Langages formels
TD n°5

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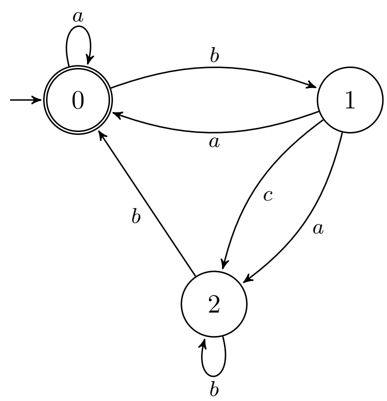
We say an MSO (Monadic Second Order logic) formula and a finite automaton are equivalent if they recognize the same language. You can use the notations $\phi \Rightarrow \psi$ for $\neg\phi \vee \psi$, $\phi \wedge \psi$ for $\neg(\neg\phi \vee \neg\psi)$, $\forall x.\ \phi$ for $\neg\exists x.\ \neg\phi$ and similarly $\forall X.\ \phi$ for $\neg\exists X.\ \neg\phi$.

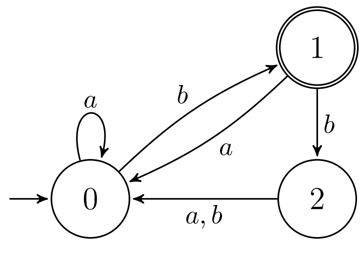
Exercise 1 : Automata $\leftrightarrow$ MSO

Give an MSO formula equivalent to the following automaton $\mathcal{A}$:



Exercise 2 : Recognizability by monoid

1. Show that a language of Σ^* is regular iff there exists a finite monoid $(M, \times)$, a morphism $\mu : (\Sigma^*, \cdot) \to (M, \times)$, and a set $P \subseteq M$ such that $L = \mu^{-1}(P)$. (*Hint for $\Rightarrow$: consider the monoid of functions of $Q \to Q$ and the composition operation*)
2. Give a finite monoid M, a morphism φ and a subset P of M which recognize the language accepted by the following automaton :



1 Working further on MSO

Exercise 3 : MSO and transitive closures

Let R be a binary relation over words which can be defined with MSO, that is such that there is an MSO formula $\varphi(x_1, x_2)$ with two free variables x_1, x_2 such that a pair of positions (i, j) of a word w is in R iff $w, x_1 \mapsto i, x_2 \mapsto j \models \varphi(x_1, x_2)$.

1. Build an MSO formula defining the reflexive transitive closure of an MSO-defined binary relation.
2. For every binary MSO-definable relation R, write a closed MSO formula which is evaluated to true over a word iff the pair of the first and last positions of the word is in the reflexive transitive closure of R.
3. Give a closed MSO formula recognizing the language of words of length a multiple a three.

2 Working further on monoids

Exercise 4 : Closure by monoids

1. Using monoids, prove that regular languages are closed by union, intersection, and complement.
2. Prove that if L is regular, the language of its kth roots $\mathfrak{R}_k(L) = \{ u : u^k \in L \}$ is also regular for $k > 1$.
3. Using monoids, show that regular languages are closed by concatenation. (*Hint : if L_1 and L_2 are recognized by $\varphi_1 : \Sigma^* \to M_1$ and $\varphi_2 : \Sigma^* \to M_2$ respectively, consider the morphism $\psi \colon \Sigma^* \to \mathfrak{P}(M_1 \times M_2)$ defined by $\psi(w) = \{ (\varphi_1(u), \varphi_2(v)) : w = uv \}$, giving $\mathfrak{P}(M_1 \times M_2)$ a suitable composition law and neutral element*)
4. Using monoids, show that regular languages are closed by quotient : if L is regular and $K \subseteq \Sigma^*$, then $K^{-1}L$ and LK^{-1} are regular.
5. Prove that the regular languages are closed by inverse substitution. That is, if A and B are two finite alphabets, $L \subseteq B^*$ a regular language and $f : A^* \to \mathfrak{P}(B^*)$ such that for every letter a, $f(a)$ is regular, then prove that $f^{-1}(L) = \{ u \in A^* : f(u) \subseteq L \}$ is regular.

Exercise 5 : Aperiodic languages

A language is *aperiodic* iff its syntactic monoid $(M, \cdot)$ is aperiodic, i.e. there exists $n \in \mathbb{N}$ such that for all $x \in M$, $x^n = x^{n+1}$.

1. A finite deterministic complete automaton has a *counter* if there exists $n > 1$, a sequence of distinct states $q_0, \ldots, q_{n-1}$ and a word $w \in \Sigma^*$ such that $\delta(q_i, w) = q_{i+1 \bmod n}$ for all $i \in \{ 0, \ldots, n-1 \}$.
   Show that $L \subseteq \Sigma^*$ is aperiodic iff its minimal automaton has no counter.
2. Show that if a language is *star-free*, i.e. in the smallest class containing the letters of the alphabet and closed by union, concatenation, and complement, then it is aperiodic. [1]
3. For the following languages, show if it is aperiodic or not :

   (a) $(ab)^*$,
   (b) $(aa)^*$,
   (c) $(a(ab)^*b)^*$,
   (d) $(ab + ba)^*$.

3 Additional exercise

Exercise 6 : Selection property

A morphism $\mu : A^* \to B^*$ has the *selection property* iff for every regular language L, there exists a regular language $K \subseteq L$ such that μ is injective over K and $\mu(K) = \mu(L)$. The goal of this exercise is to show that every morphism has the selection property.

1. Show that all injective morphisms have the selection property.
2. Show that if morphisms μ and ν have the selection property, then the morphism $\mu \circ \nu$ also has it.

We call *projection* a morphism $\pi : A^* \to B^*$ such that for every letter $a \in A$, $\pi(a) = a$ or $\pi(a) = \varepsilon$.

3. Show that for every morphism $\mu : A^* \to B^*$, there exists an alphabet C, an injective morphism $\iota : A^* \to C^*$ and a projection $\pi : C^* \to B^*$ such that $\mu = \pi \circ \iota$.

We call *elementary projection* a projection $\pi : A^* \to B^*$ such that there exists a unique letter $a \in A$ such that $\pi(a) = \varepsilon$.

4. Show that every projection is the composition of elementary projections.
5. Show that all elementary projection has the selection property.
6. Conclude.

1. The converse also holds, but is much harder to prove.