Initiation à la Vérification

Emptiness Test for Büchi automata

Stefan Schwoon

November 14, 2025

Overview

Result from the first half of the course:

Model-checking LTL reduces to checking emptiness of some Büchi automaton $\mathcal{B}$.

Reminder (for universal model-checking, existential is analogus):

$\mathcal{B}$ is the intersection of a Kripke structure $\mathcal{K}$ with a BA for the *negation* of an LTL formula ϕ.

If $\mathcal{B}$ accepts some word, we call such a word a counterexample.

$\mathcal{K} \models \phi$ iff $\mathcal{B}$ accepts the empty language.

Complexity: $\mathcal{O}(|\mathcal{K}| \cdot |\mathcal{B}_{\neg\phi}|)$

Typical instances:

- Size of $\mathcal{K}$: between several hundreds to millions of states.
- Size of $\mathcal{B}_{\neg\phi}$: exponential in $|\phi|$, but usually just a couple of states.

Typical setting:

- $\mathcal{K}$ indirectly given by some concise description (modelling or programming language); model-checking tools will generate $\mathcal{K}$ internally.
- $\mathcal{B}_{\neg\phi}$ can be generated from ϕ before start of emptiness check.

Typical setting:

$\mathcal{B}$ generated "on-the-fly" from (the description of) $\mathcal{K}$ and from $\mathcal{B}_{\neg\phi}$ and tested for emptiness *at the same time*.

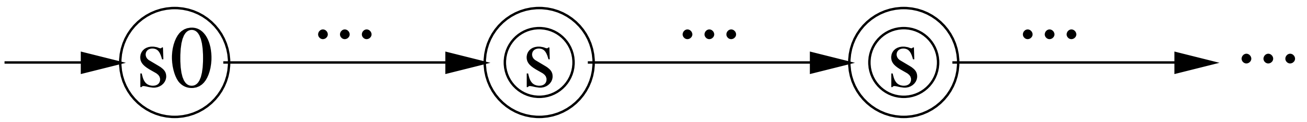
As a consequence, the size of $\mathcal{K}$ (and of $\mathcal{B}$) is not known initially!

At the beginning, only the initial state is known, and we have a function $\mathsf{succ}\colon S \to 2^S$ for computing the immediate successors of a given state (where succ implements the semantics of the description).

Naïve solution: Check for Lassos

Let $\mathcal{B} = (\Sigma, S, s_0, \delta, F)$ be a Büchi automaton.

$\mathcal{L}(\mathcal{B}) \neq \emptyset$ iff there is $s \in F$ such that $s_0 \rightarrow^* s \rightarrow^+ s$



Naïve solution:

Check for each $s \in F$ whether there is a cycle around s; let $F_\circ \subseteq F$ denote the set of states with this property.

Check whether s_0 can reach some state in $F_\circ$.

Time requirement: Each search takes linear time in the size of $\mathcal{B}$, altogether quadratic run-time $\rightarrow$ unacceptable for millions of states.

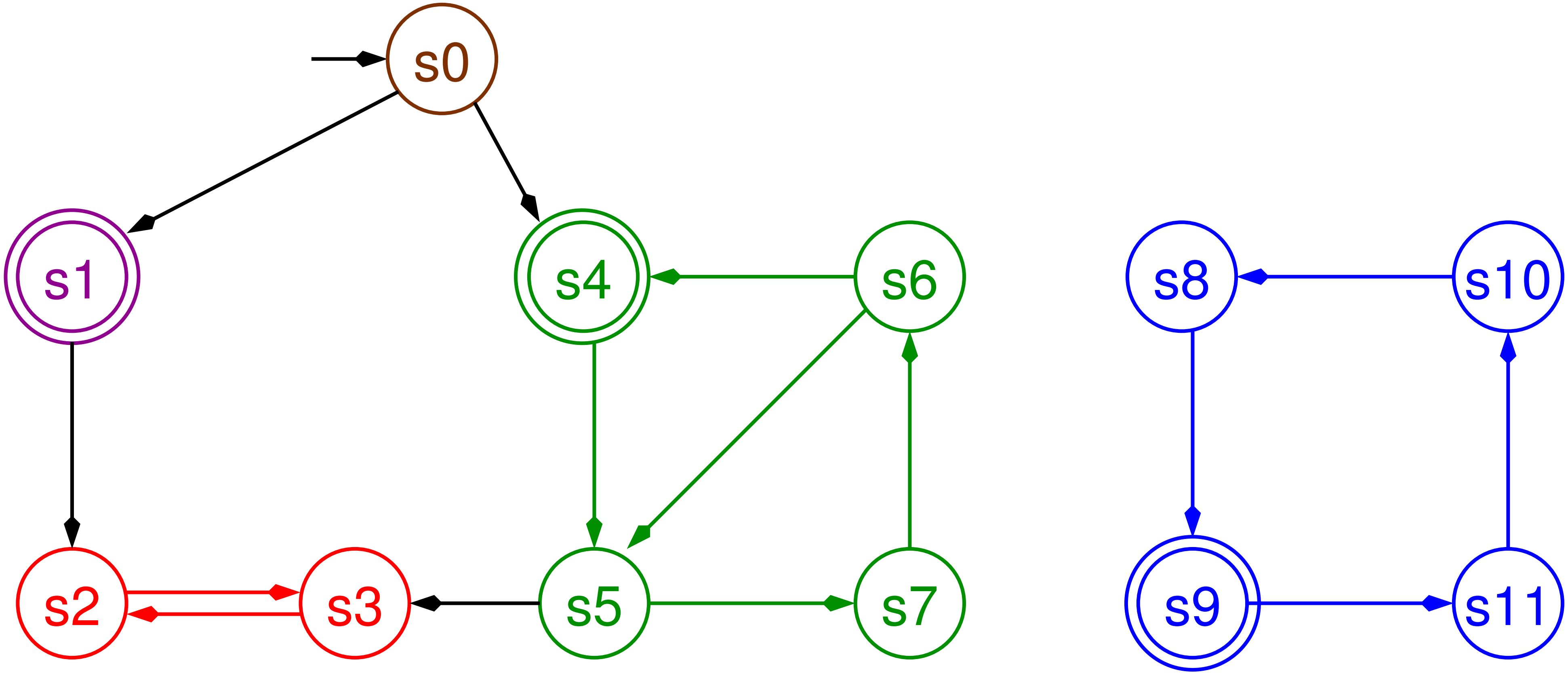
Strongly connected components

$C \subseteq S$ is called a strongly connected component (SCC) iff

- $s \rightarrow^* s'$ for all $s, s' \in C$;
- C is maximal w.r.t. the above property, i.e. there is no proper superset of C satisfying the above.

An SCC C is called trivial if $|C| = 1$ and for the unique state $s \in C$ we have $s \not\rightarrow s$ (single state without loop).

Example: SCCs



The SCCs $\{s_0\}$ and $\{s_1\}$ are trivial.

Depth-first search (basic version)

```
nr = 0;
hash = {};
dfs(s0);
exit;

dfs(s) {
   add s to hash;
   nr = nr+1;
   s.num = nr;

   for (t in succ(s)) {
      // deal with transition s -> t
      if (t not yet in hash) { dfs(t); }
   }
}
```

Memory usage

Global variables: counter *nr*, hash table for states

Auxiliary information: “DFS number” *s.num*

search path: Stack for memorizing the “unfinished” calls to *dfs*

Solution (1): based on SCCs

The algorithm of Tarjan (1972) can identify the SCCs in linear time (i.e. proportional to $|S| + |\delta|$).

Said algorithm is a slight extension of basic DFS with some additional constant-time operations on each state and transition.

Given the SCCs, one can then check if there exists a non-trivial SCC containing an accepting state.

Solution (2): nested DFS

Algorithm proposed by Courcoubetis, Vardi, Wolper, Yannakakis (1992).

The nested-DFS algorithm is an alternative requiring only two bits per state.

States are “white” initially.

A first DFS makes all the states that it visits blue.

Whenever the first (blue) DFS backtracks from an *accepting* state s, it starts a second (red) DFS to see if there is a cycle around s.

The red DFS only visits states that are not already red (including from a previous visit). Thus, every state and edge are considered at most twice.

Nested depth-first search: Algorithm

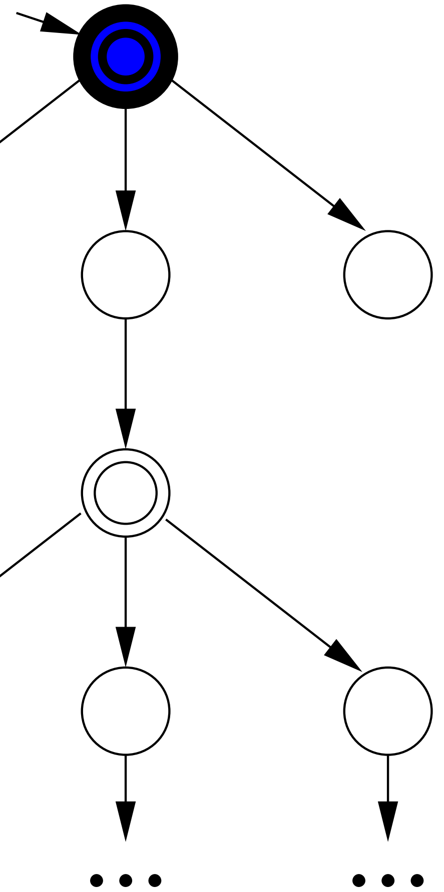
```
hash = {};
blue(s0);
report "no accepting run"

blue(s) {
   add (s,0) to hash;
   for t in succ(s)
      if (t,0) not in hash { blue(t) }
   if s is accepting and (s,1) not in hash { seed=s; red(s) }
}

red(s) {
   add (s,1) to hash;
   for t in succ(s)
      if t=seed { report "accepting run found"; exit }
      if (t,1) not in hash { red(t) }
}
```

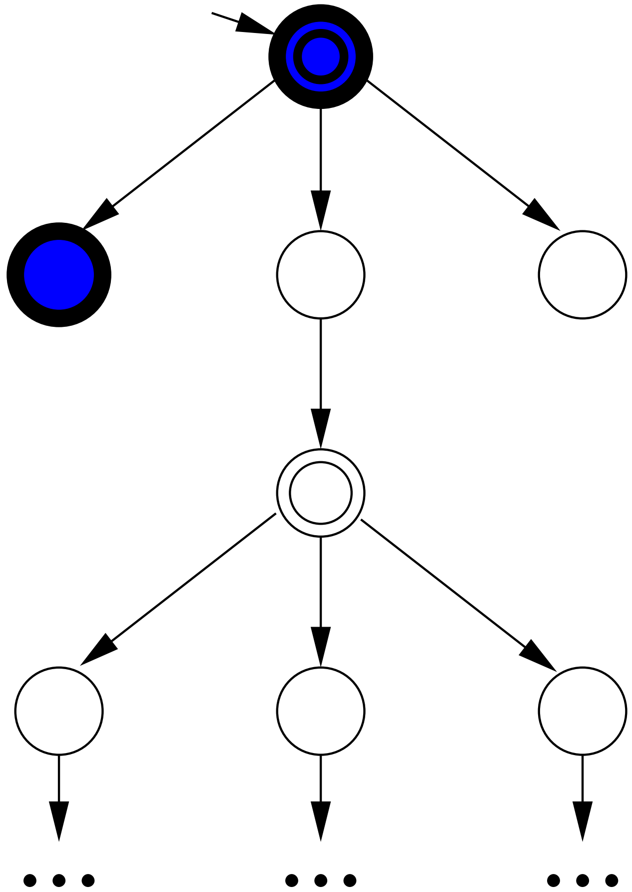
Nested DFS: Example

Blue phase: Start at initial state.



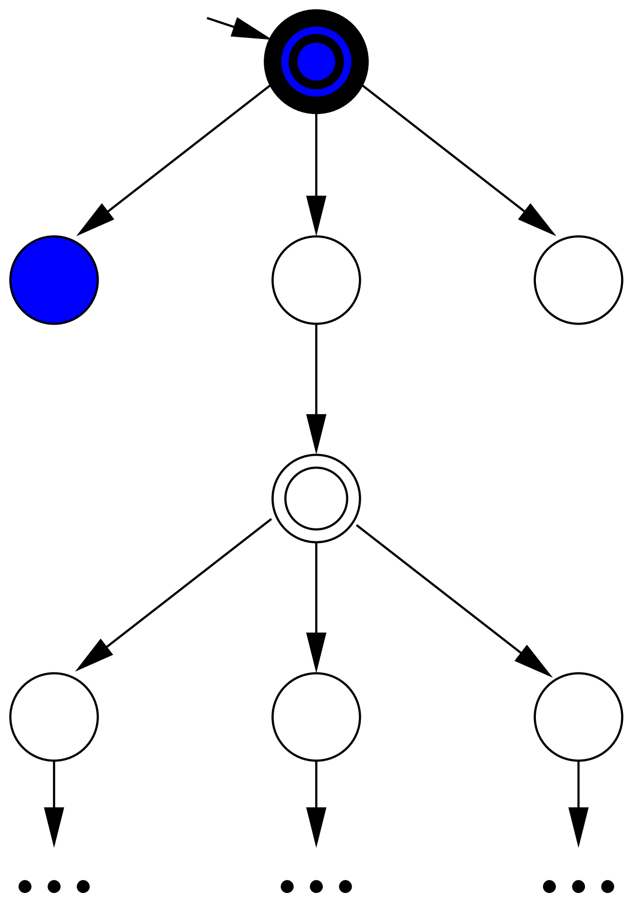
Nested DFS: Example

Visit states depth-first, colouring them blue.



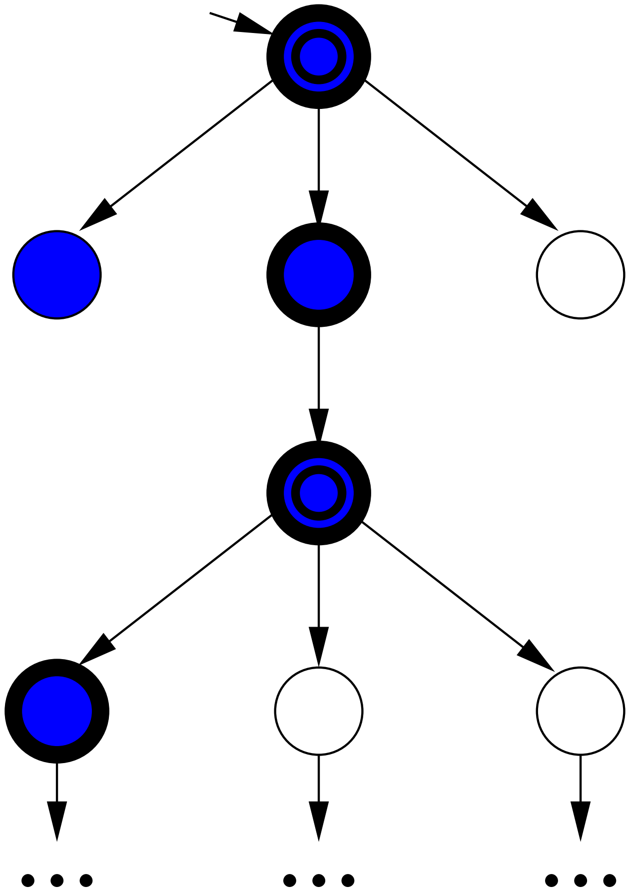
Nested DFS: Example

Simply backtrack from non-accepting states.



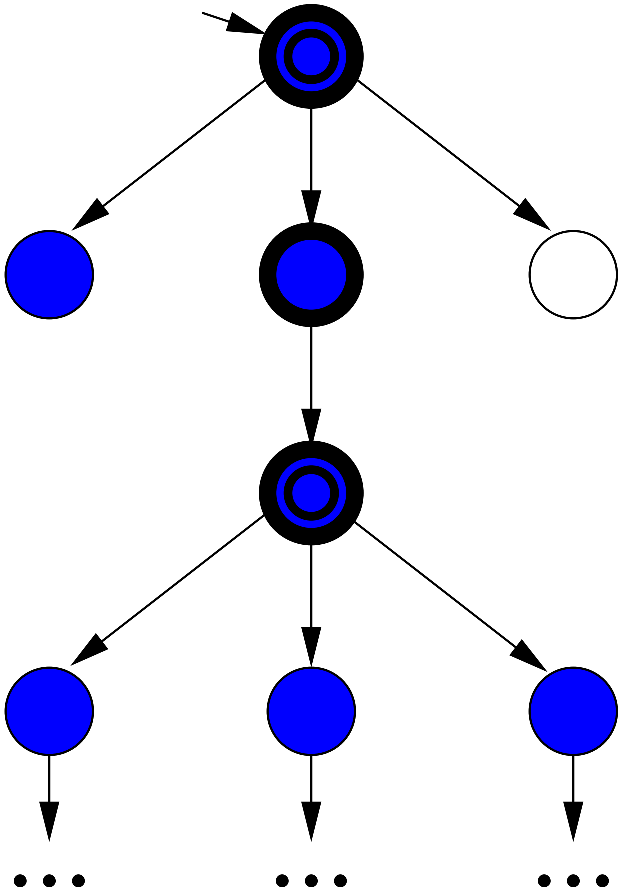
Nested DFS: Example

Continue blue search . . .



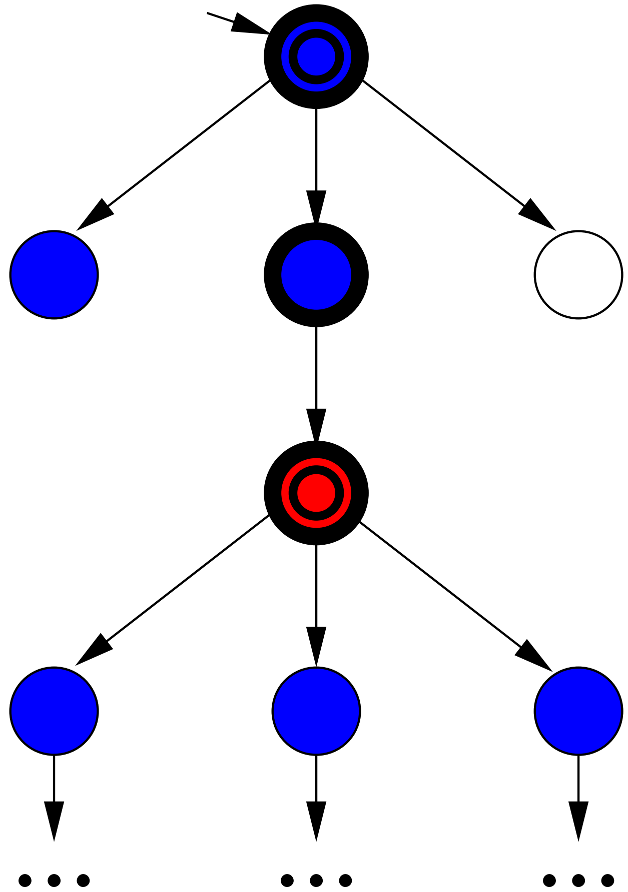
Nested DFS: Example

Continue blue search until backtracking from an accepting state.



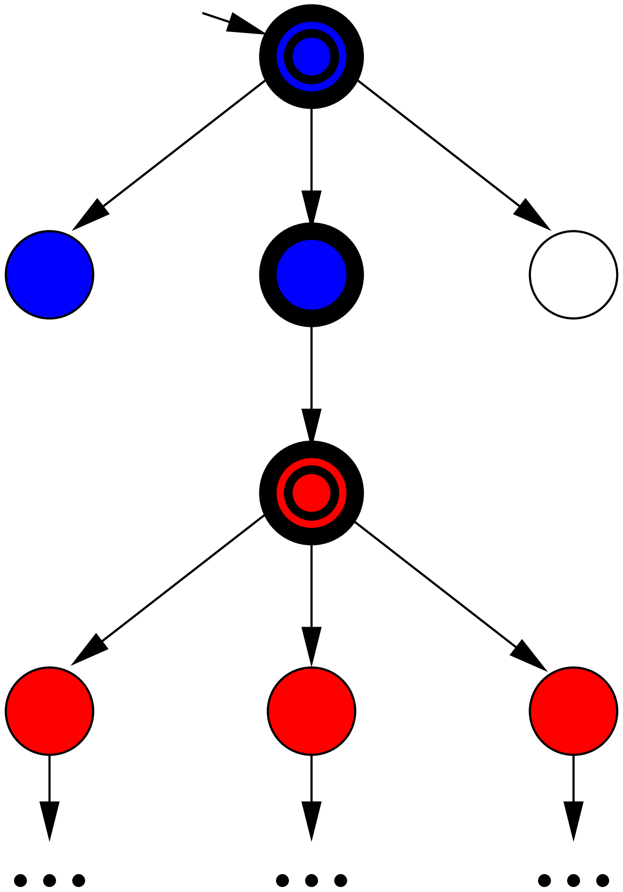
Nested DFS: Example

Before backtracking, start a “red” DFS . . .



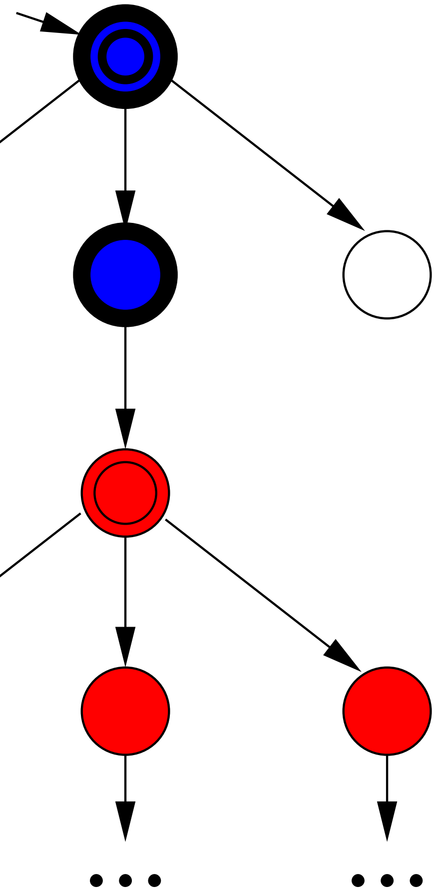
Nested DFS: Example

. . . that searches for a loop back to that accepting state.



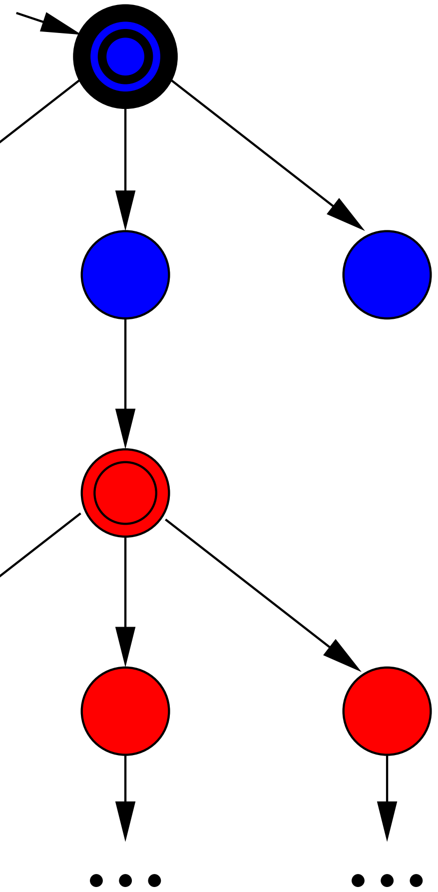
Nested DFS: Example

If red search is unsuccessful, backtrack.



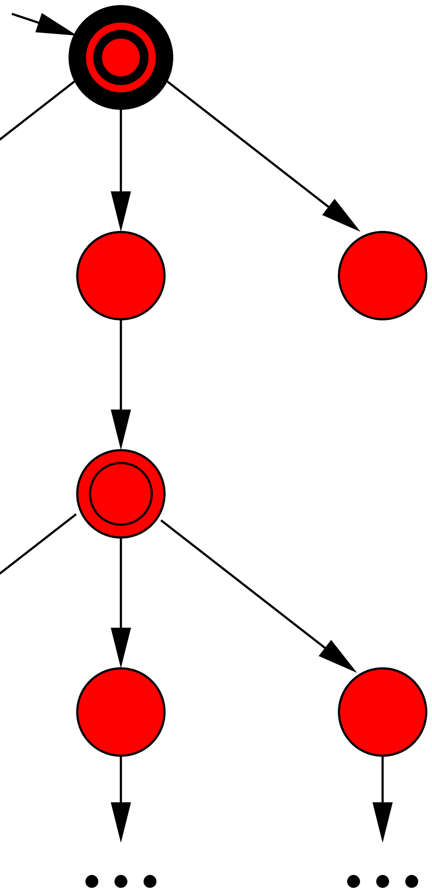
Nested DFS: Example

Carry on . . .



Nested DFS: Example

Future red searches only consider non-red states.



Properties of Nested DFS

Very economic in terms of memory

Implemented in state-of-the-art tools like Spin

Can be combined with further optimization (partial-order reduction)

Tends to prefer long counterexamples “deep down” in the state graph

$\rightarrow$ variants of Tarjan (not shown) can identify counterexamples more quickly, but are less economic on memory and more difficult to combine with other optimizations