The complexity of temporal logic with until and since over ordinals

S. Demri[1] A. Rabinovich[2]

[1]LSV, ENS Cachan, CNRS, INRIA

[2]Tel Aviv University

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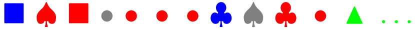
Overview

Temporal logic with until and since

- Linearly ordered set $\langle X, \leq \rangle$: reflexivity, antisymmetry, transitivity, totality.

- Models $\sigma : X \rightarrow \mathcal{P}(\mathrm{PROP})$ based on $\langle X, \leq \rangle$.

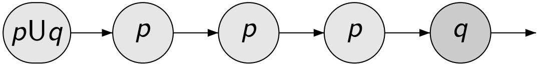


- Formulae in $\mathrm{LTL}(\mathrm{U}, \mathrm{S})$:

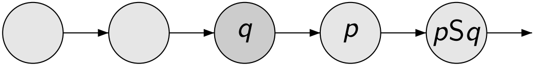
$$\phi ::= p \mid \neg\phi \mid \phi_1 \wedge \phi_2 \mid \phi_1 \mathsf{U} \phi_2 \mid \phi_1 \mathsf{S} \phi_2$$

Satisfaction relation

- $\sigma, \beta \models p$ iff $p \in \sigma(\beta)$,
- $\sigma, \beta \models \phi_1 \mathrm{U} \phi_2$ iff there is $\beta < \gamma$ such that $\sigma, \gamma \models \phi_2$ and for every $\gamma' \in (\beta, \gamma)$, we have $\sigma, \gamma' \models \phi_1$,



- $\sigma, \beta \models \phi_1 \mathrm{S} \phi_2$ iff there is $\gamma < \beta$ such that $\sigma, \gamma \models \phi_2$ and for every $\gamma' \in (\gamma, \beta)$, we have $\sigma, \gamma' \models \phi_1$.



Linear-time temporal logics

- Satisfiability and model checking for LTL with until and since over the natural numbers is PSPACE-complete. [Sisla & Clarke, JACM 85]
- Satisfiability and model checking for LTL with until and since over the reals is PSPACE-complete. [Reynolds, submitted]
- Satisfiability for LTL with until over the class of all linear orders is PSPACE-complete. [Reynolds, JCSS 03]
- $\mathrm{LTL}(\mathsf{U}, \mathsf{S})$ over the class of ordinals is as expressive as the first-order logic over the class of structures $\langle \alpha, < \rangle$ where α is an ordinal. [Kamp, PhD 68]

Well-ordered sets

- Well-ordered set $\langle X, \leq \rangle$: linearly ordered set such that each non-empty subset of X has a least element.
- Dedekind-complete $\langle X, \leq \rangle$: linearly ordered set such that every non-empty bounded subset has a least upper bound.
- Examples:
  - $\langle \mathbb{R}, \leq \rangle$ and $\langle \mathbb{N}, \leq \rangle$ are Dedekind-complete.
  - $\langle \mathbb{Q}, \leq \rangle$ and $\langle \mathbb{Z}, \leq \rangle$ are not well-ordered.
  - All the ordinals are Dedekind-complete.
- Ordinal: isomorphism class of well-ordered sets. ω is the class for $\langle \mathbb{N}, \leq \rangle$.

Two or three things about ordinals

- Every set of ordinals is well-ordered.

- Successor ordinal: existence of a maximal element

$$4 : \bullet \ \bullet \ \bullet \ \bullet \qquad \omega + 1 : \bullet \ \bullet \ \bullet \ \bullet \ \bullet \ \bullet \ \bullet \ \bullet \ \bullet \ \bullet \ \bullet \ \ldots \quad + \quad \bullet$$

- Limit ordinal: no maximal element

$$\omega^2 : \overbrace{\bullet \ \bullet \ \bullet \ \ldots}^{\omega}\overbrace{\bullet \ \bullet \ \bullet \ \ldots}^{\omega}\overbrace{\bullet \ \bullet \ \bullet \ \ldots}^{\omega}\ldots$$

$$\omega^k + \omega : \underbrace{\overbrace{\bullet \ \bullet \ \bullet \ \ldots}^{\omega^{k-1}}\overbrace{\bullet \ \bullet \ \bullet \ \ldots}^{\omega^{k-1}}\overbrace{\bullet \ \bullet \ \bullet \ \ldots}^{\omega^{k-1}}\ldots}_{\omega^k} \quad \overbrace{\bullet \ \bullet \ \bullet \ \ldots}^{\omega}$$

- ω^ω: least upper bound of $\{\omega, \omega^2, \omega^3, \ldots\}$.

Our results about $\mathrm{LTL}(\mathsf{U}, \mathsf{S})$ over ordinals

- If ϕ is satisfiable, then ϕ has an α-model with $\alpha < \omega^{|\phi|+2}$.
- The satisfiability problem for $\mathrm{LTL}(\mathsf{U}, \mathsf{S})$ over the class of countable ordinals is PSPACE-complete.
- $\{\mathsf{O}_1, \ldots, \mathsf{O}_k\}$ first-order definable operators and α countable ordinal. Satisfiability for $\mathrm{LTL}(\mathsf{O}_1, \ldots, \mathsf{O}_k)$ restricted to α-models is in PSPACE.

Uniform satisfiability is also in PSPACE

- Truncation $\mathrm{trunc}_\omega(\alpha) \in (0, \omega^\omega \times 2)$ $(\alpha > 0)$ defined by
  - $\alpha = \omega^\omega \gamma + \beta$ with $\beta \in [0, \omega^\omega)$.
  - $\mathrm{trunc}_\omega(\alpha) = \omega^\omega \times \mathit{min}(\gamma, 1) + \beta$.
- $\mathrm{trunc}_\omega(\omega^k) = \omega^k \qquad \mathrm{trunc}_\omega(\omega^{\omega^\omega} + \omega^k) = \omega^\omega + \omega^k$
- Code of α: representation of $\mathrm{trunc}_\omega(\alpha)$.
- There is a polynomial space algorithm that, given an $\mathrm{LTL}(\mathsf{U}, \mathsf{S})$ formula ϕ and the code of a countable ordinal α, determines whether ϕ has an α-model.

Models of ordinal length

- MSO (and hence LTL) over countable $\langle \alpha, < \rangle$ is decidable. [Büchi & Siefkes, LNM 73]
- Models of length $\omega \times n$ for partial approach to model checking. [Godefroid & Wolper, IC 94]
- Timed automata accepting Zeno words in order to model physical phenomena with convergent execution. [Bérard & Picaronny, 97]
- LTL with until over any countable ordinal is in EXPTIME. [Rohde, PhD 97]
- PSPACE-complete LTL over ω^k-models with unary encoding of X^β and U^β. [Demri & Nowak, IJFCS 07]

Automata on ordinals

- α-sequence $\sigma : \alpha \rightarrow \Sigma$
  (α is identified with $\{\beta : \beta < \alpha\}$.)
- Ordinal automata [Büchi, 64; Choueka, JSCC 78; Wojciechowski, 84].
- Automata on linear orderings [Bruyère & Carton, MFCS 01].
- See also [Bedon, PhD 98].

Automata-based approach

- $\phi \mapsto \mathcal{A}_\phi$ [Büchi 62; Vardi & Wolper, IC 94].
- Models of ϕ are encoded in the language accepted by $\mathcal{A}_\phi$.
- For LTL over ω-sequences, $\mathcal{A}_\phi$ is a Büchi automaton whose size is exponential in $|\phi|$.
- MSO over $\langle \mathbb{N}, \leq \rangle$ is non-elementary whereas LTL is in PSPACE.

Simple ordinal automata

- Simple ordinal automaton $\mathcal{A} = \langle X, Q, \delta_{next}, \delta_{lim} \rangle$:
  - finite set X (basis), set of locations $Q \subseteq \mathcal{P}(X)$,
  - $\delta_{next} \subseteq Q \times Q$: next-step transition relation,
  - $\delta_{lim} \subseteq \mathcal{P}(X) \times Q$: limit transition relation.
- α-path $r : \alpha \rightarrow Q$ $(\alpha > 0)$:
  - for every $\beta + 1 < \alpha$, $\langle r(\beta), r(\beta + 1) \rangle \in \delta_{next}$,
  - for every limit ordinal $\beta < \alpha$, $\exists$ a limit transition $\langle Z, q \rangle$ s.t.

$$\overbrace{(Z \cup Y) \ldots (Z \cup Y') \ldots (Z \cup Y") \text{ etc.} \quad}^{\text{always}(r,\beta)=Z} \underbrace{q}_{\text{position } \beta}$$

  Z: the set of elements of the basis that belong to every location from some $\gamma < \beta$ until β.

Acceptance conditions

- Simple ordinal automaton with acceptance conditions $\langle X, Q, I, F, \mathcal{F}, \delta_{next}, \delta_{lim} \rangle$:
  - $I \subseteq Q$ is the set of initial locations,
  - $F \subseteq Q$ is the set of final locations for accepting runs whose length is some successor ordinal,
  - $\mathcal{F} \subseteq \mathcal{P}(X)$ encodes the accepting condition for runs whose length is some limit ordinal.
- Accepting run $r : \alpha \to Q$:
  - $r(0) \in I$,
  - if α is a successor ordinal, then $r(\alpha - 1) \in F$,
  - otherwise $\mathrm{always}(r, \alpha) \in \mathcal{F}$.
- Nonemptiness problem: check whether $\mathcal{A}$ has an accepting run.

Relationships with other classes of ordinal automata

- Alternative definitions:
  - Add a finite alphabet and define δ_{next} as a subset of $Q \times \Sigma \times Q$.
  - Words of length α are accepted by runs of length $\alpha + 1$ and acceptance condition is defined from a set $F \subseteq Q$.
- With the above extensions, simple ordinal automata recognize the same languages as the Büchi ordinal automata.
  - Identify a location $q \in Q$ with $\{X \subseteq Q : q \in X\}$ (from Büchi to simple ordinal automata).
- Nonemptiness problem for Büchi ordinal automata is in P.
  [Carton, MFCS 02]
- Small runs of length $\omega^{\mathcal{O}(|Q|)}$ (standard) vs. small runs of length $\omega^{\mathcal{O}(|X|)}$ (simple).

$\mathcal{A}_\phi = \langle X, Q, I, F, \mathcal{F}, \delta_{next}, \delta_{lim} \rangle$

- $X = sub(\phi)$. and Q is the set of maximally Boolean consistent subsets of $sub(\phi)$.
- I is the set of locations that contain ϕ and no since formulae.
- F is the set of locations with no elements of the form $\psi_1 \mathsf{U} \psi_2$.
- $\mathcal{F}$ is the set of sets Y such that not $\{\psi_1, \neg\psi_2, \psi_1 \mathsf{U} \psi_2\} \subseteq Y$, for every $\psi_1 \mathsf{U} \psi_2 \in X$.
- For all $q, q' \in Q$, $\langle q, q' \rangle \in \delta_{next}$ iff the conditions below are satisfied:
  - (next$_\mathsf{U}$): for $\psi_1 \mathsf{U} \psi_2 \in sub(\phi)$, $\psi_1 \mathsf{U} \psi_2 \in q$ iff either $\psi_2 \in q'$ or $\psi_1, \psi_1 \mathsf{U} \psi_2 \in q'$,
  - (next$_\mathsf{S}$): for $\psi_1 \mathsf{S} \psi_2 \in sub(\phi)$, $\psi_1 \mathsf{S} \psi_2 \in q'$ iff either $\psi_2 \in q$ or $\psi_1, \psi_1 \mathsf{S} \psi_2 \in q$.

$\mathcal{A}_\phi = \langle X, Q, I, F, \mathcal{F}, \delta_{next}, \delta_{lim} \rangle$ (II)

For all $Y \subseteq X$ and $q \in Q$, $\langle Y, q \rangle \in \delta_{lim}$ iff the conditions below are satisfied:

- ($\mathrm{lim}_{\mathsf{U}}1$): if $\psi_1, \neg\psi_2, \psi_1 \mathsf{U} \psi_2 \in Y$, then either $\psi_2 \in q$ or $\psi_1, \psi_1 \mathsf{U} \psi_2 \in q$,
- ($\mathrm{lim}_{\mathsf{U}}2$): if $\psi_1, \psi_1 \mathsf{U} \psi_2 \in q$ and $\psi_1 \in Y$, then $\psi_1 \mathsf{U} \psi_2 \in Y$,
- ($\mathrm{lim}_{\mathsf{U}}3$): if $\psi_1 \in Y$, $\psi_2 \in q$ and $\psi_1 \mathsf{U} \psi_2$ is in the basis X, then $\psi_1 \mathsf{U} \psi_2 \in Y$,
- ($\mathrm{lim}_{\mathsf{S}}$): for every $\psi_1 \mathsf{S} \psi_2 \in sub(\phi)$, $\psi_1 \mathsf{S} \psi_2 \in q$ iff ($\psi_1 \in Y$ and $\psi_1 \mathsf{S} \psi_2 \in Y$).

Steps to get the PSPACE upper bound

- ϕ is satisfiable iff $\mathcal{A}_\phi$ has an accepting run.
- If ϕ is satisfiable, then ϕ has an α-model with $\alpha < \omega^{|\phi|+2}$.
- The nonemptiness problem for simple ordinal automata can be checked in polynomial space in $|X|$.
- The satisfiability problem for $\mathrm{LTL}(\mathsf{U}, \mathsf{S})$ over the class of ordinals is PSPACE-complete.

Conclusion

- Our main contributions:
  - Satisfiability for $\mathrm{LTL}(\mathsf{U}, \mathsf{S})$ over the class of countable ordinals is PSPACE-complete.
  - For every countable $\alpha \geq \omega^{\omega}$, satisfiability for $\mathrm{LTL}(\mathsf{U}, \mathsf{S})$ restricted to models of length α is in PSPACE.
  - Satisfiability for $\mathrm{LTL}(\mathcal{O}_{\omega})$ over the class of ω^{ω}-models is PSPACE-complete (not presented here).
  - Thanks to Kamp's theorem, the PSPACE upper bound is preserved by adding a finite amount of first-order definable temporal operators.
- Open question: what about other classes of linear orderings?