An Introduction to Capacities, Games, and Previsions

Jean Goubault-Larrecq

Supported by the ACI NIM Geocal, the INRIA ARC ProNoBis

GDR IM'07 — Feb 02, 2007

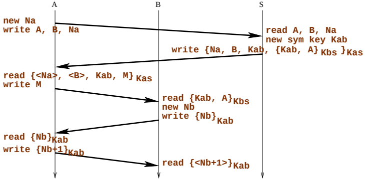
Outline

Outline

Our Goal

Verifying cryptographic protocols. E.g.,

$$\boxed{\begin{array}{lll}
1. & A \longrightarrow S : & A, B, N_a \\
2. & S \longrightarrow A : & \{N_a, B, K_{ab}, \{K_{ab}, A\}_{K_{bs}}\}_{K_{as}} \\
3. & A \longrightarrow B : & \{K_{ab}, A\}_{K_{bs}} \\
4. & B \longrightarrow A : & \{N_b\}_{K_{ab}} \\
5. & A \longrightarrow B : & \{N_b + 1\}_{K_{ab}}
\end{array}}$$



How To Verify A Protocol

The Dolev-Yao model: all agents (A, B, S) run in a context (= adversary) C.

- C can do plenty of things (encrypt, decrypt, forge, redirect, drop messages);
- C aims at reaching a so-called Bad state (e.g., where the secret K_{ab} is known to C).

How To Verify A Protocol

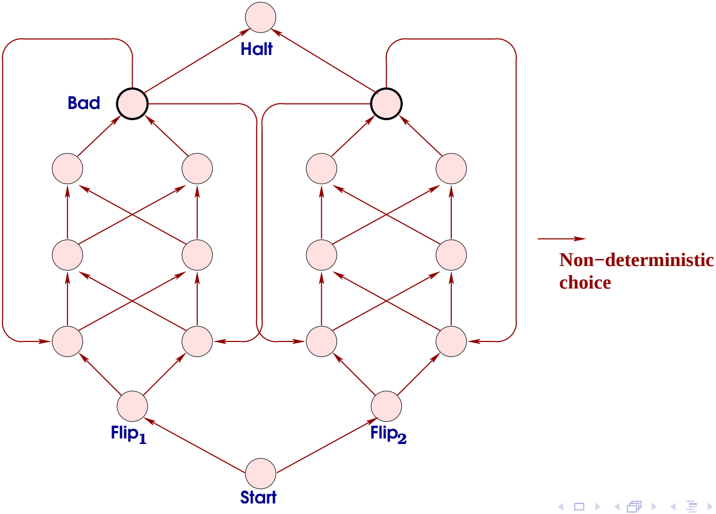
The Dolev-Yao model: all agents (A, B, S) run in a context (= adversary) C.

- C can do plenty of things (encrypt, decrypt, forge, redirect, drop messages);
- C aims at reaching a so-called Bad state (e.g., where the secret K_{ab} is known to C).

To verify:

- Draw a (big) graph.
  - *States* q are (big) tuples describing the state of the world (where each agent is currently at, what the values of local variables are, what messages C has got hold of);
  - *Transitions* $q \xrightarrow{\ell} q'$ lists when the world can evolve from q to q' (doing action ℓ).
- Check whether Bad is reachable from one of the initial states.

Non-Deterministic Choice Only: Automata

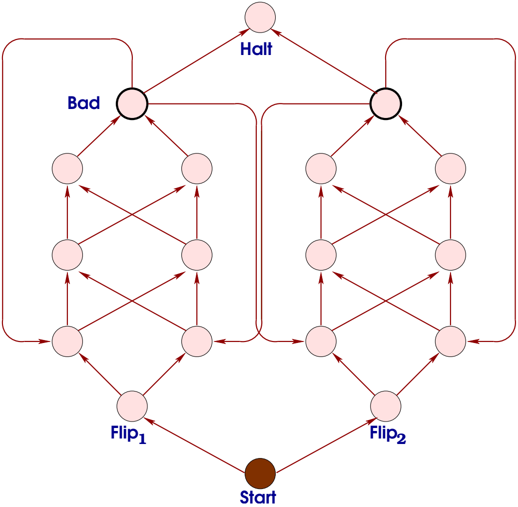


Non-Deterministic Choice: Semantics

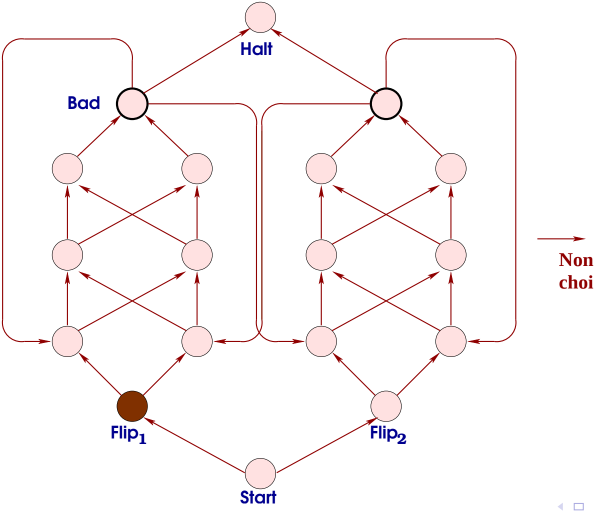
C plays as follows:

- Start at. . . Start;
- Pick some next state;
- Repeat. . .
- . . . So as to reach some set of goal states (fat circles here).

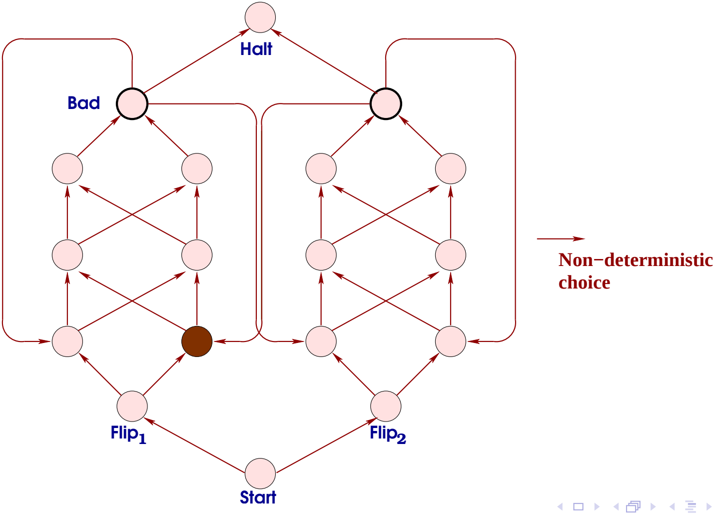
Trying to Reach a Bad State



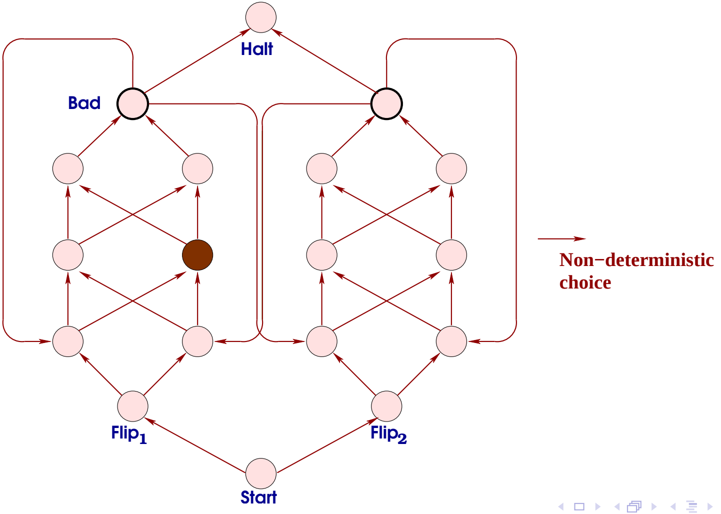
Trying to Reach a Bad State



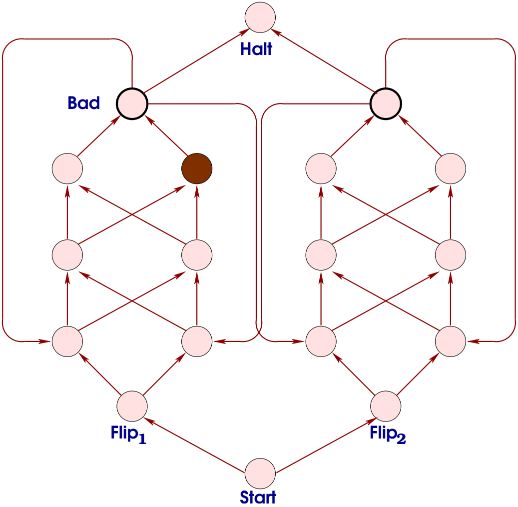
Trying to Reach a Bad State



Trying to Reach a Bad State

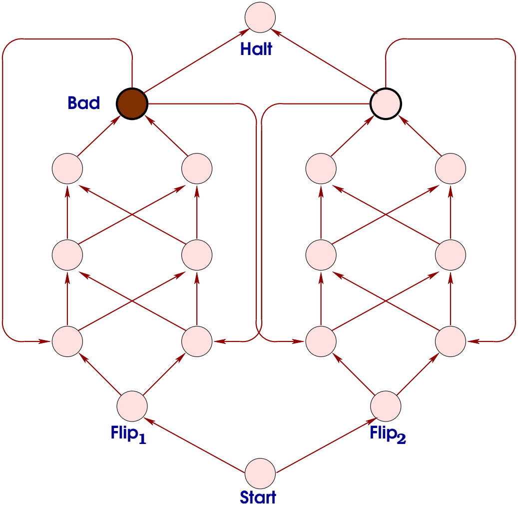


Trying to Reach a Bad State



Non−deterministic choice

Trying to Reach a Bad State



→ **Non−deterministic choice**

Remarks

- Model is relatively simple (in particular, no probabilities);
- But infinite-state: there are infinitely many states in general.

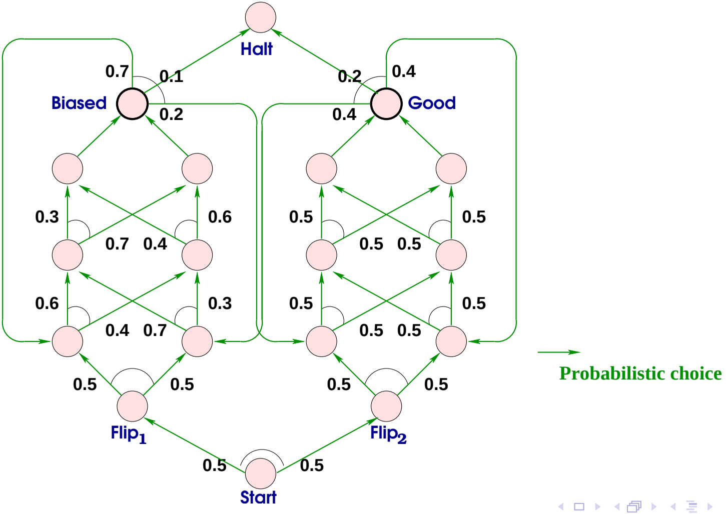
Case In Point: Probabilistic Choice

Some protocols require honest agents to draw their next move at random.

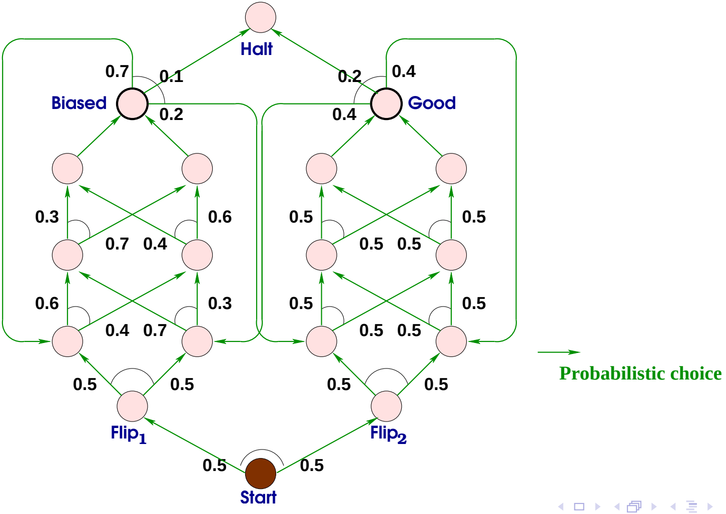
Case In Point: Probabilistic Choice

Some protocols require honest agents to draw their next move at random.
E.g., Hermann's protocol for the dining philosophers.
CSMA/CD (Ethernet).
Various self-stabilization protocols.

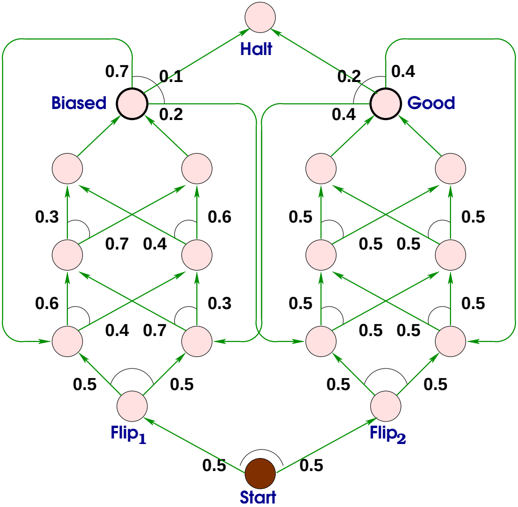
A (Finite) Markov Chain



Start

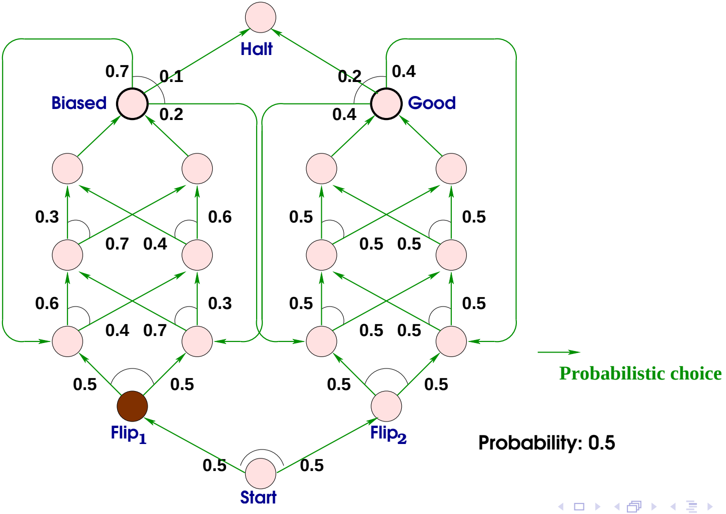

Halt
0.7
0.1
0.2
0.4
Biased
0.2
0.4
Good
0.3
0.6
0.7
0.4
0.5
0.5
0.5
0.5
0.6
0.3
0.4
0.7
0.5
0.5
0.5
0.5
0.5
0.5
0.5
0.5
Flip$_1$
Flip$_2$
0.5
0.5
Start
Probabilistic choice

Flip a Coin


Halt
0.7
0.1
0.2
0.4
Biased
0.2
0.4
Good
0.3
0.6
0.7
0.4
0.5
0.5
0.5
0.5
0.6
0.3
0.4
0.7
0.5
0.5
0.5
0.5
0.5
0.5
0.5
0.5
Flip$_1$
Flip$_2$
0.5
0.5
Start

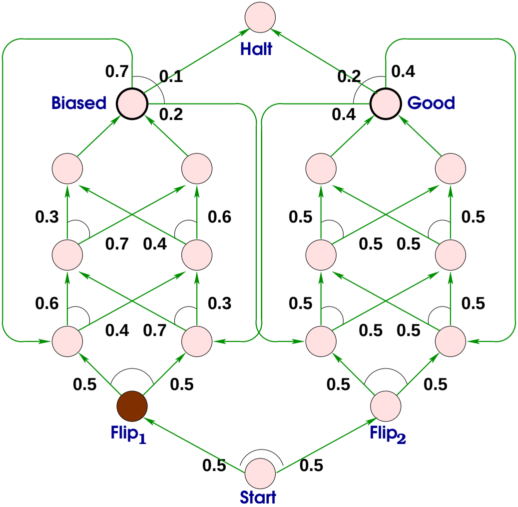

Probabilistic choice

Advance



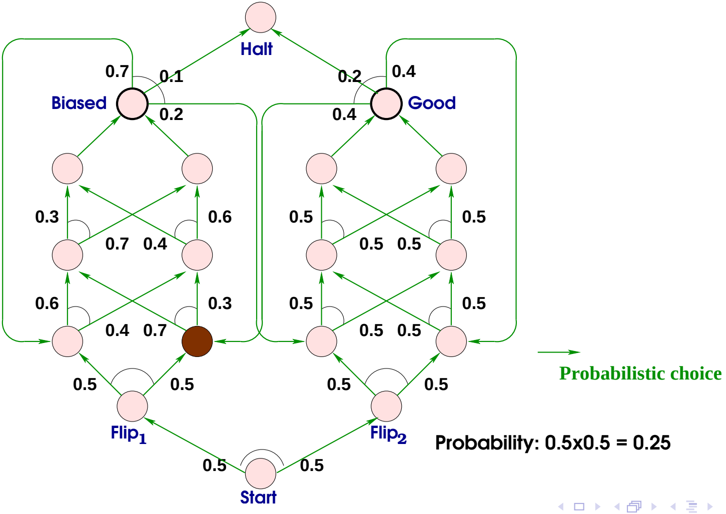
Probability: 0.5

Flip a Coin



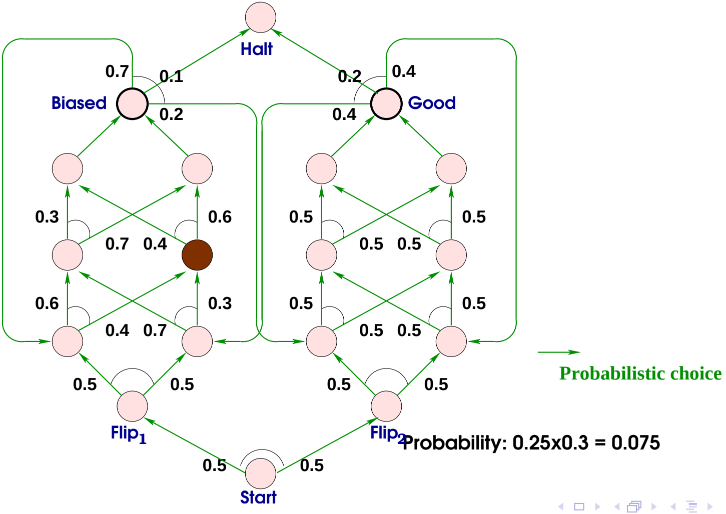

Probabilistic choice

Advance



Probability: 0.5x0.5 = 0.25

Advance



Probability: 0.25x0.3 = 0.075

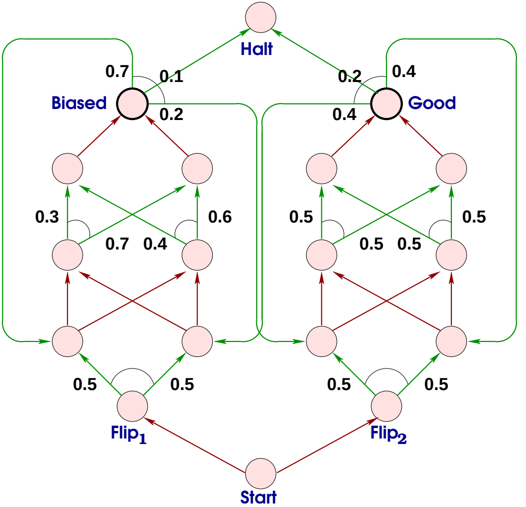
Stochastic turn-based 2-player games

In some cryptographic protocols,

- Honest agents (P) play at random (or deterministically);
- Adversaries (C) play in a demonic way (one form of non-determinism);

Also present in Arthur-Merlin games (complexity theory) and interactive proofs.

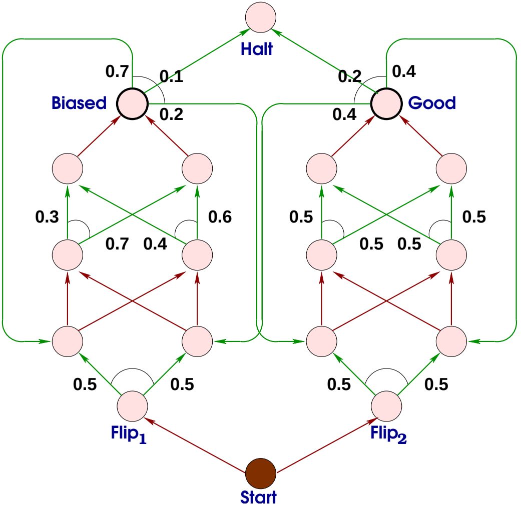
A Stochastic Game




Non−deterministic (demonic) choice (by adversary)


Probabilistic choice (by program)

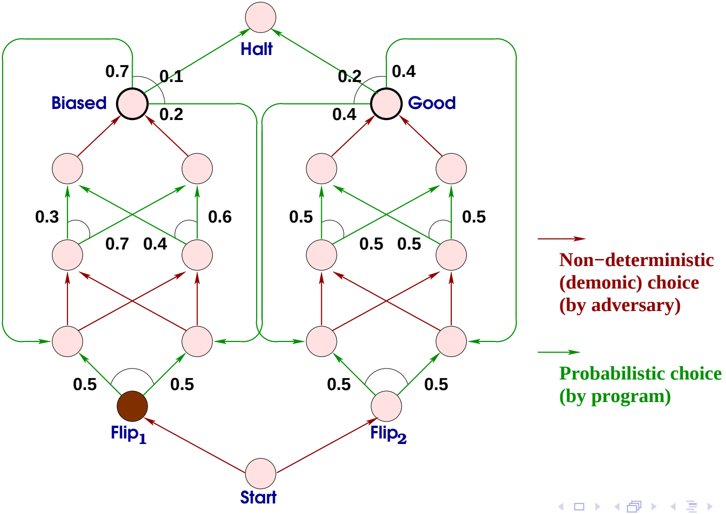
Start



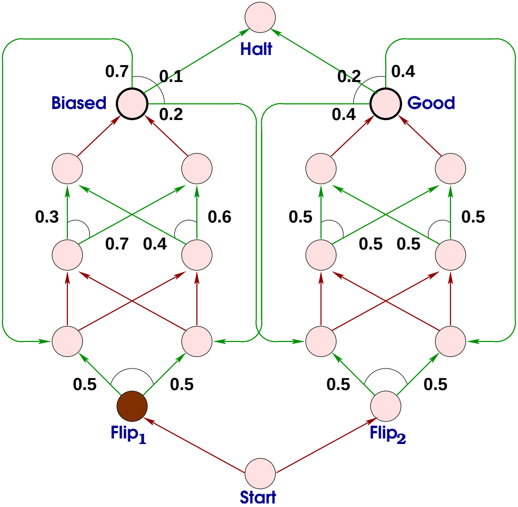
Non−deterministic (demonic) choice (by adversary)

Probabilistic choice (by program)

C's Turn: Malevolently Chooses Biased Side



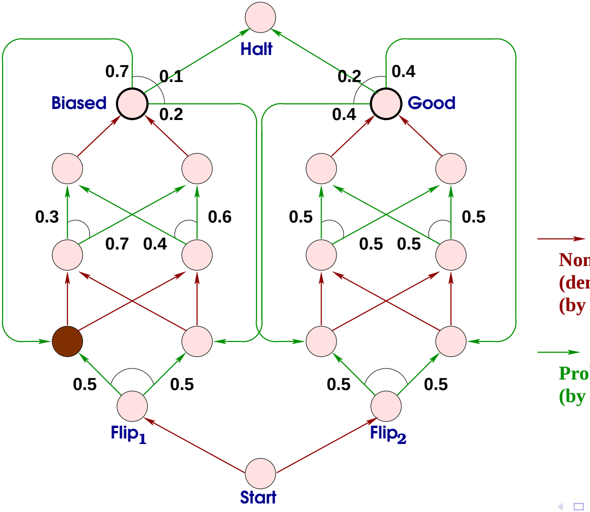
P's Turn: Flipping a Coin



Non−deterministic (demonic) choice (by adversary)

Probabilistic choice (by program)

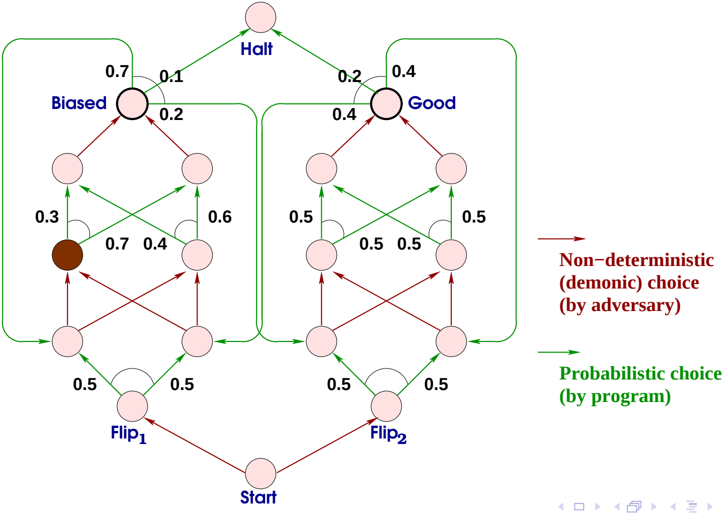
P's Turn: Advancing



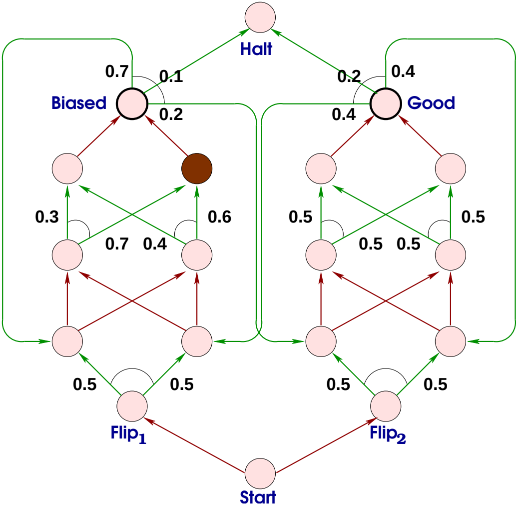
Non−deterministic
(demonic) choice
(by adversary)

Probabilistic choice
(by program)

C's Turn: Picking Most Biased Side



P's Turn



Non−deterministic (demonic) choice (by adversary)

Probabilistic choice (by program)

Our Challenge

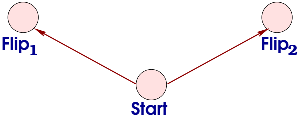
- How do you model this when state space is infinite? (E.g., a topological space, $\mathbb{R}^n$, a cpo.)
- How do you do model-checking? For what modal logic?
- How do you evaluate least average payoffs?
- How do you characterize contextual equivalence? bisimulation?

Outline

“Preprobabilities”

An idea by F. Laviolette and J. Desharnais: simulate non-deterministic choice by some form of non-additive probabilistic choice: “Preprobabilities”.

E.g., demonic choice:

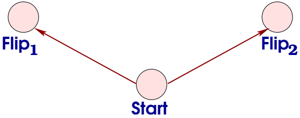


Preprobability that, from **Start**, we jump into some set U:

“Preprobabilities”

An idea by F. Laviolette and J. Desharnais: simulate non-deterministic choice by some form of non-additive probabilistic choice: “Preprobabilities”.

E.g., demonic choice:



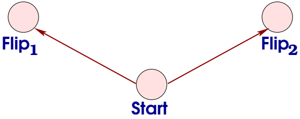
Preprobability that, from **Start**, we jump into some set U:

U	proba.	
$\emptyset$	0	
$\{\textbf{Flip}_1\}$		

"Preprobabilities"

An idea by F. Laviolette and J. Desharnais: simulate non-deterministic choice by some form of non-additive probabilistic choice: "Preprobabilities".

E.g., demonic choice:



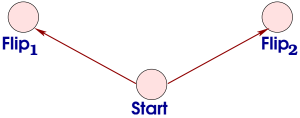
Preprobability that, from **Start**, we jump into some set U:

U	proba.	
$\emptyset$	0	
$\{$**Flip**$_1\}$	0	C can always pick **Flip**$_2$

“Preprobabilities”

An idea by F. Laviolette and J. Desharnais: simulate non-deterministic choice by some form of non-additive probabilistic choice: “Preprobabilities”.

E.g., demonic choice:



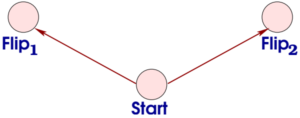
Preprobability that, from **Start**, we jump into some set U:

U	proba.	
$\emptyset$	0	
$\{$**Flip**$_1\}$	0	C can always pick **Flip**$_2$
$\{$**Flip**$_2\}$	0	C can always pick **Flip**$_1$

“Preprobabilities”

An idea by F. Laviolette and J. Desharnais: simulate non-deterministic choice by some form of non-additive probabilistic choice: “Preprobabilities”.

E.g., demonic choice:



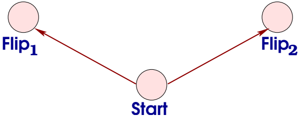
Preprobability that, from **Start**, we jump into some set *U*:

U	proba.	
$\emptyset$	0	
$\{\mathbf{Flip}_1\}$	0	C can always pick $\mathbf{Flip}_2$
$\{\mathbf{Flip}_2\}$	0	C can always pick $\mathbf{Flip}_1$
$\{\mathbf{Flip}_1, \mathbf{Flip}_2\}$		

"Preprobabilities"

An idea by F. Laviolette and J. Desharnais: simulate non-deterministic choice by some form of non-additive probabilistic choice: "Preprobabilities".

E.g., demonic choice:



Preprobability that, from **Start**, we jump into some set U:

U	proba.	
$\emptyset$	0	
$\{\mathbf{Flip}_1\}$	0	C can always pick $\mathbf{Flip}_2$
$\{\mathbf{Flip}_2\}$	0	C can always pick $\mathbf{Flip}_1$
$\{\mathbf{Flip}_1, \mathbf{Flip}_2\}$	1	C cannot escape it!

Unanimity Games

Definition

The unanimity game $\mathfrak{u}_Q$ is the set function such that:

$$\mathfrak{u}_Q(U) = \begin{cases} 1 & \text{if } Q \subseteq U \\ 0 & \text{otherwise} \end{cases}$$

Unanimity Games

Definition

The unanimity game $\mathfrak{u}_Q$ is the set function such that:

$$\mathfrak{u}_Q(U) = \begin{cases} 1 & \text{if } Q \subseteq U \\ 0 & \text{otherwise} \end{cases}$$

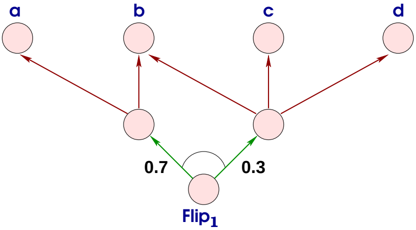
Non-deterministic (demonic) choice between Flip_1 and Flip_2:

$$\mathfrak{u}_{\{\text{Flip}_1, \text{Flip}_2\}}$$

(This notion is a special case of a “cooperative game with transferable utility function” in economics.)

Simple Belief Functions

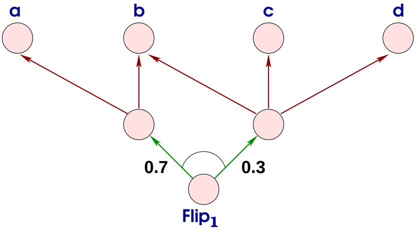
Mix (demonic) non-deterministic and probabilistic choice:



$$= 0.7\mathfrak{u}_{\{a,b\}} + 0.3\mathfrak{u}_{\{b,c,d\}}$$

Simple Belief Functions

Mix (demonic) non-deterministic and probabilistic choice:



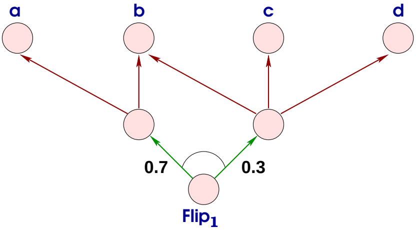
$$= 0.7\mathfrak{u}_{\{a,b\}} + 0.3\mathfrak{u}_{\{b,c,d\}}$$

Definition

A simple belief function is any $\sum_{i=1}^{n} a_i \mathfrak{u}_{Q_i}$, $a_i \in \mathbb{R}^+$, Q_i compact saturated.

Simple Belief Functions

Mix (demonic) non-deterministic and probabilistic choice:



$$= 0.7\mathfrak{u}_{\{a,b\}} + 0.3\mathfrak{u}_{\{b,c,d\}}$$

Definition

A simple belief function is any $\sum_{i=1}^{n} a_i \mathfrak{u}_{Q_i}$, $a_i \in \mathbb{R}^+$, Q_i compact saturated.

(Looks like strictly alternating probabilistic automata [SegalaLynch95], or as in [MisloveOuaknineWorrell03], except we flip first *then* choose non-deterministically.)

Axiomatization: Capacities, (Cooperative) Games

Let X be a topological space, $\Omega(X)$ its lattice of opens.
Note: we measure *opens*.

Definition

A capacity ν is a function $\Omega(X) \to \mathbb{R}^+$, with $\nu(\emptyset) = 0$.

- A game is a *monotonic* capacity: $U \subseteq V \Rightarrow \nu(U) \leq \nu(V)$;

Axiomatization: Capacities, (Cooperative) Games

Let X be a topological space, $\Omega(X)$ its lattice of opens.
Note: we measure *opens*.

Definition

A capacity ν is a function $\Omega(X) \to \mathbb{R}^+$, with $\nu(\emptyset) = 0$.

- A game is a *monotonic* capacity: $U \subseteq V \Rightarrow \nu(U) \leq \nu(V)$;
- A game is convex iff $\nu(U \cup V) \geq \nu(U) + \nu(V) - \nu(U \cap V)$;
  ($=$ for *valuations* [$\sim$ measures])

Axiomatization: Capacities, (Cooperative) Games

Let X be a topological space, $\Omega(X)$ its lattice of opens.
Note: we measure *opens*.

Definition

A capacity ν is a function $\Omega(X) \to \mathbb{R}^+$, with $\nu(\emptyset) = 0$.

- A game is a *monotonic* capacity: $U \subseteq V \Rightarrow \nu(U) \leq \nu(V)$;
- A game is totally convex (i.e., a *belief function*) iff:

$$\nu\left(\bigcup_{i=1}^{n} U_i\right) \geq \sum_{I \subseteq \{1,\ldots,n\}, I \neq \emptyset} (-1)^{|I|+1} \nu\left(\bigcap_{i \in I} U_i\right)$$

(would be $=$ for valuations: the *inclusion-exclusion principle*.)

Axiomatization: Capacities, (Cooperative) Games

Definition

A capacity ν is a function $\Omega(X) \to \mathbb{R}^+$, with $\nu(\emptyset) = 0$.

- A game is a *monotonic* capacity: $U \subseteq V \Rightarrow \nu(U) \leq \nu(V)$;
- A game is totally convex (i.e., a *belief function*) iff:

$$\nu\left(\bigcup_{i=1}^{n} U_i\right) \geq \sum_{I\subseteq\{1,\ldots,n\},I\neq\emptyset} (-1)^{|I|+1}\nu\left(\bigcap_{i\in I} U_i\right)$$

- A game is continuous iff $\nu\left(\bigcup_{i\in I}^{\uparrow} U_i\right) = \sup_{i\in I} \nu(U_i)$.

Axiomatization: Capacities, (Cooperative) Games

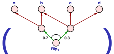
Definition

A capacity ν is a function $\Omega(X) \to \mathbb{R}^+$, with $\nu(\emptyset) = 0$.

- A game is a *monotonic* capacity: $U \subseteq V \Rightarrow \nu(U) \leq \nu(V)$;
- A game is totally convex (i.e., a *belief function*) iff:

$$\nu\left(\bigcup_{i=1}^{n} U_i\right) \geq \sum_{I \subseteq \{1,\ldots,n\}, I \neq \emptyset} (-1)^{|I|+1} \nu\left(\bigcap_{i \in I} U_i\right)$$

- A game is continuous iff $\nu\left(\bigcup^{\uparrow}_{i \in I} U_i\right) = \sup_{i \in I} \nu(U_i)$.

Lemma ()

Every simple belief function is a continuous belief function.

Conversely (1/2)

Definition

The Smyth powerdomain $\mathcal{Q}(X)$ of X is the set of all non-empty compact saturated subsets Q of X, ordered by $\supseteq$. Its Scott topology is generated by $\Box U = \{Q \in \mathcal{Q}(X) | Q \subseteq U\}$, $U \in \Omega(X)$.

$\Rightarrow$ A standard axiomatization of *demonic non-determinism*.

Conversely (1/2)

Definition

The Smyth powerdomain $\mathcal{Q}(X)$ of X is the set of all non-empty compact saturated subsets Q of X, ordered by $\supseteq$. Its Scott topology is generated by $\Box U = \{Q \in \mathcal{Q}(X) | Q \subseteq U\}$, $U \in \Omega(X)$.

Let X be a nice enough topological space (sober, locally compact; e.g., any finite space, $\mathbb{R}^n$, any continuous cpo,).

Theorem

For every continuous belief function ν on X, there is a unique continuous valuation ν^* ($\sim$ measure) on $\mathcal{Q}(X)$ such that $\nu(U) = \nu^*(\Box U)$ for all opens U.

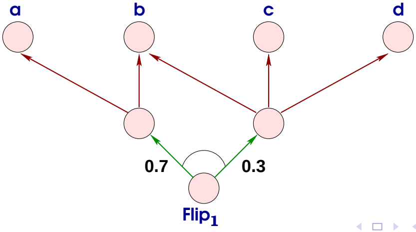
Conversely (2/2)

Let $\mathbf{V}(X)$ be the space of all continuous valuations, $\mathbf{Cd}(X)$ that of all continuous belief functions.

Corollary

$\mathbf{Cd}(X) \cong \mathbf{V}(\mathcal{Q}(X))$.

I.e., continuous belief functions $\cong$ probabilistic choice (possibly non-discrete) then demonic (possibly infinitely branching) non-deterministic choice.



The Choquet Integral [1953-54]

You can always integrate any (Scott-)continuous function $f : X \to \mathbb{R}^+$ along any game ν:

$$\oint_{x \in X} f(x) d\nu = \int_0^{+\infty} \nu(f^{-1}]t, +\infty[) dt$$

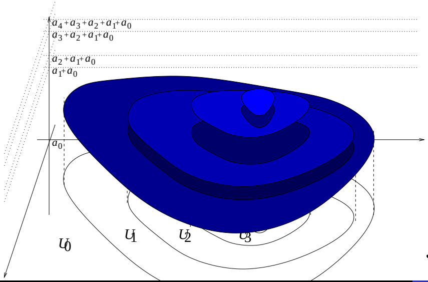
(An ordinary Riemann integral)

The Choquet Integral [1953-54]

You can always integrate any (Scott-)continuous function $f : X \to \mathbb{R}^+$ along any game ν:

$$\oint_{x \in X} f(x) d\nu \quad = \quad \int_0^{+\infty} \nu(f^{-1}]t, +\infty[) dt$$

(An ordinary Riemann integral)



$$\text{When } f \quad = \quad \sum_{i=1}^{n} a_i \chi_{U_i}$$

(a *step* function)

$$\oint_{x \in X} f(x) d\nu \quad = \quad \sum_{i=1}^{n} a_i \nu(U_i)$$

Properties of the Choquet Integral

Linear in ν

$$\oint_{x\in X} f(x)d\sum_{i=1}^{n} a_i\nu_i = \sum_{i=1}^{n} a_i \oint_{x\in X} f(x)d\nu_i$$

$$\oint_{x\in X} f(x)d\nu$$

Properties of the Choquet Integral

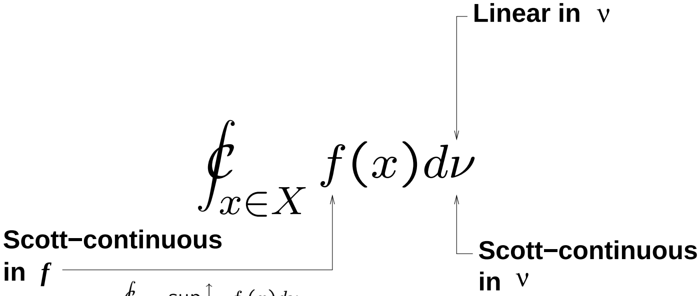
Linear in ν

$$\oint_{x\in X} f(x) d\nu$$

Scott−continuous in ν

$$\oint_{x\in X} f(x) d \sup{}^{\uparrow}_{i\in I} \nu_i = \sup{}^{\uparrow}_{i\in I} \oint_{x\in X} f(x) d\nu_i$$

Properties of the Choquet Integral



Linear in ν

$$\mathop{\int\!\!\!\!\!C}_{x\in X} f(x)\,d\nu$$

Scott−continuous in f

Scott−continuous in ν

$$\mathop{\int\!\!\!\!\!C}_{x\in X} \sup{}^{\uparrow}_{i\in I} f_i(x)\,d\nu = \sup{}^{\uparrow}_{i\in I} \mathop{\int\!\!\!\!\!C}_{x\in X} f_i(x)\,d\nu$$

(Provided ν is continuous)

Properties of the Choquet Integral

~~**Linear in *f***~~

$$\cancel{\oint_{x \in X} \sum_{i=1}^{n} a_i f_i(x) d\nu = \sum_{i=1}^{n} a_i \oint_{x \in X} f_i(x) d\nu}$$

(Unless ν is a valuation)

Linear in ν

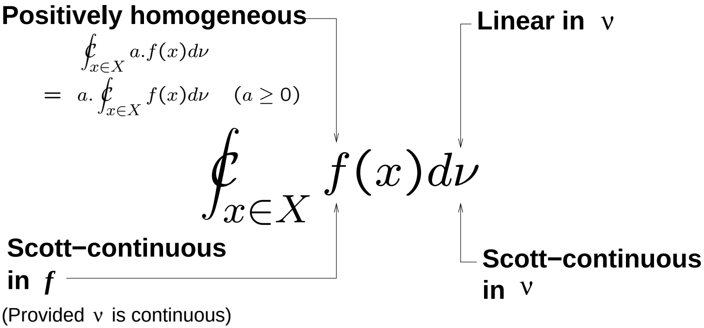
$$\oint_{x \in X} f(x) d\nu$$

Scott−continuous in *f*

(Provided ν is continuous)

Scott−continuous in ν

Properties of the Choquet Integral



Positively homogeneous

$$\oint_{x\in X} a.f(x)d\nu = a.\oint_{x\in X} f(x)d\nu \quad (a \geq 0)$$

Linear in ν

$$\oint_{x\in X} f(x)d\nu$$

Scott−continuous in f

(Provided ν is continuous)

Scott−continuous in ν

Properties of the Choquet Integral

Positively homogeneous
Additive on comonotonic f, g

$$\int\!\!\!\!\!C_{x\in X} f(x)+g(x)d\nu \qquad (f \smile g)$$

$$= \int\!\!\!\!\!C_{x\in X} f(x)d\nu + \int\!\!\!\!\!C_{x\in X} g(x)d\nu$$

Linear in ν

$$\int\!\!\!\!\!C_{x\in X} f(x)d\nu$$

Scott−continuous
in f

(Provided ν is continuous)

Scott−continuous
in ν

Properties of the Choquet Integral

Positively homogeneous
Additive on comonotonic f, g
Superadditive

(Provided ν is convex)

$$\oint_{x\in X} f(x)+g(x)d\nu \geq \oint_{x\in X} f(x)d\nu + \oint_{x\in X} g(x)d\nu$$

Linear in ν

$$\oint_{x\in X} f(x)d\nu$$

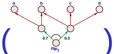
Scott−continuous in f

(Provided ν is continuous)

Scott−continuous in ν

Integrating Along a Belief Function

Lemma (



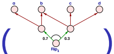
)

Let $\nu = \sum_{i=1}^{n} a_i \mathfrak{u}_{Q_i}$ *a simple belief function. Then:*

$$\mathrm{C}\!\!\!\int_{x \in X} f(x) d\nu \quad = \quad \sum_{i=1}^{n} a_i \min_{x \in Q_i} f(x)$$

In other words:

Integrating Along a Belief Function

Lemma (

)

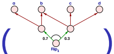
Let $\nu = \sum_{i=1}^{n} a_i \mathfrak{u}_{Q_i}$ *a simple belief function. Then:*

$$\mathrm{C}\!\!\!\int_{x \in X} f(x) d\nu \quad = \quad \sum_{i=1}^{n} a_i \min_{x \in Q_i} f(x)$$

In other words:

- P draws i at random, with probability a_i;

Integrating Along a Belief Function

Lemma ( **)**

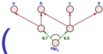
Let $\nu = \sum_{i=1}^{n} a_i \mathfrak{u}_{Q_i}$ *a simple belief function. Then:*

$$\mathrlap{\;C}\int_{x \in X} f(x) d\nu = \sum_{i=1}^{n} a_i \min_{x \in Q_i} f(x)$$

In other words:

- P draws i at random, with probability a_i;
- C then picks x from Q_i so as to minimize payoff $f(x)$.

Integrating Along a Belief Function

Theorem (

)

Let X be sober, locally compact, $\nu \in \mathbf{Cd}(X)$*.*

$$\mathop{\mathrm{C}\!\!\!\!\int}_{x \in X} f(x) d\nu \quad = \quad \mathop{\mathrm{C}\!\!\!\!\int}_{Q \in \mathcal{Q}(X)} \min_{x \in Q} f(x) d\nu^*$$

In other words:

- P draws $Q \in \mathcal{Q}(X)$ at random, with probability ν^*;
- C then picks x from Q so as to minimize payoff $f(x)$.

Further Developments

- One can also deal with angelic non-determinism, where C now helps (maximizes payoff);

Further Developments

- One can also deal with angelic non-determinism, where C now helps (maximizes payoff);
- $\Rightarrow$ plausibilities: $\mathbf{Pb}(X) \cong \mathbf{V}(\mathcal{H}_u(X))$, where $\mathcal{H}_u(X)$ is the Hoare (angelic) powerdomain;

Further Developments

- One can also deal with angelic non-determinism, where C now helps (maximizes payoff);
- $\Rightarrow$ plausibilities: $\mathbf{Pb}(X) \cong \mathbf{V}(\mathcal{H}_u(X))$, where $\mathcal{H}_u(X)$ is the Hoare (angelic) powerdomain;
- Also with chaotic non-determinism: estimates and (Heckmann's version of) the Plotkin powerdomain.

Ludic Transition Systems

Definition

A ludic transition system σ is a family of continuous maps $\sigma_\ell : X \to \mathbf{J}_{\leq 1\, wk}(X)$, $\ell \in L$.

Ludic Transition Systems

Definition

A ludic transition system σ is a family of continuous maps $\sigma_\ell : X \to \mathbf{J}_{\leq 1\,wk}(X)$, $\ell \in L$.

- L is a set of actions that P has control over;

Ludic Transition Systems

Definition

A ludic transition system σ is a family of continuous maps $\sigma_\ell : X \to \mathbf{J}_{\leq 1\ wk}(X)$, $\ell \in L$.

- L is a set of actions that P has control over;
- $\sigma_\ell(x)(U)$ is the preprobability that, from state x, by playing $\ell \in L$, P will move to $y \in U$;

Ludic Transition Systems

Definition

A ludic transition system σ is a family of continuous maps $\sigma_\ell : X \to \mathbf{J}_{\leq 1\, wk}(X)$, $\ell \in L$.

- L is a set of actions that P has control over;
- $\sigma_\ell(x)(U)$ is the preprobability that, from state x, by playing $\ell \in L$, P will move to $y \in U$;
- $\mathbf{J}_{\leq 1}(X)$ is the space of continuous games (not just belief functions) over X, with $\nu(X) \leq 1$ ($\sim$ subprobabilities);

Ludic Transition Systems

Definition

A ludic transition system σ is a family of continuous maps $\sigma_\ell : X \to \mathbf{J}_{\leq 1\ wk}(X)$, $\ell \in L$.

- L is a set of actions that P has control over;
- $\sigma_\ell(x)(U)$ is the preprobability that, from state x, by playing $\ell \in L$, P will move to $y \in U$;
- $\mathbf{J}_{\leq 1}(X)$ is the space of continuous games (not just belief functions) over X, with $\nu(X) \leq 1$ ($\sim$ subprobabilities);
- $\mathbf{J}_{\leq 1\ wk}(X)$ is the same, except with the weak topology (nicer theoretically, and more general).

Evaluating Average-Min Payoffs

As in Markov Decision Processes ($1\frac{1}{2}$-player games), let:

Evaluating Average-Min Payoffs

As in Markov Decision Processes ($1\frac{1}{2}$-player games), let:

- P plays according to a finite-state program Π; internal states q, transitions $q \xrightarrow{\ell} q'$;

Evaluating Average-Min Payoffs

As in Markov Decision Processes ($1\frac{1}{2}$-player games), let:

- P plays according to a finite-state program Π; internal states q, transitions $q \xrightarrow{\ell} q'$;
- Reward functions $r_{q \xrightarrow{\ell} q'} : X \to \mathbb{R}$ (bounded, continuous);

Evaluating Average-Min Payoffs

As in Markov Decision Processes ($1\frac{1}{2}$-player games), let:

- P plays according to a finite-state program Π; internal states q, transitions $q \xrightarrow{\ell} q'$;
- Reward functions $r_{q \xrightarrow{\ell} q'} : X \to \mathbb{R}$ (bounded, continuous);
- Discounts $\gamma_{q \xrightarrow{\ell} q'} \in]0, 1]$;

Evaluating Average-Min Payoffs

As in Markov Decision Processes ($1\frac{1}{2}$-player games), let:

- P plays according to a finite-state program Π; internal states q, transitions $q \xrightarrow{\ell} q'$;
- Reward functions $r_{q \xrightarrow{\ell} q'} : X \to \mathbb{R}$ (bounded, continuous);
- Discounts $\gamma_{q \xrightarrow{\ell} q'} \in]0, 1]$;

The average payoff at state x when in internal state q:

$$V_q(x) \quad = \quad \sup_{\ell, q' / q \xrightarrow{\ell} q'} \left[r_{q \xrightarrow{\ell} q'}(x) + \gamma_{q \xrightarrow{\ell} q'} \oint_{y \in X} V_{q'}(y) d\sigma_\ell(x) \right]$$

Evaluating Average Payoffs—$2\frac{1}{2}$-Player Games

$$V_q(x) = \sup_{\ell, q' / q \xrightarrow{\ell} q'} \left[r_{q \xrightarrow{\ell} q'}(x) + \gamma_{q \xrightarrow{\ell} q'} \oint_{y \in X} V_{q'}(y) d\sigma_\ell(x) \right]$$

E.g., when $\sigma_\ell(x)$ is a simple belief function $\sum_{i=1}^{n_\ell} a_{i\ell x} \mathfrak{u}_{Q_{i\ell x}}$, then:

Evaluating Average Payoffs—$2\frac{1}{2}$-Player Games

$$V_q(x) \quad = \quad \sup_{\ell, q'/q \xrightarrow{\ell} q'} \left[r_{q \xrightarrow{\ell} q'}(x) + \gamma_{q \xrightarrow{\ell} q'} \, \mathcal{C}\!\!\!\int_{y \in X} V_{q'}(y) d\sigma_\ell(x) \right]$$

E.g., when $\sigma_\ell(x)$ is a simple belief function $\sum_{i=1}^{n_\ell} a_{i\ell x} \mathfrak{u}_{Q_{i\ell x}}$, then:

P **maximizes its average payoff**

C **minimizes payoff**

$$V_q(x) = \sup_{\ell, q'/q \xrightarrow{\ell} q'} \left[r_{q \xrightarrow{\ell} q'}(x) + \gamma_{q \xrightarrow{\ell} q'} \sum_{i=1}^{n_\ell} a_{i\ell x} \min_{y \in Q_{i\ell x}} V_{q'}(y) \right]$$

take weighted average

Evaluating Average Payoffs—$2\frac{1}{2}$-Player Games

$$V_q(x) = \sup_{\ell, q' / q \xrightarrow{\ell} q'} \left[r_{q \xrightarrow{\ell} q'}(x) + \gamma_{q \xrightarrow{\ell} q'} \oint_{y \in X} V_{q'}(y) d\sigma_\ell(x) \right]$$

Theorem

The equation above has a unique solution when:

- *[Finite Horizon] Π terminates, or;*
- *[Discounted Case]* $\gamma_{q \xrightarrow{\ell} q'} \leq \gamma$ *for some* $\gamma < 1$ *+ mild assumptions (e.g.,* $\sigma_\ell(x)(X) = 1$*)*

Modal Logic

Logic $\mathcal{L}^{\top\wedge\vee}_{\mathrm{open}}$:

$$\begin{array}{lllllll}
F & ::= & \top & \text{true} & [\![\top]\!]_\sigma & = & X \\
 & | & F \wedge F & \text{conjunction} & [\![F_1 \wedge F_2]\!]_\sigma & = & [\![F_1]\!]_\sigma \cap [\![F_2]\!]_\sigma \\
 & | & F \vee F & \text{disjunction} & [\![F_1 \vee F_2]\!]_\sigma & = & [\![F_1]\!]_\sigma \cup [\![F_2]\!]_\sigma \\
 & | & [\ell]_{>r} F & \text{modality} & [\![[\ell]_{>r} F]\!]_\sigma & = & \{x \in X \mid \delta_\ell(x)([\![F]\!]_\sigma) > r\}
\end{array}$$

Theorem (à la Desharnais-Edalat-Panangaden)

$\mathcal{L}^{\top\wedge\vee}_{\mathit{open}}$ *characterizes simulation.*

Modal Logic

Logic $\mathcal{L}^{\top\wedge\vee}_{\text{open}}$:

$$\begin{array}{llll rcl}
F & ::= & \top & \text{true} & [\![\top]\!]_\sigma & = & X \\
 & | & F \wedge F & \text{conjunction} & [\![F_1 \wedge F_2]\!]_\sigma & = & [\![F_1]\!]_\sigma \cap [\![F_2]\!]_\sigma \\
 & | & F \vee F & \text{disjunction} & [\![F_1 \vee F_2]\!]_\sigma & = & [\![F_1]\!]_\sigma \cup [\![F_2]\!]_\sigma \\
 & | & [\ell]_{>r} F & \text{modality} & [\![[\ell]_{>r} F]\!]_\sigma & = & \{x \in X | \delta_\ell(x)([\![F]\!]_\sigma) > r\}
\end{array}$$

Theorem (à la Desharnais-Edalat-Panangaden)

$\mathcal{L}^{\top\wedge\vee}_{open}$ *characterizes simulation.*

Or rather. . . simulation *topologies*: $\mathcal{O} \subseteq \Omega(X)$ such that δ_ℓ is continuous from $X : \mathcal{O}$ to $\mathbf{J}_{\leq 1\ wk}(X : \mathcal{O})$.

Modal Logic

Logic $\mathcal{L}^{\top\wedge\vee}_{\mathrm{open}}$:

$$\begin{array}{lllllll} F & ::= & \top & \text{true} & [\![\top]\!]_\sigma & = & X \\ & | & F \wedge F & \text{conjunction} & [\![F_1 \wedge F_2]\!]_\sigma & = & [\![F_1]\!]_\sigma \cap [\![F_2]\!]_\sigma \\ & | & F \vee F & \text{disjunction} & [\![F_1 \vee F_2]\!]_\sigma & = & [\![F_1]\!]_\sigma \cup [\![F_2]\!]_\sigma \\ & | & [\ell]_{>r} F & \text{modality} & [\![[\ell]_{>r} F]\!]_\sigma & = & \{x \in X | \delta_\ell(x)([\![F]\!]_\sigma) > r\} \end{array}$$

Theorem (à la Desharnais-Edalat-Panangaden)

$\mathcal{L}^{\top\wedge\vee}_{open}$ characterizes simulation.

Or rather…simulation *topologies*: $\mathcal{O} \subseteq \Omega(X)$ such that δ_ℓ is continuous from $X : \mathcal{O}$ to $\mathbf{J}_{\leq 1\ wk}(X : \mathcal{O})$.

Let $\preceq_{\mathcal{O}}$ (*simulation*) the specialization quasi-ordering of $\mathcal{O}$, $\equiv_{\mathcal{O}}$ its associated equivalence. One can then *lump* together equivalent states.

Outline

Our Goal

Find a semantics for higher-order functional languages with both:

- probabilistic choice;
- non-deterministic choice.

Our Goal

Find a semantics for higher-order functional languages with both:

- probabilistic choice;
- non-deterministic choice.

Several proposals already exist: [Varacca02], [Mislove00], [TixKeimelPlotkin05].

Our Goal

Find a semantics for higher-order functional languages with both:

- probabilistic choice;
- non-deterministic choice.

Several proposals already exist: [Varacca02], [Mislove00], [TixKeimelPlotkin05].
We present a simple one based on continuous previsions [Walley91].

Representation Theorems

Well-known in measure theory:

Theorem (Riesz)

Let X be compact Hausdorff. Then:

$$\nu \text{ measure} \quad \mapsto \quad \lambda f : X \to \mathbb{R} \cdot \int_{x \in X} f(x) d\nu$$

is a bijection from the space of (bounded) measures on X to the space of bounded, linear and positive functionals from $\langle X \to \mathbb{R} \rangle$ to $\mathbb{R}$.

A Representation Theorem for Valuations

Theorem (Tix)

Let X be a topological space. Let $\langle X \to \mathbb{R}^+\rangle$ be the space of all bounded, (Scott-)continuous functions from X to $\mathbb{R}^+$. Then:

$$\nu \in \mathbf{V}(X) \quad \mapsto \quad \lambda f \in \langle X \to \mathbb{R}^+\rangle \cdot \oint_{x \in X} f(x) d\nu$$

is an isomorphism between $\mathbf{V}(X)$ (continuous valuations) and the space of functionals F from $\langle X \to \mathbb{R}\rangle$ to $\mathbb{R}$ such that:

- *F is positively homogeneous: $F(af) = aF(f)$ ($a \geq 0$);*
- *F is monotonic: if $f \leq g$ then $F(f) \leq F(g)$;*
- *F is (Scott-)continuous: $F(\sup^{\uparrow}_{i \in I} f_i) = \sup^{\uparrow}_{i \in I} F(f_i)$;*
- *F is additive: $F(f+g) = F(f) + F(g)$.*

Properties of the Choquet Integral (Remember?)

Positively homogeneous
Additive on comonotonic f, g
Superadditive

(Provided ν is convex)

$$\oint_{x \in X} f(x) + g(x) d\nu \geq \oint_{x \in X} f(x) d\nu + \oint_{x \in X} g(x) d\nu$$

$$\oint_{x \in X} f(x) d\nu$$

Linear in ν

Scott−continuous in f

(Provided ν is continuous)

Scott−continuous in ν

Previsions

Definition

A *prevision* F is a functional from $\langle X \rightarrow \mathbb{R}^+\rangle$ to $\mathbb{R}^+$ such that:

- F is positively homogeneous: $F(af) = aF(f)$ $(a \geq 0)$;
- F is monotonic: if $f \leq g$ then $F(f) \leq F(g)$;

I.e., we drop additivity: $F(f+g) = F(f) + F(g)$.

Previsions

Definition

A *colinear* *prevision* F is a functional from $\langle X \to \mathbb{R}^+\rangle$ to $\mathbb{R}^+$ such that:

- F is positively homogeneous: $F(af) = aF(f)$ ($a \geq 0$);
- F is monotonic: if $f \leq g$ then $F(f) \leq F(g)$;
- F is colinear: if $f \asymp g$ then $F(f+g) = F(f) + F(g)$;

I.e., we drop additivity: $F(f+g) = F(f) + F(g)$.

Previsions

Definition

A *continuous colinear prevision F* is a functional from $\langle X \rightarrow \mathbb{R}^+ \rangle$ to $\mathbb{R}^+$ such that:

- F is positively homogeneous: $F(af) = aF(f)$ $(a \geq 0)$;
- F is monotonic: if $f \leq g$ then $F(f) \leq F(g)$;
- F is colinear: if $f \smile g$ then $F(f+g) = F(f) + F(g)$;
- F is (Scott-)continuous: $F(\sup^{\uparrow}_{i \in I} f_i) = \sup^{\uparrow}_{i \in I} F(f_i)$;

I.e., we relax additivity: $F(f+g) = F(f) + F(g)$.

Previsions

Definition

A *continuous* *linear* *prevision* F is a functional from $\langle X \to \mathbb{R}^+\rangle$ to $\mathbb{R}^+$ such that:

- F is positively homogeneous: $F(af) = aF(f)$ $(a \geq 0)$;
- F is monotonic: if $f \leq g$ then $F(f) \leq F(g)$;
- F is colinear: if $f \smile g$ then $F(f+g) = F(f) + F(g)$;
- F is (Scott-)continuous: $F(\sup^{\uparrow}_{i \in I} f_i) = \sup^{\uparrow}_{i \in I} F(f_i)$;
- F is linear: $F(f+g) = F(f) + F(g)$;

Previsions

Definition

A *continuous* lower *prevision* F is a functional from $\langle X \to \mathbb{R}^+ \rangle$ to $\mathbb{R}^+$ such that:

- F is positively homogeneous: $F(af) = aF(f)$ $(a \geq 0)$;
- F is monotonic: if $f \leq g$ then $F(f) \leq F(g)$;
- F is colinear: if $f \frown g$ then $F(f+g) = F(f) + F(g)$;
- F is (Scott-)continuous: $F(\sup^{\uparrow}_{i \in I} f_i) = \sup^{\uparrow}_{i \in I} F(f_i)$;
- F is lower: $F(f+g) \geq F(f) + F(g)$.

A Dictionary of Representation Theorems

Continuous Games	Continuous Previsions
Valuations	Linear previsions [Tix99]

A Dictionary of Representation Theorems

Continuous Games	Continuous Previsions
Valuations Games	Linear previsions [Tix99] Colinear previsions

A Dictionary of Representation Theorems

Continuous Games	Continuous Previsions
Valuations	Linear previsions [Tix99]
Games	Colinear previsions
Convex games	Colinear lower previsions [$\sim$ Schmeidler92]

A Dictionary of Representation Theorems

Continuous Games	Continuous Previsions
Valuations	Linear previsions [Tix99]
Games	Colinear previsions
Convex games	Colinear lower previsions [$\sim$ Schmeidler92]
Concave games	Colinear upper previsions

A Dictionary of Representation Theorems

Continuous Games	Continuous Previsions
Valuations	Linear previsions [Tix99]
Games	Colinear previsions
Convex games	Colinear lower previsions [$\sim$ Schmeidler92]
Concave games	Colinear upper previsions
Belief functions	Colinear pessimistic previsions
Plausibilities	Colinear optimistic previsions

A Probabilistic Non-Deterministic Lambda-Calculus

In a nutshell:

- Take Moggi's monadic λ-calculus [Mog91];
- Requires a strong monad $(\boldsymbol{T}, \boldsymbol{\eta}, \boldsymbol{\mu}, \boldsymbol{t})$;

A Probabilistic Non-Deterministic Lambda-Calculus

In a nutshell:

- Take Moggi's monadic λ-calculus [Mog91];
- Requires a strong monad $(\boldsymbol{T}, \boldsymbol{\eta}, \boldsymbol{\mu}, \boldsymbol{t})$;
- Take $\boldsymbol{T}X = \mathbf{V}(X)$ [Jones90]: models probabilistic choice, no non-determinism;

A Probabilistic Non-Deterministic Lambda-Calculus

In a nutshell:

- Take Moggi's monadic λ-calculus [Mog91];
- Requires a strong monad $(\boldsymbol{T}, \boldsymbol{\eta}, \boldsymbol{\mu}, \boldsymbol{t})$;
- Take $\boldsymbol{T} X = \mathcal{Q}(X)$: models (demonic) non-determinism, no probabilities;

A Probabilistic Non-Deterministic Lambda-Calculus

In a nutshell:

- Take Moggi's monadic λ-calculus [Mog91];
- Requires a strong monad $(\boldsymbol{T}, \boldsymbol{\eta}, \boldsymbol{\mu}, \boldsymbol{t})$;
- Take $\boldsymbol{T}X = \mathbf{Cd}(X)$: models both...

A Probabilistic Non-Deterministic Lambda-Calculus

In a nutshell:

- Take Moggi's monadic λ-calculus [Mog91];
- Requires a strong monad $(\boldsymbol{T}, \boldsymbol{\eta}, \boldsymbol{\mu}, \boldsymbol{t})$;
- Take $\boldsymbol{T}X = \mathbf{Cd}(X)$: models both...
  but not a monad (cannot define $\boldsymbol{\mu}$).

A Probabilistic Non-Deterministic Lambda-Calculus

In a nutshell:

- Take Moggi's monadic λ-calculus [Mog91];
- Requires a strong monad $(\boldsymbol{T}, \boldsymbol{\eta}, \boldsymbol{\mu}, \boldsymbol{t})$;
- Take $\boldsymbol{T}X = \{$continuous convex games$\}$: models both...

A Probabilistic Non-Deterministic Lambda-Calculus

In a nutshell:

- Take Moggi's monadic λ-calculus [Mog91];
- Requires a strong monad $(\boldsymbol{T}, \boldsymbol{\eta}, \boldsymbol{\mu}, \boldsymbol{t})$;
- Take $\boldsymbol{T}X = \{\text{continuous convex games}\}$:
  models both. . .
  but not a monad (cannot define $\boldsymbol{\mu}$).

A Probabilistic Non-Deterministic Lambda-Calculus

In a nutshell:

- Take Moggi's monadic λ-calculus [Mog91];
- Requires a strong monad $(\boldsymbol{T}, \boldsymbol{\eta}, \boldsymbol{\mu}, \boldsymbol{t})$;
- Take $\boldsymbol{T}X = \{$continuous colinear lower previsions$\}$: models both. . .

A Probabilistic Non-Deterministic Lambda-Calculus

In a nutshell:

- Take Moggi's monadic λ-calculus [Mog91];
- Requires a strong monad $(\boldsymbol{T}, \boldsymbol{\eta}, \boldsymbol{\mu}, \boldsymbol{t})$;
- Take $\boldsymbol{T}X = \{$continuous colinear lower previsions$\}$: models both. . .
  but not a monad (cannot define $\boldsymbol{\mu}$).

A Probabilistic Non-Deterministic Lambda-Calculus

In a nutshell:

- Take Moggi's monadic λ-calculus [Mog91];
- Requires a strong monad $(\boldsymbol{T}, \boldsymbol{\eta}, \boldsymbol{\mu}, \boldsymbol{t})$;
- Take $\boldsymbol{T}X = \{$continuous ~~*colinear*~~ lower previsions$\}$: models both…

  and works!

A Probabilistic Non-Deterministic Lambda-Calculus

$$\begin{array}{rcll}
M, N & ::= & x & \text{variable} \\
 & | & \texttt{flip} \mid \texttt{amb} \mid \ldots & \text{constants} \\
 & | & MN & \text{application} \\
 & | & \lambda x \cdot M & \text{abstraction} \\
 & | & () & \text{empty tuple} \\
 & | & (M, N) & \text{pair} \\
 & | & \mathrm{fst}\ M & \text{first proj.} \\
 & | & \mathrm{snd}\ M & \text{second proj.} \\
 & | & \texttt{val}\, M & \text{trivial comp.} \\
 & | & \texttt{let}\,\texttt{val}\, x = M\ \texttt{in}\ N & \text{sequence}
\end{array}$$

$$\begin{array}{rcll}
\tau & ::= & \texttt{bool} \mid \texttt{int} & \text{base types} \\
 & | & \mathtt{u} & \text{type of ()} \\
 & | & \tau \times \tau & \text{product} \\
 & | & \tau \to \tau & \text{function types} \\
 & | & T\tau & \text{computation} \\
 & & & \text{types}
\end{array}$$

A Continuation Semantics

In an environment ρ, with continuation $h : [\![\tau]\!] \to \mathbb{R}^+$,

$$\begin{aligned}
[\![\mathtt{val}\, M]\!]\, \rho(h) &= h([\![M]\!]\, \rho) \\
[\![\mathtt{let}\, \mathtt{val}\, x = M\ \mathtt{in}\ N]\!]\, \rho(h) &= [\![M]\!]\, \rho(\lambda v \cdot [\![N]\!]\, (\rho[x := v])(h)) \\
[\![\mathtt{case}]\!]\, \rho(b, v_0, v_1) &= \begin{cases} v_0 & \text{if } b = \text{false} \\ v_1 & \text{if } b = \text{true} \end{cases}
\end{aligned}$$

A Continuation Semantics

In an environment ρ, with continuation $h : [\![\tau]\!] \to \mathbb{R}^+$,

$$\begin{array}{rcl}
[\![\mathtt{val}\, M]\!]\,\rho(h) & = & h([\![M]\!]\,\rho) \\
[\![\mathtt{let}\,\mathtt{val}\, x = M\ \mathtt{in}\ N]\!]\,\rho(h) & = & [\![M]\!]\,\rho(\lambda v \cdot [\![N]\!]\,(\rho[x := v])(h)) \\
[\![\mathtt{case}]\!]\,\rho(b, v_0, v_1) & = & \left\{ \begin{array}{ll} v_0 & \text{if } b = \text{false} \\ v_1 & \text{if } b = \text{true} \end{array} \right. \\
[\![\mathtt{flip} : \mathtt{Tbool}]\!]\,\rho(h) & = & 1/2(h(\text{false}) + h(\text{true})) \\
 & & \text{(take mean payoff)}
\end{array}$$

A Continuation Semantics

In an environment ρ, with continuation $h : [\![\tau]\!] \to \mathbb{R}^+$,

$$\begin{array}{rcl}
[\![\mathtt{val}\, M]\!]\, \rho(h) & = & h([\![M]\!]\, \rho) \\
[\![\mathtt{let}\, \mathtt{val}\, x = M\ \mathtt{in}\ N]\!]\, \rho(h) & = & [\![M]\!]\, \rho(\lambda v \cdot [\![N]\!]\, (\rho[x := v])(h)) \\
[\![\mathtt{case}]\!]\, \rho(b, v_0, v_1) & = & \begin{cases} v_0 & \text{if } b = \text{false} \\ v_1 & \text{if } b = \text{true} \end{cases} \\
[\![\mathtt{flip} : \mathtt{Tbool}]\!]\, \rho(h) & = & 1/2(h(\text{false}) + h(\text{true})) \\
 & & \text{(take mean payoff)} \\
[\![\mathtt{amb} : \mathtt{Tbool}]\!]\, \rho(h) & = & \min(h(\text{false}), h(\text{true})) \\
 & & \text{(take min payoff)}
\end{array}$$

Angelic, Chaotic Non-Determinism

- Can also deal with angelic non-determinism (Hoare): take $\boldsymbol{T}X = \{$continuous upper previsions$\}$;

$$\begin{aligned} [\![\text{amb} : \texttt{Tbool}]\!]\,\rho(h) &= \max(h(\text{false}), h(\text{true})) \\ &\quad \text{(take max payoff)} \end{aligned}$$

Angelic, Chaotic Non-Determinism

- Can also deal with angelic non-determinism (Hoare): take $\boldsymbol{T} X = \{\text{continuous upper previsions}\}$;

$$\begin{array}{rcl} [\![\texttt{amb} : \texttt{Tbool}]\!] \rho(h) & = & \max(h(\text{false}), h(\text{true})) \\ & & \text{(take max payoff)} \end{array}$$

- Can also deal with chaotic non-determinism (Plotkin): take $\boldsymbol{T} X = \{\text{continuous forks}\}$, where a fork is any pair $F = (F^-, F^+)$ with:
  - F^- a lower prevision;
  - F^+ an upper prevision;
  - $F^-(h + h') \leq F^-(h) + F^+(h') \leq F^+(h + h')$.

$$\begin{array}{rcl} [\![\texttt{amb}]\!] \rho & = & (\lambda h \cdot \min(h(0), h(1)), \lambda h \cdot \max(h(0), h(1))) \\ & & \text{(take both min and max payoff)} \end{array}$$

Completeness

- Prevision models are sound: any mixture of (demonic, angelic, chaotic) non-determinism with probabilistic choice is accounted for.

Completeness

- Prevision models are sound: any mixture of (demonic, angelic, chaotic) non-determinism with probabilistic choice is accounted for.
- We show completeness: there is no junk—prevision models are no more than mixtures of non-determinism with probabilistic choice.

Shapley's [1965] and Rosenmuller's [1971] Theorems

Fundamental theorems in economy (for finite X, colinear F).

Definition

The core of a game ν is the set of measures p such that:

- $\nu(U) \leq p(U)$ for any U;
- $\nu(X) = p(X)$.

Shapley's [1965] and Rosenmuller's [1971] Theorems

Fundamental theorems in economy (for finite X, colinear F).

Definition

The core of a game ν is the set of measures p such that:

- $\nu(U) \leq p(U)$ for any U;
- $\nu(X) = p(X)$.

Theorem (Shapley)

Every convex game has a non-empty core.

Entails existence of economic equilibria.

Shapley's [1965] and Rosenmuller's [1971] Theorems

Fundamental theorems in economy (for finite X, colinear F).

Definition

The core of a game ν is the set of measures p such that:

- $\nu(U) \leq p(U)$ for any U;
- $\nu(X) = p(X)$.

Theorem (Rosenmuller)

A game is convex iff:

- *it has a non-empty core;*
- *and for every $f : X \to \mathbb{R}^+$,*

$$\oint_{x \in X} f(x) d\nu = \min_{p \text{ in the core of } \nu} \oint_{x \in X} f(x) dp$$

The Heart of a Continuous Prevision

Use normalized previsions ($\sim$ non-additive probabilities).

Definition (Heart)

The heart $CCoeur_1(F)$ of $F : \langle X \to \mathbb{R}^+ \rangle \to \mathbb{R}^+$ is the set of continuous linear normalized previsions G such that $F \leq G$.

The Heart of a Continuous Prevision

Use normalized previsions ($\sim$ non-additive probabilities).

Definition (Heart)

The heart $CCoeur_1(F)$ of $F : \langle X \to \mathbb{R}^+\rangle \to \mathbb{R}^+$ is the set of continuous linear normalized previsions G such that $F \leq G$.

Theorem (à la Rosenmuller, topological; no colinearity needed)

Let X be nice enough (stably locally compact), F a continuous normalized prevision on X.
Then F is lower iff:

- $CCoeur_1(F) \neq \emptyset$;
- *and for every* $f \in \langle X \to \mathbb{R}^+\rangle$, $F(f) = \inf_{G \in CCoeur_1(F)} G(f)$.

The the inf is attained: $F(f) = \min_{G \in CCoeur_1(F)} G(f)$.

Completeness

Define the weak topology on the space $\mathbf{P}(X)$ ($\bigtriangledown\,\mathbf{P}(X)$, $\bigtriangleup\,\mathbf{P}(X)$) of all continuous (lower, upper) previsions on X, as the coarsest that makes $F \mapsto F(f)$ continuous, for each $f \in \langle X \to \mathbb{R}^+\rangle$.

Theorem

Let X be nice enough (stably compact), F a normalized continuous lower prevision.
Then $CCoeur_1(F)$ is a non-empty saturated compact convex subset of $\mathbf{P}^{\bigtriangleup}_{1\,wk}(X)$.

Completeness

Define the weak topology on the space $\mathbf{P}(X)$ ($\nabla\,\mathbf{P}(X)$, $\triangle\,\mathbf{P}(X)$) of all continuous (lower, upper) previsions on X, as the coarsest that makes $F \mapsto F(f)$ continuous, for each $f \in \langle X \to \mathbb{R}^+\rangle$.

Theorem

Let X be nice enough (stably compact), F a normalized continuous lower prevision.
Then $CCoeur_1(F)$ is a non-empty saturated compact convex subset of $\mathbf{P}^{\triangle}_{1\,wk}(X)$.

Corollary

$CCoeur_1 \dashv \prod$ is a continuous Galois injection ("almost an isomorphism") of $\nabla\,\mathbf{P}_1(X)$ into $\mathcal{Q}(\mathbf{P}^{\triangle}_{1\,wk}(X))$.

I.e., $\nabla\,\mathbf{P}_1(X)$ contains no junk:

Completeness

Theorem

Let X be nice enough (stably compact), F a normalized continuous lower prevision.
Then $CCoeur_1(F)$ is a non-empty saturated compact convex subset of $\mathbf{P}^{\triangle}_{1\,wk}(X)$.

Corollary

$CCoeur_1 \dashv \sqcap$ is a continuous Galois injection ("almost an isomorphism") of $\bigtriangledown\mathbf{P}_1(X)$ into $\mathcal{Q}(\mathbf{P}^{\triangle}_{1\,wk}(X))$.

I.e., $\bigtriangledown\mathbf{P}_1(X)$ contains no junk:
Every normalized continuous lower prevision is essentially one non-deterministic choice *then* one probabilistic choice (à la [SegalaLynch95, Mislove00, MisloveOuaknineWorrell03, TixKeimelPlotkin05]; the converse of belief functions).

Completeness (cont'd)

- In the angelic case, $\bigsqcup \dashv CPeau_1$ is a continuous Galois surjection of $\mathcal{H}(\mathbf{P}^{\triangle}_{1\ wk}(X))$ onto $\nabla\,\mathbf{P}_1(X)$.

Completeness (cont'd)

- In the angelic case, $\bigsqcup \dashv CPeau_1$ is a continuous Galois surjection of $\mathcal{H}(\mathbf{P}^{\triangle}_{1\,wk}(X))$ onto $\nabla\,\mathbf{P}_1(X)$.
- In the chaotic case, for any fork $F = (F^-, F^+)$, $CCoeur_1(F^-) \cap CPeau_1(F^+)$ is a lens, i.e., an element of the Plotkin powerdomain of $\mathbf{P}^{\triangle}_{1\,wk}(X)$.

Completeness (cont'd)

- In the angelic case, $\bigsqcup \dashv CPeau_1$ is a continuous Galois surjection of $\mathcal{H}(\mathbf{P}^{\triangle}_{1\,wk}(X))$ onto $\bigtriangledown \mathbf{P}_1(X)$.
- In the chaotic case, for any fork $F = (F^-, F^+)$, $CCoeur_1(F^-) \cap CPeau_1(F^+)$ is a lens, i.e., an element of the Plotkin powerdomain of $\mathbf{P}^{\triangle}_{1\,wk}(X)$.
- Our prevision models are "almost isomorphic" to models of compact convex subsets (resp. closed convex subsets, convex lenses) of probability valuations [Mislove00, TixKeimelPlotkin05].

Outline

- Stochastic Games
  - Non-Deterministic Choice
  - Probabilistic Choice: Markov Chains
  - Mixing Non-Determinism and Probabilities
- Capacities, Games, Belief Functions
  - Unanimity Games
  - Belief Functions
  - The Choquet Integral
  - Ludic Transition Systems
- Previsions
  - Representation Theorems
  - A Probabilistic Non-Deterministic Lambda-Calculus
  - Completeness
- Conclusion

Conclusion

- I love realizing new problems have old solutions: here I use and extend theories of capacities [Choquet53-54], cooperative games [Shapley65], belief functions [Dempster67], previsions [Walley91].

Conclusion

- I love realizing new problems have old solutions: here I use and extend theories of capacities [Choquet53-54], cooperative games [Shapley65], belief functions [Dempster67], previsions [Walley91].
- Ludic transition systems: a smart formulation of $2\frac{1}{2}$-player games that smells of $1\frac{1}{2}$-player games (Markov decision processes).

Conclusion

- I love realizing new problems have old solutions: here I use and extend theories of capacities [Choquet53-54], cooperative games [Shapley65], belief functions [Dempster67], previsions [Walley91].
- Ludic transition systems: a smart formulation of $2\frac{1}{2}$-player games that smells of $1\frac{1}{2}$-player games (Markov decision processes).
- Previsions: an elegant and simple semantics for probabilistic and non-deterministic higher-order functional languages.