Finitary decision structures in infinite games

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Introduction

Infinite duration games are useful:

- verification;
- synthesis.

Goal: transform infinite duration games into finite duration ones.

Zero-sum games are well known. It's not the case of non-zero-sum games.

Introduction: the problem with non-zero-sum games

Impossibility to make assumptions about the strategies used by the players

Players could:

- play as in zero-sum games;
- cooperate;
- voluntary loose...

This raises problems for our transformation. The solution is our main contribution: the power matrix.

About games

A *game* is constituted of a *game graph* and conditions that the players want to ensure.

A *play* is a sequence of positions in a game.

A *strategy* for a player is a function which let the player know what he has to do at his turn.

Plan

1. First intuition: use outcome equivalence
2. Too many equivalence classes
   - Fundamental lemma
   - Proof and remarks
3. Our solution: the power matrix
   - Power matrix and power graph
   - How to use power matrix

Intuition

Represent a 2-player game G as a matrix where:

- the rows and the columns are strategies;
- the entries are the associated outcomes.

Each player chooses a strategy. The outcome is the entry corresponding to the chosen row and the chosen column.

In fact, the strategies in this matrix correspond to equivalence classes over strategies in G.

For n players the matrix will have n dimensions.

Outcome equivalence

Strategies should be equivalence classes for the following equivalence relation:

Outcome equivalence

Two strategies s^i and r^i are *outcome equivalent* $\iff \forall s^{-i}$, the only play consistent with the profile of strategies (s^i, s^{-i}) has exactly the same outcome as the only play consistent with the profile of strategies (r^i, s^{-i}).

To be usable, the matrix has to be finite.

A problematic observation

Lemma

There exist games with infinitely many strategies that are not outcome equivalent.

Due to this, it is, in general, impossible to construct a finite matrix as described before.

Proof and remarks

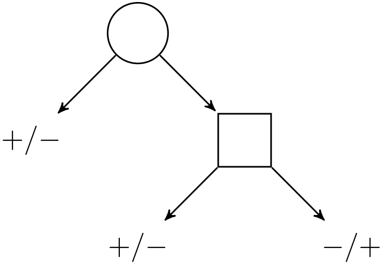
The idea of the proof is to construct:

- an infinite sequence of strategies for player 0;
- a corresponding sequence of strategies for player 1;

such that it is easy to show that all these strategies are in different equivalence classes.

The proof is only based on the structure of the game graph.

The case of zero-sum games



Power matrix

The *power matrix* of a game G is a matrix:

- the rows and the columns are called power strategies;
- the entries are sets of outcomes associated to these strategies.

Power strategies correspond to equivalence classes for a given equivalence.

Power matrix

The equivalence used is based on sets of outcomes, not only singletons.

Lemma

The power matrix of every game is finite.

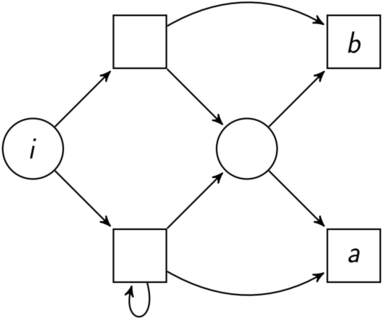
Proved just by counting the sets of outcomes.

Power graph

A *power* graph is a graph representation of a power matrix.

- *Imperfect information* represents that the players choose their strategies at the same time.
- *Non-determinism* is used to choose an outcome in the final set of outcomes.

Power matrix and power graph: example



	$\{a, b\}$	$\{\infty, b\}$	$\{\infty, a, b\}$
$\{a, \infty\}$	$\{a\}$	$\{\infty\}$	$\{\infty\}$
$\{a, \infty, b\}$	$\{a, b\}$	$\{\infty\}$	$\{\infty\}$
$\{b\}$	$\{b\}$	$\{b\}$	$\{b\}$
$\{a, b\}$	$\{a, b\}$	$\{b\}$	$\{a\}$

Power matrix and power graph: example

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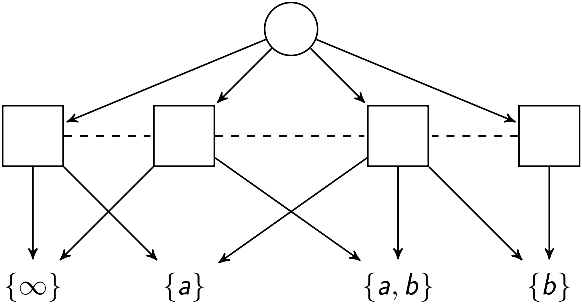
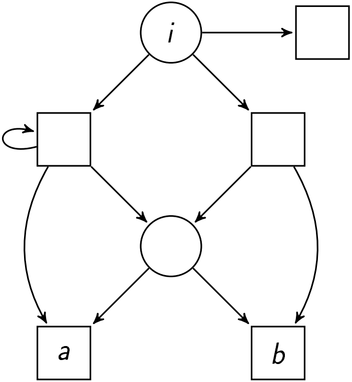
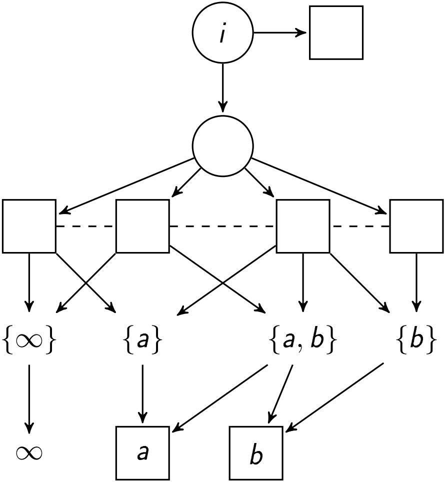


Replacing components

Components of a game can be replaced by power graphs.



The new game graph



Our contribution

We introduced:

- the concept of power matrix and power graph;
- the method of replacing components of a game.

We also applied these to *guarantee games*.

The *dominance relation* between strategies is preserved by the replacement of components.

Further work

Replace components in:

- non-zero-sum games, like parity games;
- games with imperfect information.