Natural homology

HOMOTOPY IN CONCURRENCY AND REWRITING

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joint work with
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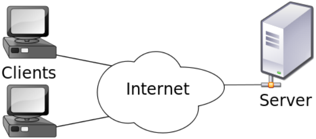
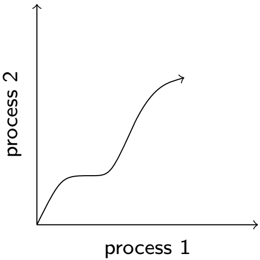
11th June, 2015

General context : verification of concurrent systems



Models of true concurrency



- Petri nets **[Petri 62]**
- progress graphs **[Dijkstra 68]**
- trace theories **[Mazurkiewicz 70s]**
- event structures **[Winskel 80s]**
- higher dimensional automata (HDA) **[Pratt 91]**

Plan

I. Geometry of true concurrency

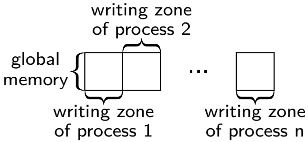
II. Classical homology

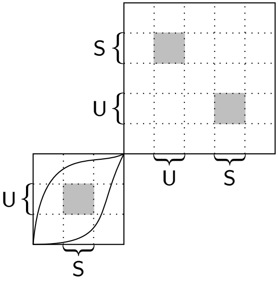
III. A candidate of directed homology : natural homology

I.
Geometry of true concurrency

A toy language : SU-programs **[Afek et al. 90]**

- shared global memory
- atomic operations :
  - S : scan ALL the memory
  - U : update ONLY its OWN part of the memory
- synchronization • (rendez-vous)
- S and U non independent





$$(S|U) \bullet (U.S|U.S)$$

Pospaces **[Nachbin 65]**, dipaths, traces **[Raussen 09]**

- X pospace = space **+ order**

- **di**path = **increasing** path = **increasing** continuous function $p : [0,1] \longrightarrow X$
  = « execution with memory of the time between actions »
- **di**path space : $\overrightarrow{\mathfrak{P}}(X)(a,b) = \{p : a \longrightarrow b\}$

- trace $\langle p \rangle$ = **di**path p modulo **increasing** reparametrization
  = « execution where only organization of actions is significant »
- trace space : $\overrightarrow{\mathfrak{T}}(X)(a,b) = \{\langle p \rangle \text{ with } p : a \longrightarrow b\}$

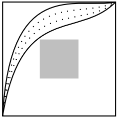
Dihomotopy

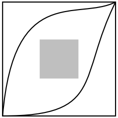
Dihomotopy :

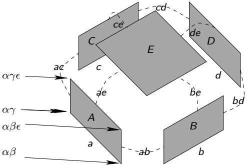
f and g **di**paths from a to b in X.

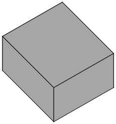
- $H : [0,1] \times [0,1] \longrightarrow X$ **di**homotopy from f to g if :
  - H continuous and **increasing in the second coordinate**
  - $H(0,.) = f$, $H(1,.) = g$, $H(.,0) = a$ et $H(.,1) = b$
- f and g **di**homotopic if there exists a **di**homotopy from one to the other

dihomotopic = « deforming continuously one to the other **while staying a dipath** »









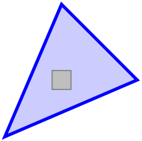
Objective

- study those concurrent systems through their geometry (dipaths, traces, dihomotopies)
- homology = essential notion, computable abstraction of homotopy

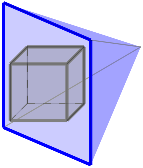
  $\Rightarrow$ defining a directed analogue of homology

II.
Classical homology

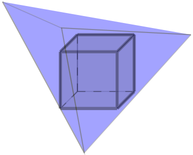
Homology = counting holes



hole of dimension 1



no hole of dimension 1 ...



... but a hole of dimension 2

In first approximation : $H_n(X) \simeq \mathbb{Z}^{number\ of\ holes\ of\ dimension\ n}$

In general : $H_n(X) \simeq \prod_{T_i\ hole\ of\ dimension\ n} \mathbb{Z}/k_i\mathbb{Z}$

Particular case : $H_0(X) \simeq \mathbb{Z}^{number\ of\ path-connected\ components}$

Properties of homology

- **(sound)** invariant by homotopy, correction
- **(precis)** not too much loss of information (Hurewicz), partial completeness (Whitehead)
- **(mod)** modularity = homology of a space expressible from homology of smaller spaces (Mayer-Vietoris)
- **(calc)** computability in the case of finitely presented spaces (simplicial, pre-cubical sets)

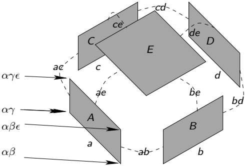
Objective :

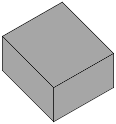
- study those concurrent systems through their geometry (dipaths, traces, dihomotopies)
- homology = essential notion, computable abstraction of homotopy

  $\Rightarrow$ defining a directed analogue of homology with the same kind of properties

Existing works

	(sound)	(precis)	(mod)	(calc)
[Goubault 95]	-	×	-	✓
[Grandis 04]	✓	×	-	-
[Farhenberg 04]	✓	×	-	-
[Kahl 13]	(✓)	×	-	-





Existing works

	(sound)	(precis)	(mod)	(calc)
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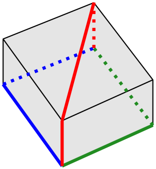
[Grandis 04]	✓	×	-	-
[Farhenberg 04]	✓	×	-	-
[Kahl 13]	(✓)	×	-	-



Existing works

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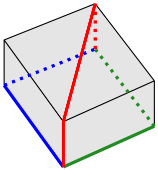
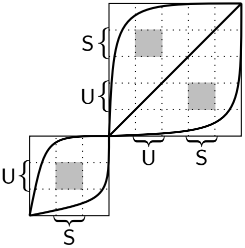
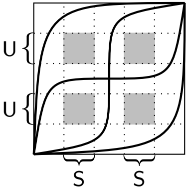
[Farhenberg 04]	✓	×	-	-
[Kahl 13]	(✓)	×	-	-
[D.G.G.]	(✓)	✓	(✓)	✓

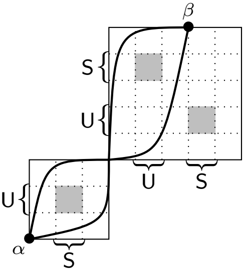


III.
A candidate of directed homology : natural homology

trace spaces vs evolution of trace spaces



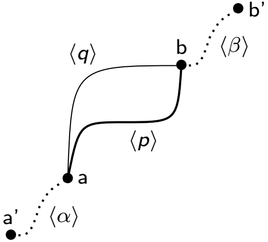




(geometric) Natural homology

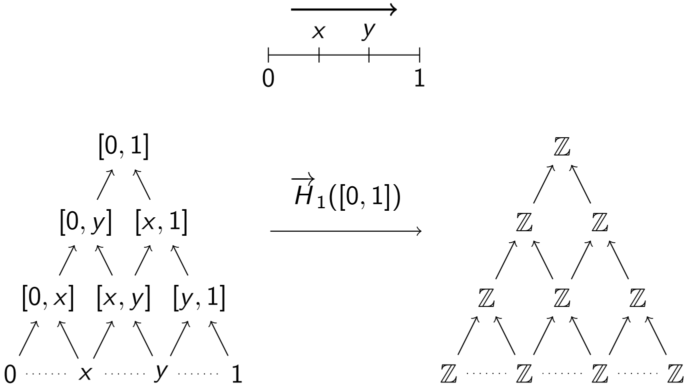
Natural homology $\overrightarrow{H}_n(X)$ $(n \geq 1)$:

trace $\langle p \rangle$ p dipath from a to b	$\longmapsto$	$H_{n-1}(\overrightarrow{\mathfrak{T}}(X)(a,b))$
extension $(\langle \alpha \rangle, \langle \beta \rangle)$ α from a' to a, β from b to b'	$\longmapsto$	$H_{n-1}((\langle q \rangle \in \overrightarrow{\mathfrak{T}}(X)(a,b) \mapsto$ $\langle \alpha \star q \star \beta \rangle \in \overrightarrow{\mathfrak{T}}(X)(a',b')))$

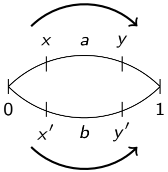


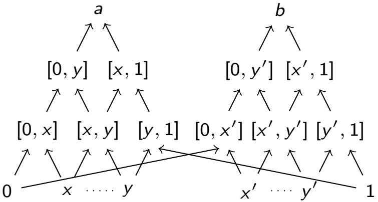
$\overrightarrow{H}_n(X)$ = functor from the category of factorization of the category of traces to **Ab** = natural system on the category of traces

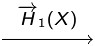
Example I

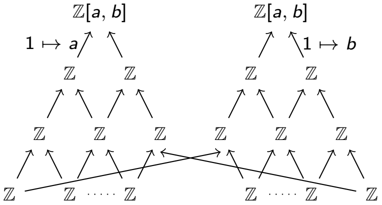


Example II


x
a
y
0
1
x′
b
y′


a
b
[0, y]
[x, 1]
[0, y′]
[x′, 1]
[0, x]
[x, y]
[y, 1]
[0, x′]
[x′, y′]
[y′, 1]
0
x
y
x′
y′
1


$\overrightarrow{H}_1(X)$


ℤ[a, b]
ℤ[a, b]
1 ↦ a
1 ↦ b
ℤ

Study of (**mod**)

Proposition :

- $\overrightarrow{H}_n$ is a functor from **PoTop** to the category of functors **Fun**(**Ab**).
- **Fun**(**Ab**) is not abelian but is homological in the sense of **[Grandis 91]**.

Proof :

- morphisms from $F : \mathcal{C} \longrightarrow$ **Ab** to $G : \mathcal{D} \longrightarrow$ **Ab** : pairs (Φ, σ) where :
  - $\Phi : \mathcal{C} \longrightarrow \mathcal{D}$
  - $\sigma : F \longrightarrow G \circ \Phi$
- null morphisms : (Φ, σ) with σ_c are zero
- kernels : $c \mapsto Ker \sigma_c$
- cokernels : a bit tricky (because colimits in **Fun**(**Ab**) are more complicated)
- $+$ some morphisms are exact (because **Ab** is abelian and its morphisms are exact)

Study of (**mod**)

Proposition :

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- **Fun**(**Ab**) is not abelian but is homological in the sense of **[Grandis 91]**.

Theorem (**mod**) **[Grandis 91]** :

Let $\mathcal{A}$ be a homological category.
For every short exact sequence in $C_\bullet(\mathcal{A})$:

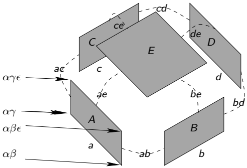
$$\mathrm{U} \xrightarrow{\quad \mathrm{m} \quad} \mathrm{V} \xrightarrow{\quad \mathrm{p} \quad} \mathrm{W}$$

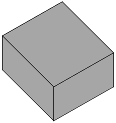
there exists a long sequence of order two in $\mathcal{A}$:

$$\cdots \longrightarrow H_n(V) \xrightarrow{H_n(p)} H_n(W) \xrightarrow{\partial_n} H_{n-1}(U) \xrightarrow{H_{n-1}(m)} H_{n-1}(V) \longrightarrow \cdots$$

natural in the short exact sequence.
Moreover, there are conditions to turn the long sequence to an exact sequence.

Study of (**precis**)





$H_0(\overrightarrow{\mathfrak{T}}(X)(\alpha\beta,\gamma\delta)) \simeq \mathbb{Z}^2$

$\Rightarrow$ there exists two dipaths that are not dihomotopic

$\Rightarrow$ we can see it in $\overrightarrow{H}_1(X)$!

Theorem (precis) [D.G.G.] :

- if X 0-diconnected, $\overrightarrow{H}_1(X) \simeq \mathit{Free} \circ \overrightarrow{\Pi}_1(X)$
- if X 1-diconnected, $\overrightarrow{H}_2(X) \simeq \mathit{Ab} \circ \overrightarrow{\Pi}_2(X)$
- if X $(n-1)$-diconnected $(n \geq 3)$, $\overrightarrow{H}_n(X) \simeq \overrightarrow{\Pi}_n(X)$

Computability

Theorem (calc) [D.G.G. 15] :

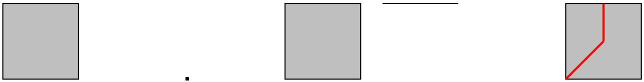
Given a finite pre-cubical complex, there exists $\overrightarrow{h}_n(X)$ (discrete natural homology) :

- computable
- equivalent to $\overrightarrow{H}_n(X)$

Proof (construction)

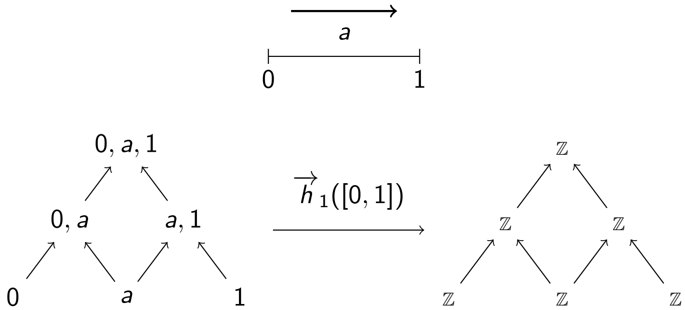
X a pre-cubical complex

- a discrete trace from $x \in X$ to $y \in X$: sequence $c_0, \dots, c_n \in X$ such that $c_0 = x$, $c_n = y$ and for all i :
  - either c_{i-1} is of the form $\delta^0_{i_k} \circ \cdots \circ \delta^0_{i_0}(c_i)$
  - either c_i is of the form $\delta^1_{i_k} \circ \cdots \circ \delta^1_{i_0}(c_{i-1})$
- we map each discrete trace to a geometric one :

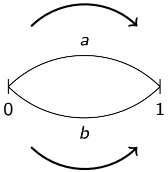


- $\overrightarrow{h}_n(X)$ is the restriction of $\overrightarrow{H}_n(X)$ to those traces

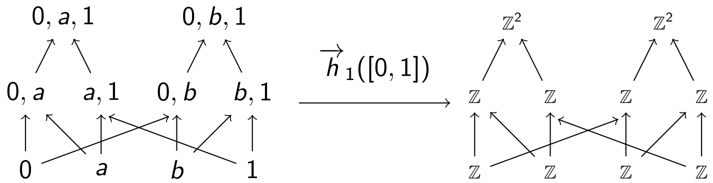
Example I



Example II



a
0
1
b



0, a, 1
0, b, 1
0, a
a, 1
0, b
b, 1
0
a
b
1
$\vec{h}_1([0,1])$
$\mathbb{Z}^2$
$\mathbb{Z}^2$
$\mathbb{Z}$

Proof (computability)

- enumeration of discrete traces
- construction of a finite representation (prod-simplical complex) of each trace space **[Raussen 09]**
- computation of classical homology

Which notion of equivalence ?

problem : $\overrightarrow{h}_n(X)$ and $\overrightarrow{H}_n(X)$ non isomorphic by cardinality

solution : existence of a morphism with some lifting properties

equivalence : existence of a span of such morphisms

$\mathcal{P}$-open maps **[Joyal et al. 94]**

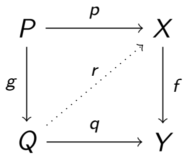
$\mathcal{P}$ = sub-category = category of paths

paths : $G_1 \xrightarrow{f_1} G_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} G_n$

extensions of paths :

$$\begin{array}{ccccccccccc}
G_1 & \xrightarrow{f_1} & G_2 & \xrightarrow{f_2} & \cdots & \xrightarrow{f_{k-1}} & G_k & & & & \\
{\scriptstyle id}\downarrow & & {\scriptstyle id}\downarrow & & & & {\scriptstyle id}\downarrow & & & & \\
G_1 & \xrightarrow[f_1]{} & G_2 & \xrightarrow[f_2]{} & \cdots & \xrightarrow[f_{k-1}]{} & G_k & \xrightarrow[f_k]{} & \cdots & \xrightarrow[f_{n-1}]{} & G_n
\end{array}$$

$\mathcal{P}$-open = has the lifting property with respect to those extensions :



Bisimilarity - Computer science point of view

Two functors $F : \mathcal{C} \longrightarrow \mathbf{Ab}$ and $G : \mathcal{D} \longrightarrow \mathbf{Ab}$ are bisimilar if there exists a set

$$R \subseteq \{(c, f, d) \mid c \in \mathcal{C} \wedge d \in \mathcal{D} \wedge f \in Ab(F(c), G(d)) \textit{ isomorphism}\}$$

such that :

- for all $c \in \mathcal{C}$, there exists d, f such that $(c, f, d) \in R$
- for all $d \in \mathcal{D}$, there exists c, f such that $(c, f, d) \in R$
- for all $(c, f, d) \in R$ and $i : c \longrightarrow c'$ there exists $j : d \longrightarrow d'$ and $g : F(c') \longrightarrow G(d')$ iso such that $(c', g, d') \in R$ and $f \circ F(i) = G(j) \circ g$
- for all $(c, f, d) \in R$ and $j : d \longrightarrow d'$ there exists $i : c \longrightarrow c'$ and $g : F(c') \longrightarrow G(d')$ iso such that $(c', g, d') \in R$ and $f \circ F(i) = G(j) \circ g$

Bisimilarity

A open map from $H : E \longrightarrow Ab$ to $G : D \longrightarrow Ab$ is :

- a functor $\Phi : E \longrightarrow D$ satisfying :
  - Φ is surjective on objects
  - Φ has the lifting property for any morphism of D : for every morphism $j : d \longrightarrow d'$ of D and every object e of E such that $\Phi(e) = d$ there exists a morphism $i : e \longrightarrow e'$ such that $\Phi(i) = j$
- a natural isomorphism $\sigma : H \longrightarrow G \circ \Phi$

We say that $F : C \longrightarrow Ab$ and $G : D \longrightarrow Ab$ are bisimilar if there exists a span of open maps between them.

Proof (equivalence)

Proposition :

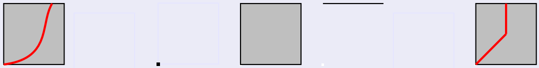
There exists an open map *Carrier* : $\overrightarrow{H}_n(X) \longrightarrow \overrightarrow{h}_n(X)$.

Proof (construction) :

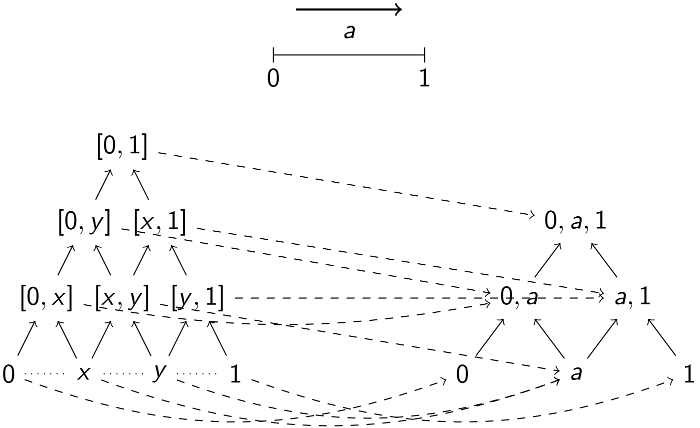
construction a functor

$$Carrier : \text{geometric traces} \longrightarrow \text{discrete traces}$$

$Carrier(trace)$ = « the sequence of hypercubes crossed by this trace »



Example



Proof (equivalence)

Proposition :

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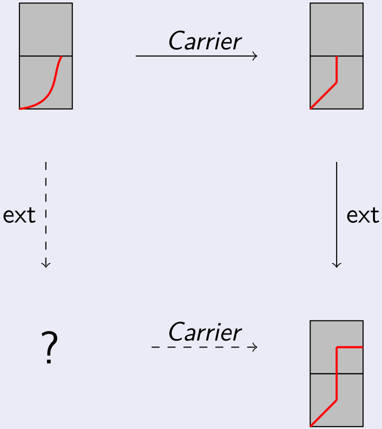
Proof (lifting property) :

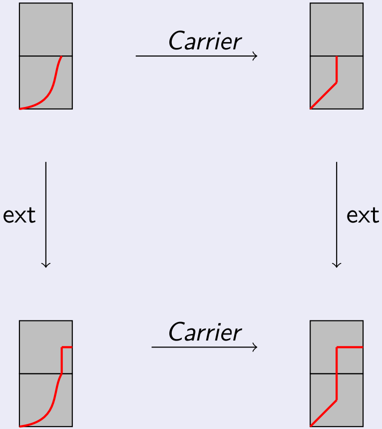


Proof (equivalence)

Proposition :

Il existe une open map *Carrier* : $\overrightarrow{H}_n(X) \longrightarrow \overrightarrow{h}_n(X)$.

Proof (lifting property) :



Proof (equivalence)

Proposition :

There exists an open map *Carrier* : $\overrightarrow{H}_n(X) \longrightarrow \overrightarrow{h}_n(X)$.

Corollary :

If X' is a barycentric subdivision of X, $\overrightarrow{h}_n(X)$ and $\overrightarrow{h}_n(X')$ are $\mathcal{P}$-bisimilar.

Results and future works

Results :

(sound) invariance by dihomeomorphism, subdivision (action refinement)
(precis) Hurewicz-like theorem
(mod) existence of long sequence in homology by the theory of homological category of **[Grandis 91]**
(calc) notion of bisimulation of functors, equivalence with a discrete computable natural homology

Future works :

- link with bisimulations **[Fahrenberg, Legay 13]**, observational equivalences **[Plotkin, Pratt 90]**, temporal properties **[Baldan, Crafa 10]** in true concurrency
- improve the algorithmic
  - better representation of trace spaces
  - decidability of bisimilarity using matrix algorithmic
- link with persistence homology **[Carlsson 09]**
- applications in higher order rewriting