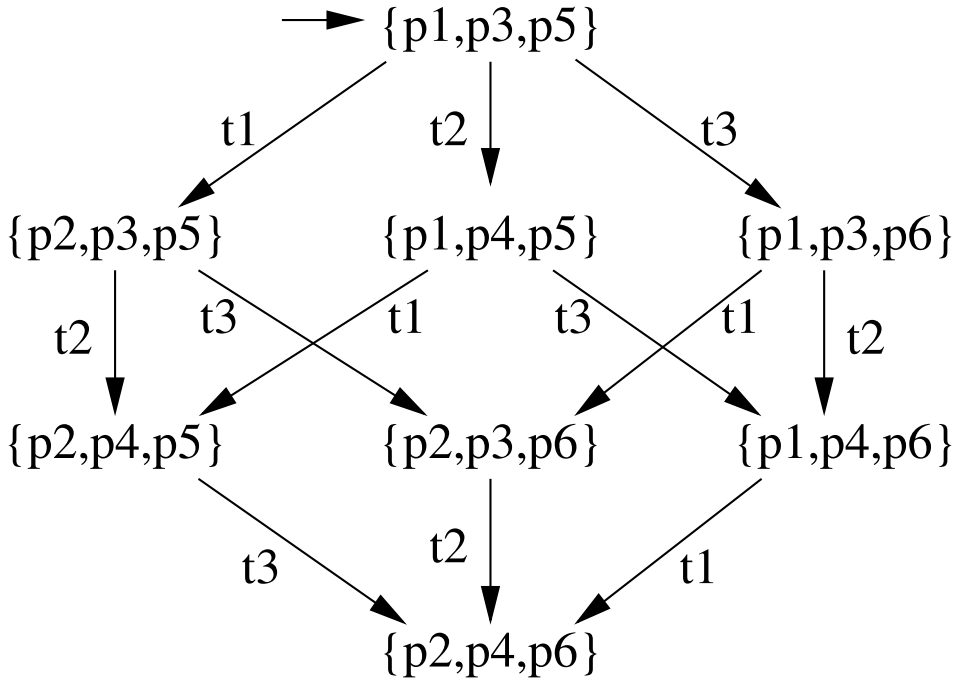
if $a, b \in en(s)$, $s \xrightarrow{a} t$, and $s \xrightarrow{b} u$,
then there exists v such that $a \in en(u)$, $b \in en(t)$, $t \xrightarrow{b} v$ and $u \xrightarrow{a} v$.

Independence



Independence: Example

In the example below all pairs of actions are independent.



Remark: In general, independence relations are not transitive!

(In-)Visibility

$U \subseteq A$ is called an invisibility set, if all $a \in U$ have the following property:

for all $s \xrightarrow{a} s'$ we have: $\nu(s) = \nu(t)$.

I.e., no action in U ever changes the validity of an atomic proposition.

Motivation: Interleavings of visible actions may not be eliminated because they might influence the validity of LTL properties.

Remarks

Sources for I and U: “external” knowledge about the model and the actions possible in it

e.g. addition is commutative, structural information about a Petri net,. . .

will *not* be obtained from first constructing all of $\mathcal{K}$!

Every (symmetric) subset of an independence relation remains an independence relation, every subset of an invisibility set remains an invisibility set.

$\rightarrow$ conservative approximation possible

But: The bigger I and U are, the more information we have at hand to improve the reduction.

Stuttering equivalence

Definition: Let σ, ρ be infinite sequences over 2^{AP}. We call σ and ρ stuttering equivalent (noted $\sigma \approx \rho$) iff there are integer sequences

$$0 = i_0 < i_1 < i_2 < \cdots \text{ and } 0 = k_0 < k_1 < k_2 < \cdots,$$

such that for all $\ell \geq 0$:

$$\sigma(i_\ell) = \sigma(i_\ell + 1) = \cdots = \sigma(i_{\ell+1} - 1) =$$
$$\rho(k_\ell) = \rho(k_\ell + 1) = \cdots = \rho(k_{\ell+1} - 1)$$

I.e., σ and ρ can be partitioned into "blocks" of possibly differing sizes, but with the same valuations. We can generalize this notion to finite sequences in the obvious way.

We also extend this notion to Kripke structures:

Let $\mathcal{K}, \mathcal{K}'$ be two Kripke structures with the same set of atomic propositions AP.

Let $\sigma \in \mathcal{L}(\mathcal{K})$ and $\tau \in \mathcal{L}(\mathcal{K}')$. We write $\sigma \approx \tau$ iff $\nu(\sigma) = \nu'(\tau)$.

$\mathcal{K}$ and $\mathcal{K}'$ are called stuttering-equivalent iff for every $\sigma \in \mathcal{L}(\mathcal{K})$ there exists $\tau \in \mathcal{L}(\mathcal{K}')$ such that $\sigma \approx \tau$.

In other words, $\mathcal{K}$ and $\mathcal{K}'$ contain the same equivalence classes.

Invariance under stuttering

Let ϕ be an LTL formula. We call ϕ invariant under stuttering iff for all stuttering-equivalent pairs of sequences σ and ρ:

$$\sigma \in [\![\phi]\!] \quad \text{iff} \quad \rho \in [\![\phi]\!].$$

Put differently: ϕ cannot distinguish stuttering-equivalent sequences. (And neither stuttering-equivalent Kripke structures.)

Theorem: Any LTL formula without $\mathbf{X}$ is invariant under stuttering.
Proof: Exercise.

Strategy

We replace $\mathcal{K}$ by a stuttering-equivalent, smaller structure $\mathcal{K}'$.

Then we check whether $\mathcal{K}' \models \phi$, which is equivalent to $\mathcal{K} \models \phi$
(if ϕ does not contain any $\mathbf{X}$).

We generate $\mathcal{K}'$ by performing a DFS on $\mathcal{K}$, and in each step eliminating certain successor states, based on the knowledge about properties of actions that is imparted by I and U.

The method presented here is called the ample set method.

Ample sets

For every state s we compute a set $red(s) \subseteq en(s)$, where $red(s)$ contains the actions whose corresponding successor states will be explored.

In other words, $\mathcal{K}'$ is obtained from $\mathcal{K}$ by deleting, in each state s, the outgoing transitions with $en(s) \setminus red(s)$. We also delete all states that now become unreachable from r.

(partially conflicting) goals:

- $red(s)$ must be chosen in such a way that stuttering equivalence is guaranteed.
- The choice of $red(s)$ should reduce $\mathcal{K}$ strongly.
- The computation of $red(s)$ should be efficient.

Conditions for Ample Sets

C0: $red(s) = \emptyset$ iff $en(s) = \emptyset$

C1: Every path of $\mathcal{K}$ starting at a state s satisfies the following: no action that depends on some action in $red(s)$ occurs before an action from $red(s)$.

C2: If $red(s) \neq en(s)$ then all actions in $red(s)$ are invisible.

C3: For all cycles in $\mathcal{K}'$ the following holds: if $a \in en(s)$ for some state s in the cycle, then $a \in red(s')$ for some (possibly other) state s' in the cycle.

Idea

C0 ensures that no additional deadlocks are introduced.

C1 and C2 ensure that every stuttering-equivalence class of runs is preserved.

C3 ensures that enabled actions cannot be omitted forever.

Example

Pseudocode program with two concurrent processes:

int x,y **init** 0;
cobegin $\{P \parallel Q\}$ **coend**

$P =$ `p0:` $x := x + 1;$ (action a)
`p1:` $y := y + 1;$ (action b)
`p2:` **end**

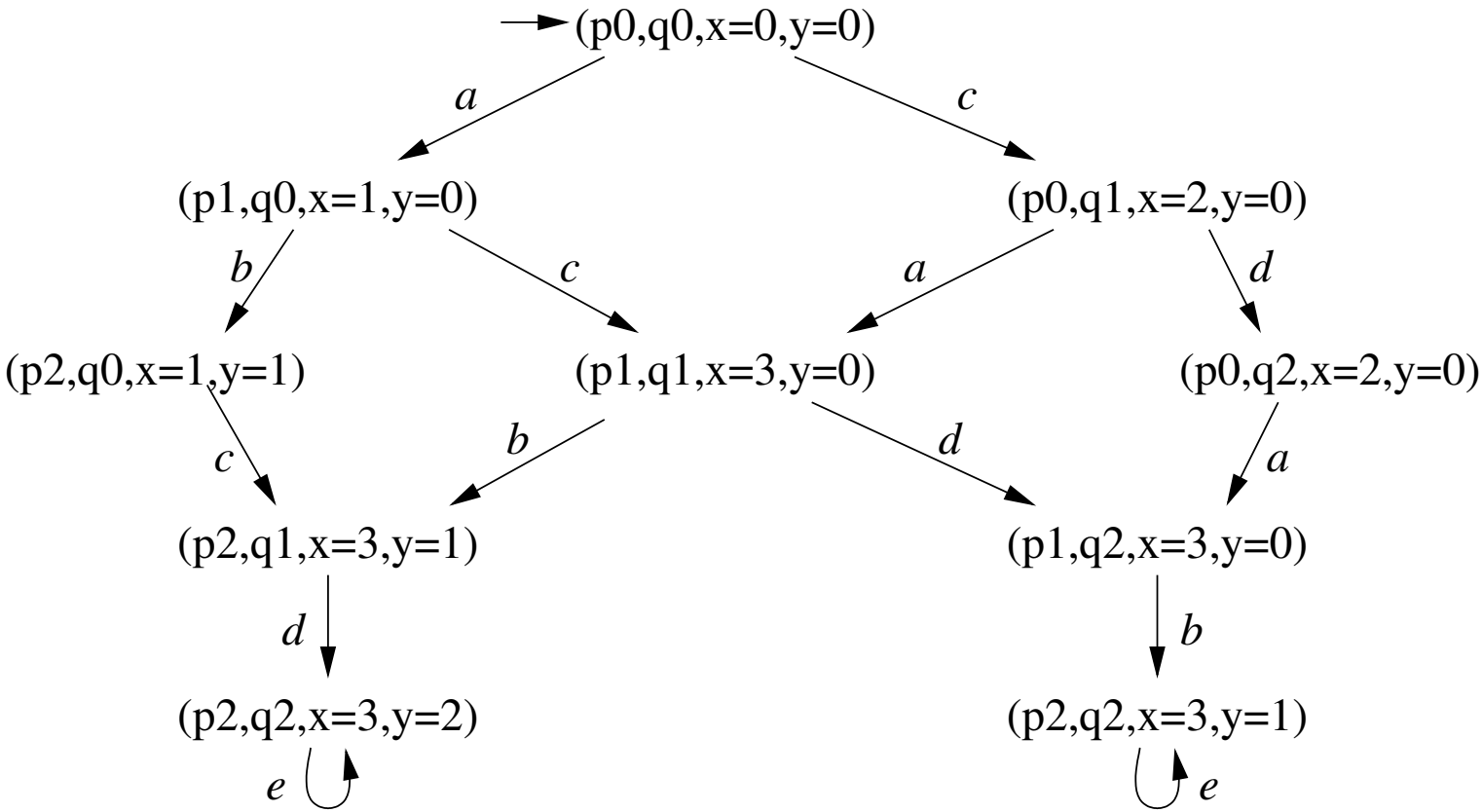
$Q =$ `q0:` $x := x + 2;$ (action c)
`q1:` $y := y * 2;$ (action d)
`q2:` **end**

Two remarks:

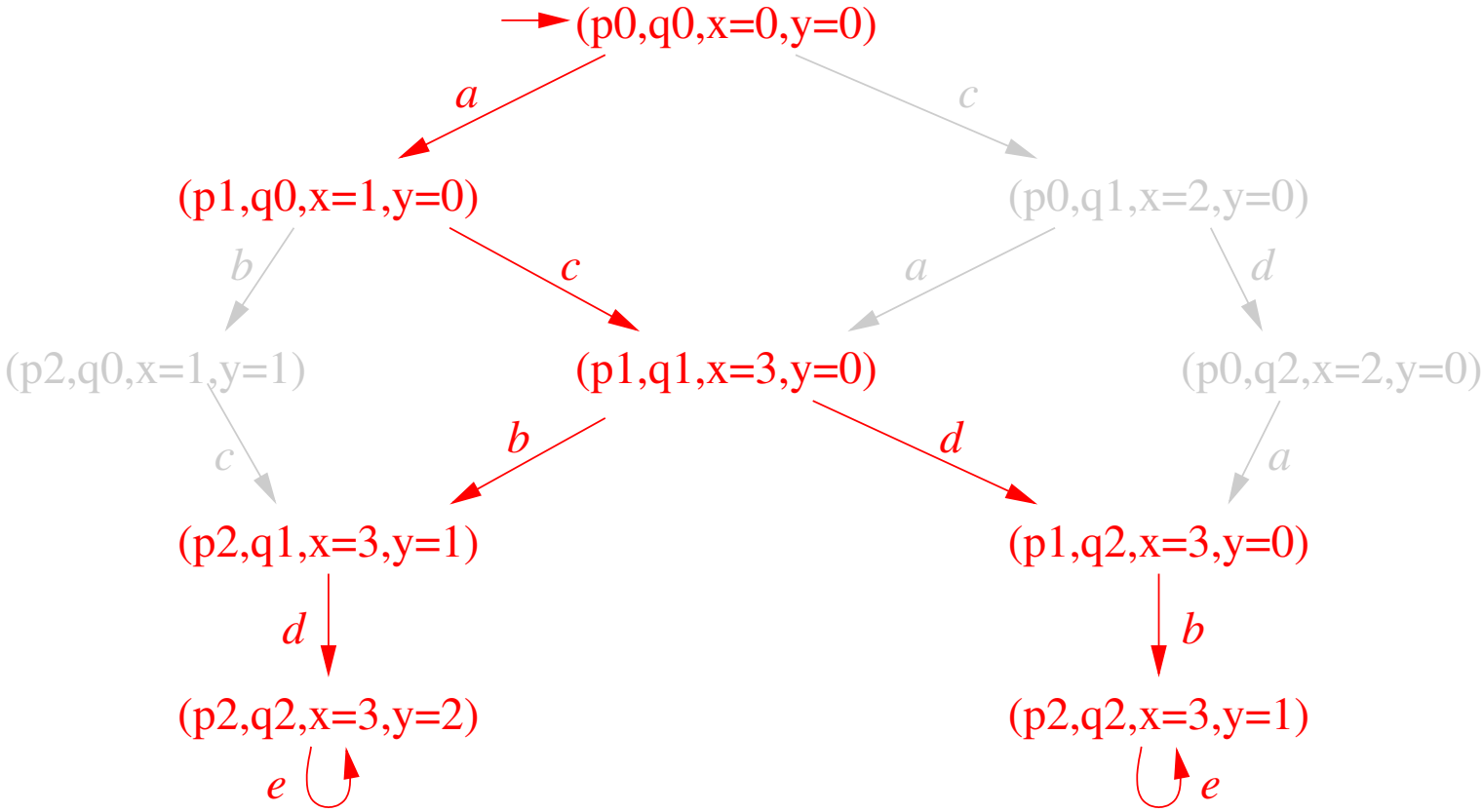
In order to make all runs infinite, we assume that **end** corresponds to an infinite loop with action e.

The laws of mathematics imply that b and d cannot be independent.

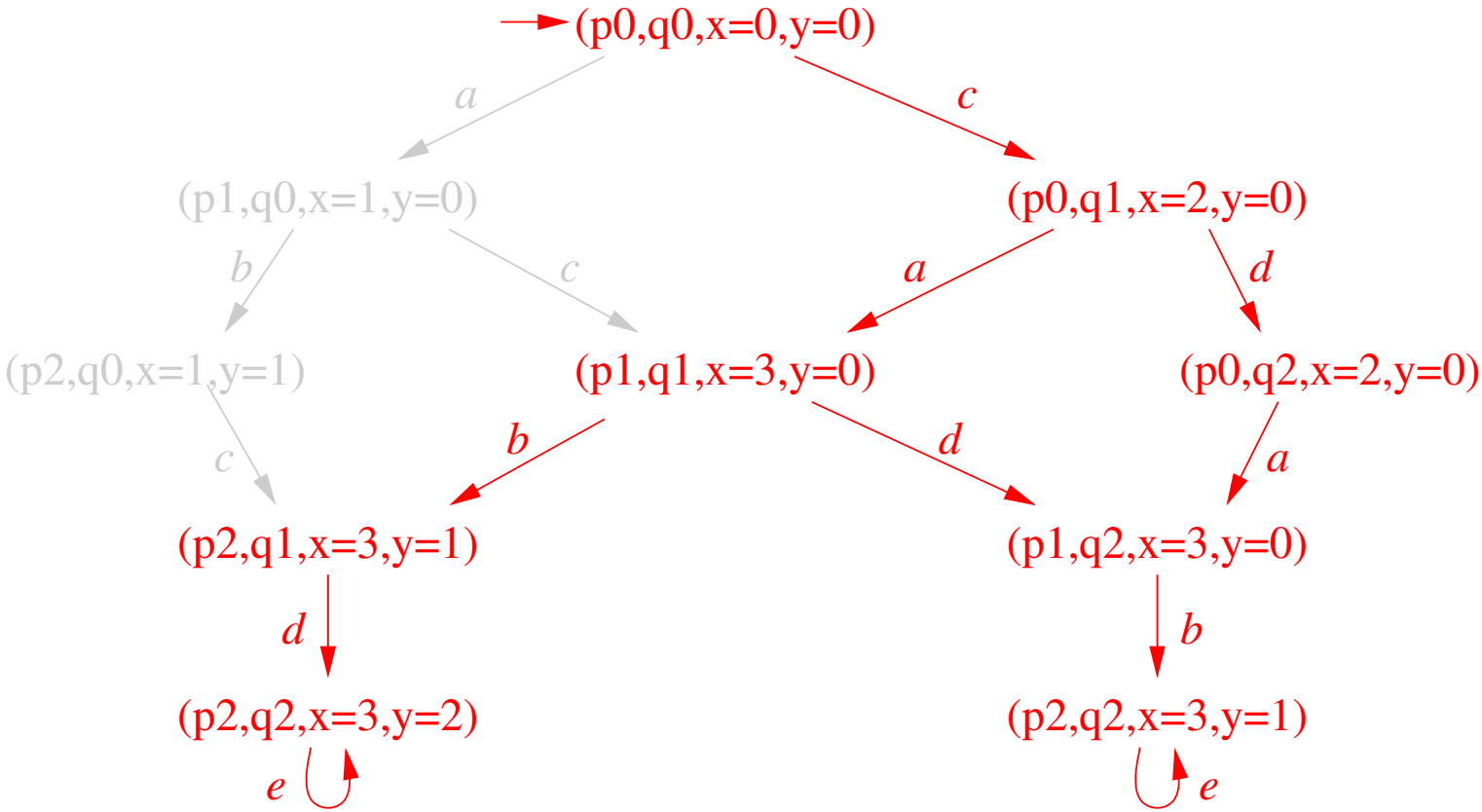
Kripke structure of the example:



Possible reduced structure if all actions are invisible:



Possible reduced structure if *a*, *d* are visible:



Correctness

Claim: If red satisfies conditions C0 through C3, then $\mathcal{K}'$ is stuttering-equivalent to $\mathcal{K}$.

Proof: Let $\sigma \in \mathcal{L}(\mathcal{K})$. We shall construct a sequence of infinite words $\sigma = \sigma_0, \sigma_1, \ldots$ such that:

(i) $\sigma_i \approx \sigma$ for all i;

(ii) $\sigma_i \in \mathcal{L}(\mathcal{K})$ for all i;

(iii) $\sigma_{i|i} \in \mathcal{L}^*(\mathcal{K}')$ and $\sigma_{i+1|i} = \sigma_{i|i}$ for all i;

Let τ be the limit of this construction defined by $\tau(i) = \sigma_{i+1}(i)$.

In other words, τ is constructed by chosing a path through $\mathcal{K}'$, one transition at a time, while assuring that the chosen path reads a word that remains equivalent to σ. Clearly, $\tau \in \mathcal{L}(\mathcal{K}')$.

Construction of σ_i

Base: $\sigma_0 = \sigma$ satisfies conditions (i) to (iii).

Step: For σ_{i+1}, let s be the stated reached from r by $\sigma_{i|i}$ and $a_1 = \sigma_i(i)$.

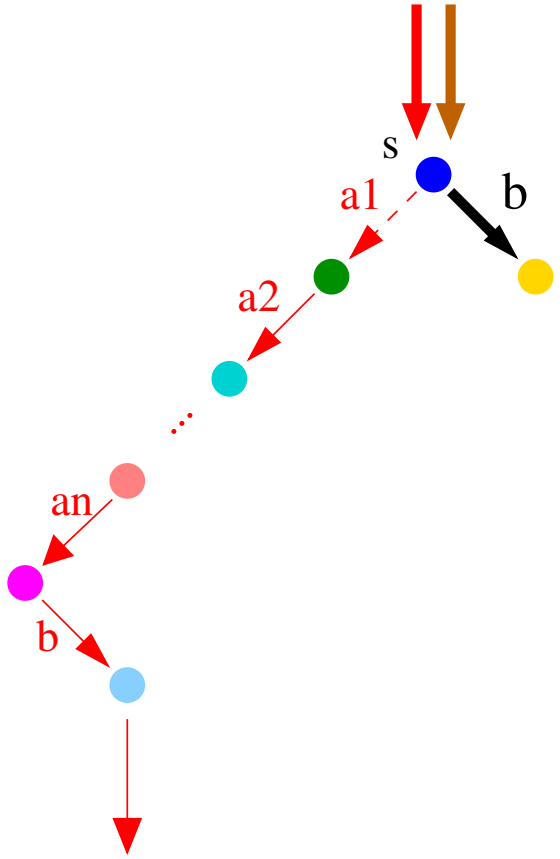
Case 0: If $a_1 \in red(s)$, we simply choose $\sigma_{i+1} = \sigma_i$, and we are done.

Otherwise, we distinguish two cases:

Case 1: σ_i^i contains an action from $red(s)$.

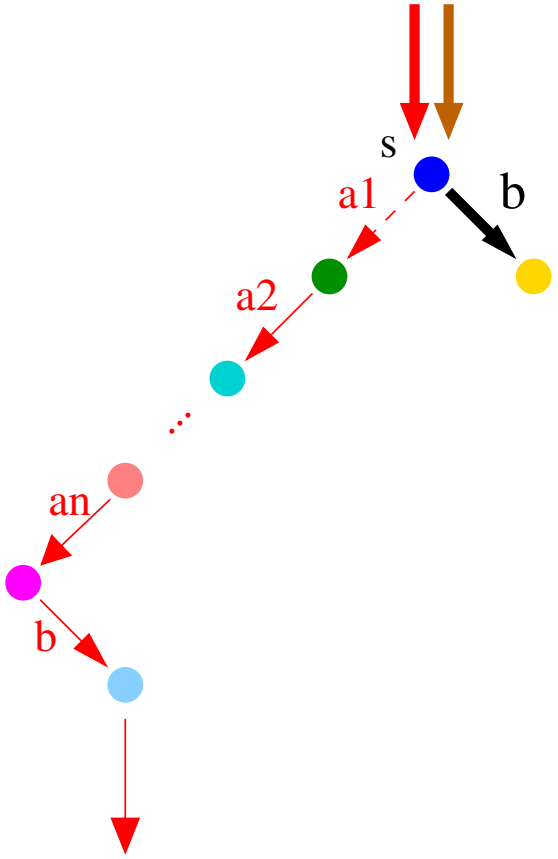
Case 2: σ_i^i does not contain an action from $red(s)$.

Case 1



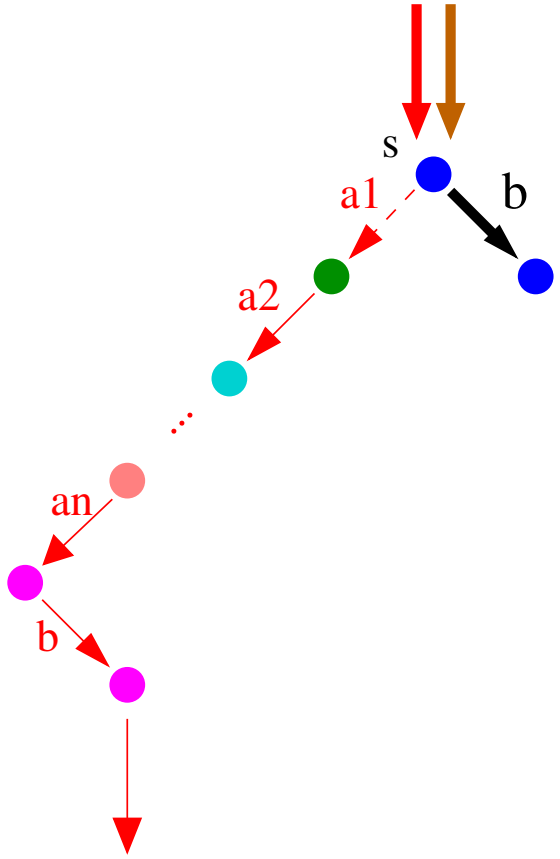
Let $\sigma_i^j = a_1 \cdots a_n b \cdots$ such that $b \in red(s)$ and $a_j \notin red(s)$ for $j = 1, \ldots, n$. Transitions of σ_i are shown in red, those of σ_{i+1} in brown, and states have the same colours if they are known to have the same atomic propositions.

Case 1



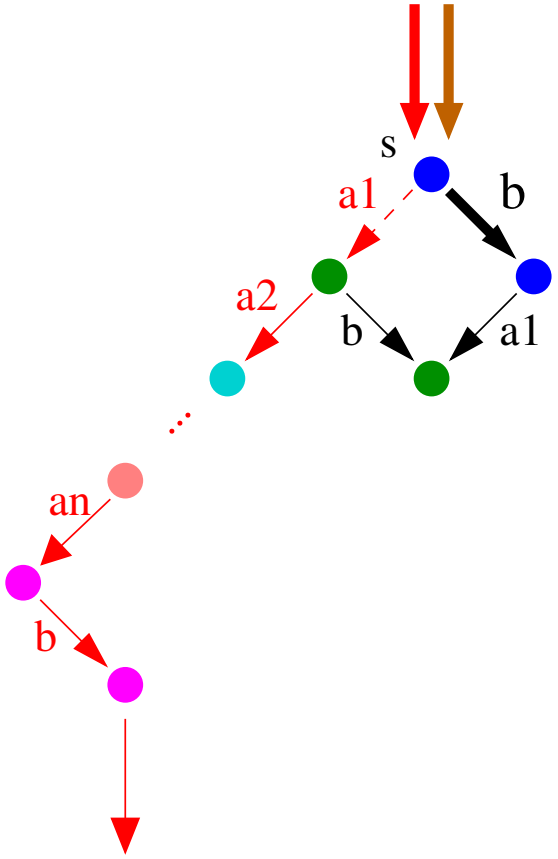
Also, transitions are *bold* when known to be part of $\mathcal{K}'$ and *dashed* when known *not* to be in $\mathcal{K}'$.

Case 1



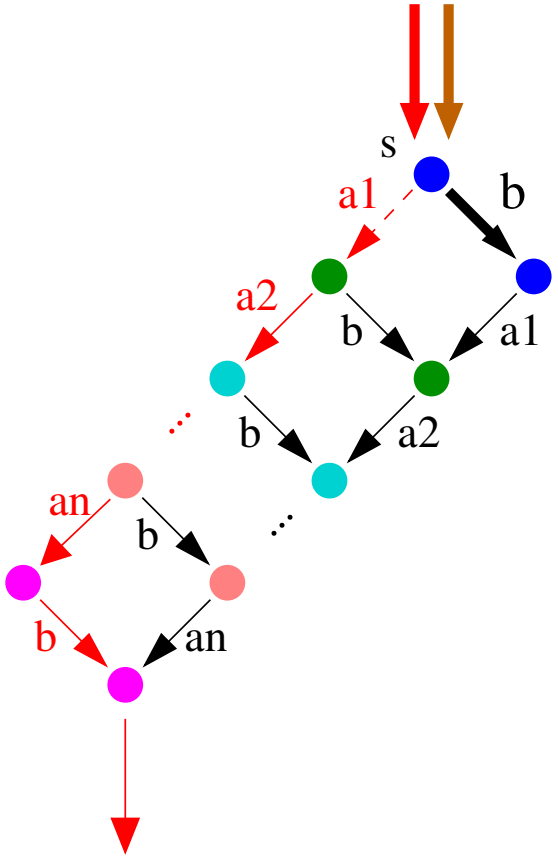
Since $a_1 \in en(s) \setminus red(s)$ and $b \in red(s)$, condition C2 implies that b must be invisible.

Case 1



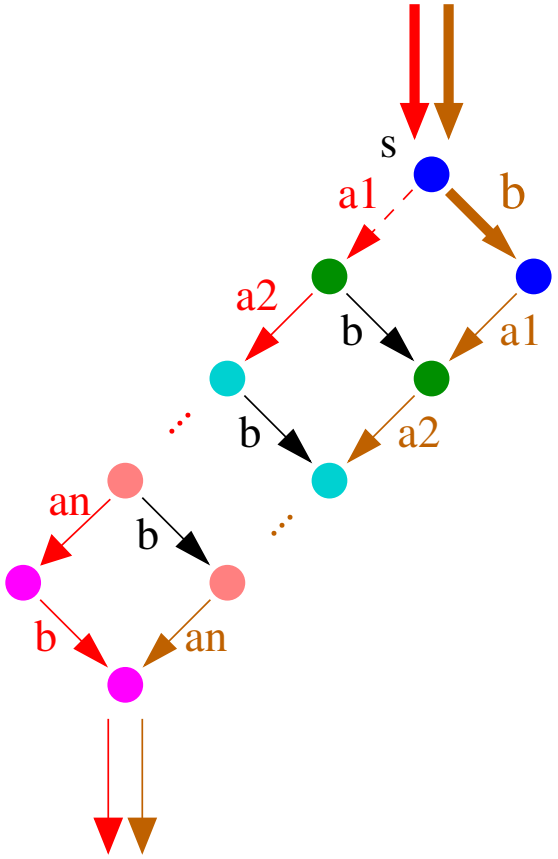
Because $a_1, \ldots, a_n \notin red(s)$, condition C1 implies that b must be independent from all these actions. Therefore, there exists a “diamond” between a_1 and b.

Case 1



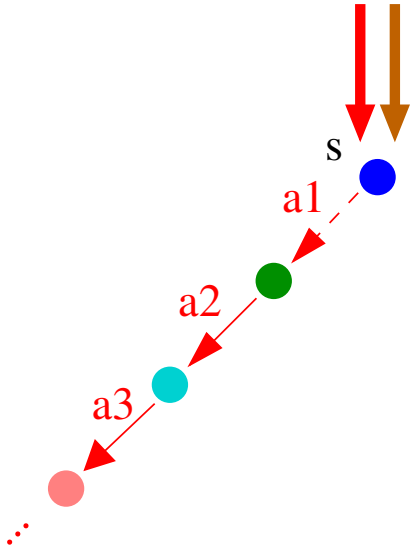
Continuing with this logic, we can construct further diamonds.

Case 1



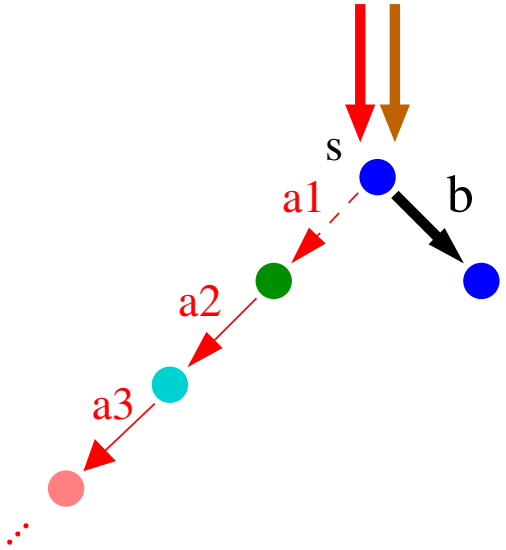
We can now choose σ_{i+1} as indicated. Indeed, this choice satisfies the requirements (i), (ii), and (iii).

Case 2



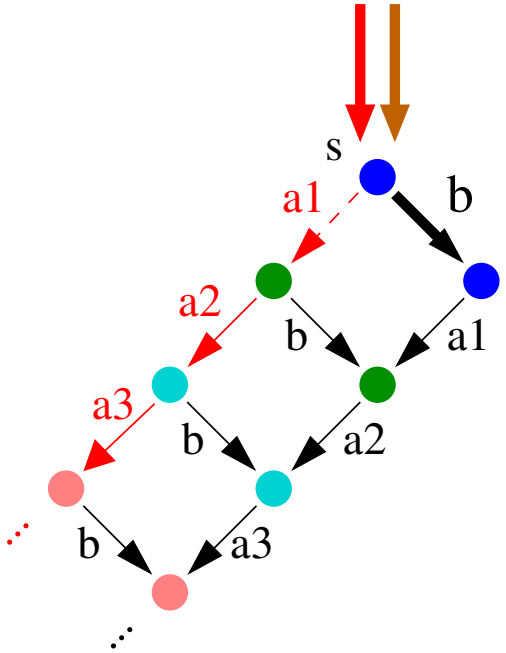
Suppose now that $\sigma_i^j = a_1 a_2 a_3 \cdots$ such that $a_j \notin red(s)$ for all $j \geq 1$.

Case 2



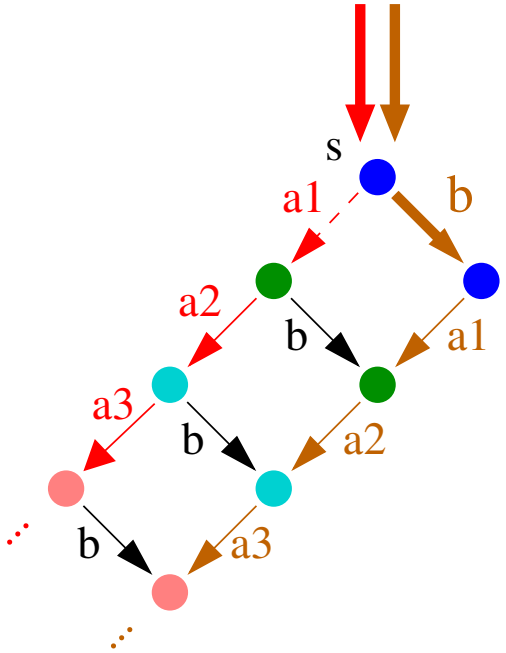
Since $a_1 \in en(s) \setminus red(s)$, condition C0 implies that there must be some other action $b \in en(s) \cap red(s)$, and because of C2, again b must be invisible.

Case 2



By the same reasoning as in Case 1, we can construct diamonds between the a_j and b.

Case 2



Now we can choose σ_{i+1} as indicated, which again satisfies conditions (i), (ii), and (iii).

Conclusion

We therefore see that σ_i can be chosen, for any $i \geq 0$, so that $\sigma_i \in \mathcal{L}(\mathcal{K})$ $\sigma_i \approx \sigma$ and thus τ, the limit of the construction, is also in $\mathcal{L}(\mathcal{K})$.

We still need to prove that $\sigma \approx \tau$.

> Indeed, if for all $j \geq i$, for some $i \geq 0$, all σ_j are chosen using Cases 1 or 2, then τ would stutter indefinitely.
>
> It suffices to show that Case 0 applies infinitely often, i.e. no action is deferred infinitely often in the choice of $red(s)$.
>
> Suppose that this was the case, i.e. σ_i^i begins with $a \in en(s) \setminus red(s)$, and the choice of all σ_j, for $j \geq i$, is made through Cases 1 or 2. Consider run $s_{\mathcal{K}'}(\tau^i)$, and notice that a is enabled along all its states.

(Continuation of conclusion)

By pigeonhole principle, at some point this run needs to repeat one of the finitely many states in $\mathcal{K}$, so it contains a cycle.

Thanks to condition C3, a must be in $red(s'')$ for some state s'' along the cycle, at which point we would have continued with Case 0.

Thus, we have reached a contradiction, and as a consequence, we have indeed $\sigma \approx \tau$.

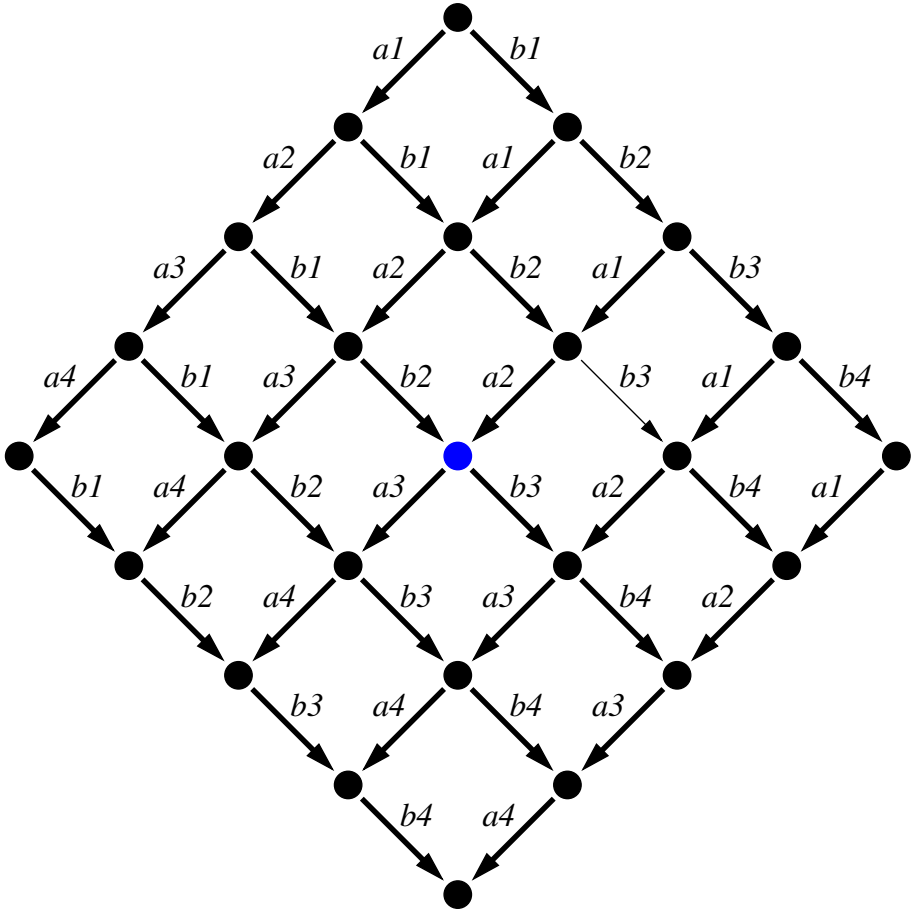
(Non-)Optimality

An ideal reduction would retain only one execution from each stuttering equivalence class.

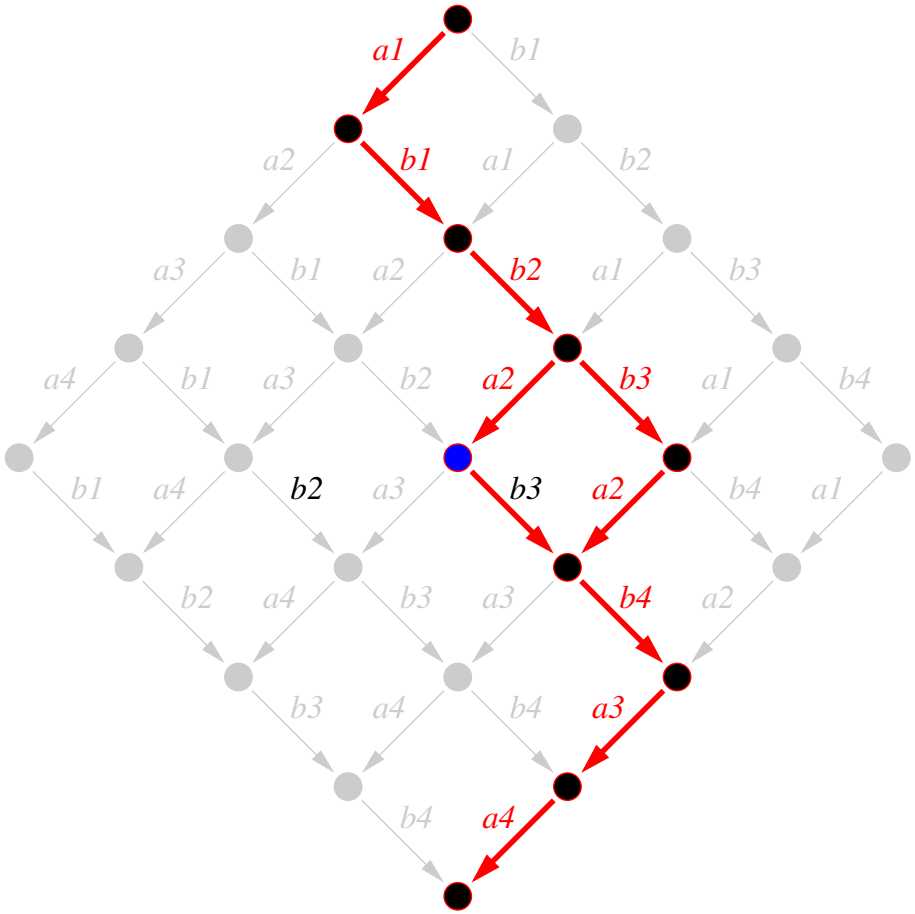
C0–C3 do *not* ensure such an ideal reduction, i.e. the resulting reduced structure is not minimal in general.

Example (see next slide): two parallel processes with four actions each ($a_1, \ldots, a_4$ or $b_1, \ldots, b_4$, resp.), all independent.

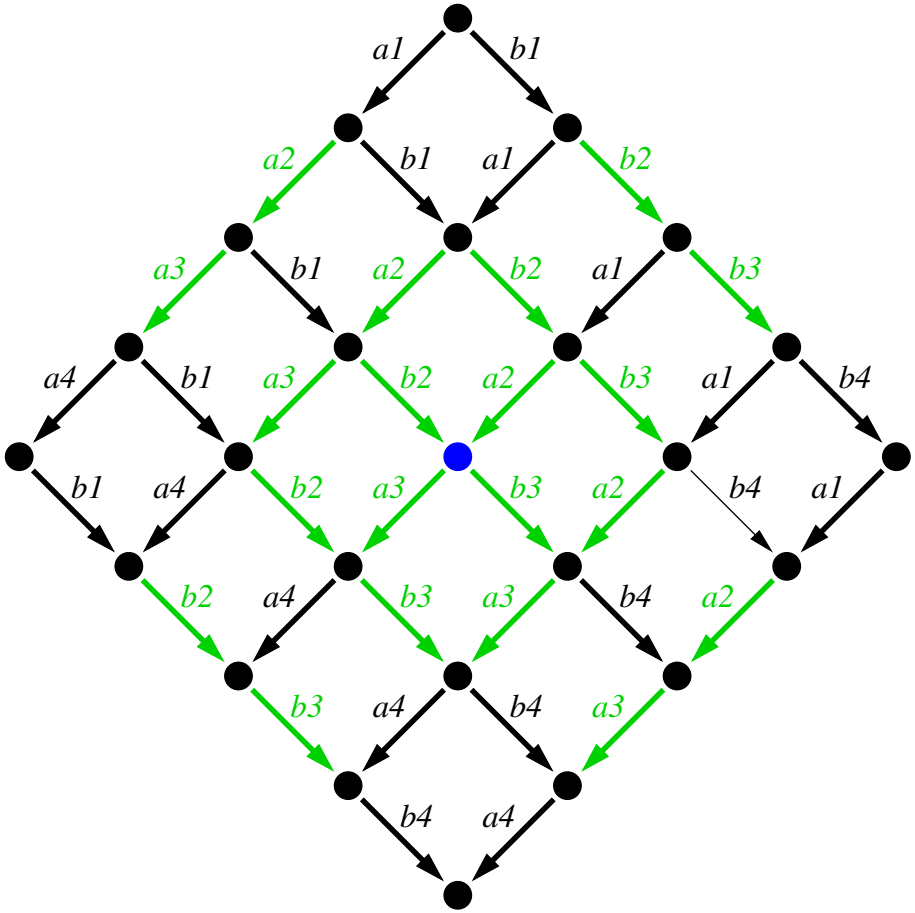
Assume that the valuation of the blue state differs from the others:



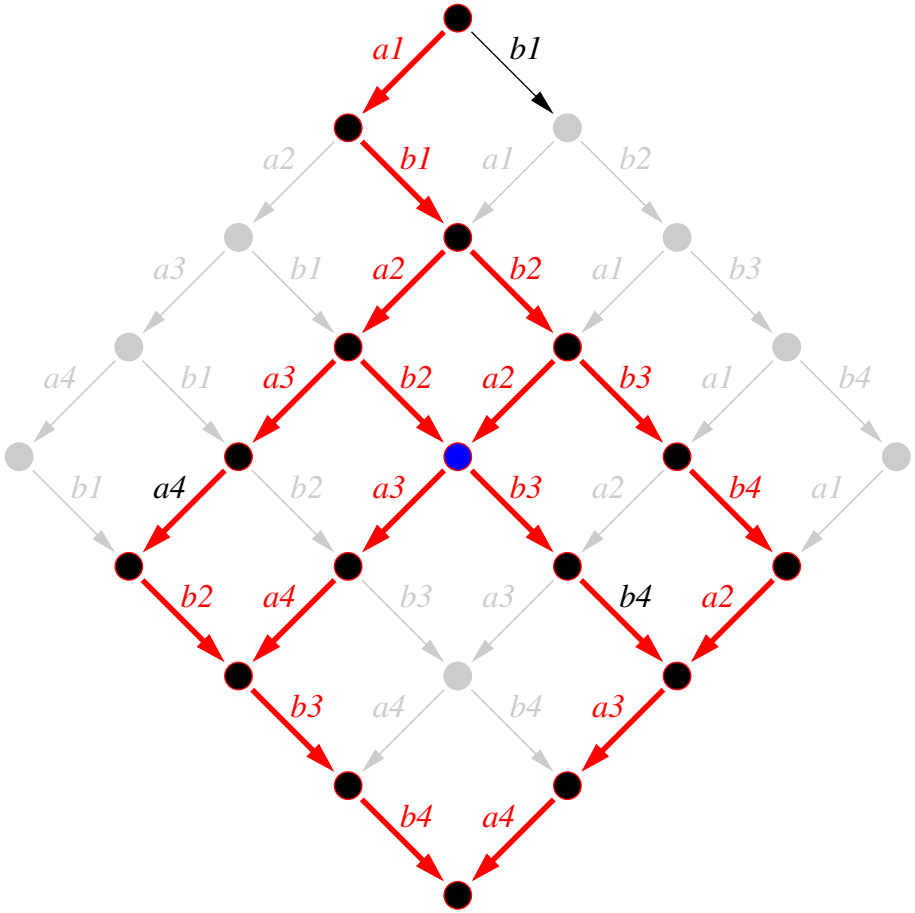
Minimal stuttering-equivalent structure:



Visible actions: a_2, a_3, b_2, b_3 (in green):



Smallest structure satisfying C0–C3:



Implementation issues

C0 and C2: obvious

C1 and C3 depend on the complete state graph

We shall find conditions that are stronger than C1 and C3. These will exclude certain reductions but can be efficiently implemented during DFS.

Replace C3 by C3':

> If $red(s) \neq en(s)$ for some state s, then no action of $red(s)$ may lead to a state that is currently on the search stack of the DFS.

Heuristic for C1

Depends on the underlying description of the system; here, we will discuss what Spin is doing for Promela models.

In general, a Promela model will contain multiple concurrent processes $P_1, \ldots, P_n$ communicating via global variables and message channels.

Let $pc_i(s)$ be the control-flow label of process P_i in state s.

Heuristic for C1 (in Spin)

Let $pre(a)$ be the set of actions that might activate a, i.e. (some overapproximation of) the set

$$\{\, b \mid \exists s\colon\ a \notin en(s),\ b \in en(s),\ a \in en(b(s)) \,\}$$

In Spin: Let a be an action in process P_i; then $pre(a)$ contains, e.g.,

- all actions of P_i leading to a control-flow label in which a can be executed;
- if the guard of a uses global variables, all actions in other processes that modify these variables;
- if a reads from a message channel q or writes into it, then all actions in other processes that do the opposite (write/read) on q.

Heuristic for C1 (in Spin)

$dep(a) := \{ b \mid (a, b) \notin I \}$ contains the actions dependent on a.

In Spin, if a is an action in process P_i, then $dep(a)$ will be overapproximated by:

all other actions in P_i;

actions in other processes that write to a variable from which a reads, or vice versa;

if a reads from a message channel q or writes to it, then all actions in other processes doing the same to q.

Heuristic for C1 (in Spin)

Let A_i denote the possible actions in process P_i.

$E_i(s)$ denotes the actions of process P_i activated in s:

$$E_i(s) = en(s) \cap A_i$$

$C_i(s)$ denotes the actions possible in P_i at label $pc_i(s)$:

$$C_i(s) = \bigcup_{\{\, s' | pc_i(s) = pc_i(s') \,\}} E_i(s')$$

(Some of these may not be activated in s itself because guards are not satisfied...)

The approach in Spin

Spin will test the sets $E_i(s)$, for $i = 1, \ldots, n$, as candidates for $red(s)$. If all these candidates fail, it takes $red(s) = en(s)$.

A violation of C1 implies that some action a depending on $E_i(s)$ might be executed before an action from $E_i(s)$ itself. To check whether such a violation is possible, Spin checks the following two conditions:

Either a belongs to some other process P_j, $j \neq i$.
$\Rightarrow$ check whether $A_j \cap dep(E_i(s)) \neq \emptyset$

Otherwise $a \in C_i(s) \setminus E_i(s)$.
$\Rightarrow$ check whether $C_i(s) \setminus E_i(s)$ may be activated by some other process.

Test for C1

function $check\_C1(s, P_i)$
  for all $P_j \neq P_i$ **do**
    if $dep(E_i(s)) \cap A_j \neq \emptyset \vee$
      $pre(C_i(s) \setminus E_i(s)) \cap A_j \neq \emptyset$ **then**
        return $False$;
    end if;
  end for all;
  return $True$;
end function