On the Complexity of Information Logics

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Outline

- Information systems.
- Information logics.
- Tableaux-like decision procedures in PSPACE.
- Tree automata-based decision procedures.

Information systems

- An information system $\mathcal{IS}$ is a structure of the form $\langle OB, AT, (VAL_a)_{a \in AT}, f \rangle$, where
  - OB is a non-empty set of objects,
  - AT is a non-empty set of attributes,
  - VAL_a is a non-empty set of *values* of the attribute a,
  - f is a total function $OB \times AT \rightarrow \bigcup_{a \in AT} \mathcal{P}(VAL_a)$ such that for every $\langle x, a \rangle \in OB \times AT$, $f(x, a) \subseteq VAL_a$.

- $\mathcal{IS}$ is total $\overset{\text{def}}{\Leftrightarrow}$ for every $a \in AT$ and for every $x \in OB$, $f(x, a) \neq \emptyset$.

- $D(AT) \overset{\text{def}}{=} \{x \in OB : \mathrm{card}(a(x)) \leq 1 \text{ for every } a \in AT\}$.

- Structures introduced in [Lipski76,Pawlak82].

Derived relations

- Derived relations make explicit properties in information systems.

- Some standard relations:

  (indiscernibility) $o_1 \; ind(A) \; o_2$ iff for every $a \in A$, $a(o_1) = a(o_2)$,

  (complementarity) $o_1 \; comp(A) \; o_2$ iff for every $a \in A$, $a(o_1) = Val_a \setminus a(o_2)$,

  (similarity) $o_1 \; sim(A) \; o_2$ iff for every $a \in A$, $a(o_1) \cap a(o_2) \neq \emptyset$,

  (forward inclusion) $o_1 \; fin(A) \; o_2$ iff for every $a \in A$, $a(o_1) \subseteq a(o_2)$,

  (backward inclusion) $o_1 \; bin(A) \; o_2$ iff for every $a \in A$, $a(o_2) \subseteq a(o_1)$.

- Since information systems are first-order definable structures, first-order logic provides a means to define much more relations.

Some properties

- Each $ind(A)$ is an equivalence relation.
  $\rightarrow \langle OB, ind(AT) \rangle$ is a rough set.
- If $\mathcal{IS}$ is total, then $sim(A)$ is reflexive and symmetric.
- $fin(A)$ and $bin(A)$ are reflexive and transitive.
- For every $R \in \{ind, fin, bin\}$,
  - $R(\emptyset) = OB \times OB$,
  - $R(A \cup A') = R(A) \cap R(A')$,
  - $A \subseteq A'$ implies $R(A') \subseteq R(A)$.

Frames

- Relative frame: $\langle W, (R^1_P)_{P \subseteq PAR}, \ldots, (R^n_P)_{P \subseteq PAR} \rangle$.
- Plain frame: $\langle W, R^1, \ldots, R^n \rangle$.
- Derived relative frame from “indiscernibility specification”: $\langle OB, (ind(A))_{A \subseteq AT} \rangle$.
- Derived plain frame from “indiscernibility specification”: $\langle OB, ind(AT) \rangle$.
- In full generality, frames can be derived from any first-order specification.

Informational representability

- Informational representability: adequacy between a class of (abstract) frames and a class of frames derived from information systems.

- **Theorem.** [Vakarelov89] The class of plain frames derived from information systems with the indiscernibility specification is precisely the class of S5 frames, i.e. structures of the form $\langle W, R \rangle$ such that R is an equivalence relation.

- **Theorem.** [Vakarelov89] The class of plain frames derived from information systems with the forward inclusion specification is precisely the class of S4 frames, i.e. structures of the form $\langle W, R \rangle$ such that R is reflexive and transitive.

- Basis for the models of information logics.

Approximation operators

- Lower $ind(A)$-approximation of $X \subseteq OB$:
  $L_{ind(A)}(X) = \bigcup\{|x|_{ind(A)} : x \in OB, |x|_{ind(A)} \subseteq X\}$.
- Upper $ind(A)$-approximation of $X \subseteq OB$:
  $U_{ind(A)}(X) = \bigcup\{|x|_{ind(A)} : x \in OB, |x|_{ind(A)} \cap X \neq \emptyset\}$.
- $L_{ind(A)}(X) \subseteq X \subseteq U_{ind(A)}(X)$.
- Knowledge operator:
  $K_{ind(A)}(X) = L_{ind(A)}(X) \cup (OB \setminus U_{ind(A)}(X))$.
- These operators are closely related to modal operators.

Information logics

- Information logics are logical systems developed for the reasoning with data from information systems.
- Here, the information logics are modal logics in a broad sense.
- Classes of models defined either from plain frames or from relative frames.
- Some features of information logics:
  - Complicated conditions between accessibility relations.
  - Boolean structure of attribute expressions.
  - Presence of intersection on relations.
- Specific instantiations of known proof techniques are needed.

Logic NIL

- NIL introduced in [Orlowska&Pawlak84,Vakarelov87].
- Formulae: $\phi ::= p \mid \phi \wedge \phi \mid \neg\phi \mid [\sigma]\phi \mid [\leq]\phi \mid [\geq]\phi$.
- $[\sigma]$: “similarity” modality.
- $[\leq]$, $[\geq]$: “forward” and “backward” modality, respectively.
- NIL-model $\mathcal{M} = \langle W, R_{\leq}, R_{\geq}, R_{\sigma}, m \rangle$:
  - W non-empty set and $m : W \rightarrow \mathcal{P}(\mathrm{PROP})$,
  - $R_{\leq}$ is the converse of $R_{\geq}$,
  - $R_{\leq}$ is reflexive and transitive (S4 modality),
  - R_{σ} is reflexive and symmetric (B modality),
  - $R_{\geq} \circ R_{\sigma} \circ R_{\leq} \subseteq R_{\sigma}$.

Satisfaction relation

- **Theorem.** [Vakarelov87] The class of NIL frames is exactly the set of structures $\langle OB, fin(AT), bin(AT), sim(AT) \rangle$ derived from total information systems.

- $\mathcal{M}, w \models p$ iff $w \in m(p)$,
  $\mathcal{M}, w \models \phi_1 \wedge \phi_2$ iff $\mathcal{M}, w \models \phi_1$ and $\mathcal{M}, w \models \phi_2$,

- $\mathcal{M}, w \models [\alpha]\phi$ iff for every $w' \in R_\alpha(w)$, $\mathcal{M}, w' \models \phi$ with
  - $\alpha \in \{\sigma, \leq, \geq\}$,
  - $R_\alpha(w) = \{w' \in W : \langle w, w' \rangle \in R_\alpha\}$.

- NIL satisfiability is PSPACE-hard (by easy reduction from modal logic S4, restriction of NIL to $[\leq]$).

- NIL satisfiability can be easily translated into first-order logic.

Logic IL [Vakarelov91]

- Formulae: $\phi ::= \mathtt{D} \mid p \mid \phi \land \phi \mid \neg\phi \mid [\sigma]\phi \mid [\leq]\phi \mid [\equiv]\phi$.
- $[\equiv]$: "indiscernibility" modality.
- $\mathtt{D}$: deterministic objects.
- IL-model $\mathcal{M} = \langle W, R_{\equiv}, R_{\leq}, R_{\sigma}, D, m \rangle$:
  - $m(\mathtt{D}) = D$,
  - $R_{\equiv}$ is an equivalence relation,
  - $R_{\leq}$ is reflexive and transitive,
  - R_{σ} is weakly reflexive and symmetric,
  - $y \in D$ and $\langle x, y \rangle \in R_{\sigma}$ imply $x \in D$,
  - + many other conditions, some of them not being modally definable.

Satisfaction relation

- **Theorem.** [Vakarelov91] The class of IL frames is exactly the set of structures $\langle OB, ind(AT), fin(AT), sim(AT), D(AT)\rangle$ derived from information systems.

- $\mathcal{M}, w \models \mathtt{D}$ iff $w \in D$,

- IL satisfiability is PSPACE-hard.

- IL satisfiability is in NEXPTIME by using a sophisticated filtration construction [Vakarelov 91].

Logic DAL [Fariñas&Orłowska85]

- Modal expressions: $a ::= c \mid a \cap a \mid a \cup^* a$.
- Formulae: $\phi ::= p \mid \phi \wedge \phi \mid \neg\phi \mid [a]\phi$.
- $[a]$: “indiscernibility” modality.
- DAL-model $\mathcal{M} = \langle W, (R_a)_{a \in M}, m \rangle$:
  - W non-empty set and $m : W \rightarrow \mathcal{P}(\mathrm{PROP})$,
  - each R_a is an equivalence relation,
  - $R_{a \cap a'} = R_a \cap R_{a'}$, $R_{a \cup^* a'} = (R_a \cup R_{a'})^*$.
- DAL satisfiability is decidable [Lutz05].

Logic DALLA [Gargov86]

- Same language as DAL.
- Relations $R, R' \subseteq W \times W$ are in local agreement $\stackrel{\text{def}}{\Leftrightarrow}$ for every $x \in W$, either $R(x) \subseteq R'(x)$ or $R'(x) \subseteq R(x)$.
- For all equivalence relations R and R', R and R' are in local agreement iff $R \cup R'$ is transitive.
- DALLA-model $\mathcal{M} = \langle W, (R_a)_{a \in M}, m \rangle$:
  - each R_a is an equivalence relation,
  - $R_{a \cap a'} = R_a \cap R_{a'}$, $R_{a \cup^* a'} = R_a \cup R_{a'}$.
- DALLA$'$: restriction of DALLA to modalities $[c]$ (no $\cap$ and $\cup^*$).

LA-logics

- Same language as DALLA$'$, M_0: set of modal constants.
- Each LA-logic $\mathcal{L}$ is characterized by some set $lin(\mathcal{L})$ of linear orderings over M_0.
- $\mathcal{L}$-model $\mathcal{M} = \langle W, (R_a)_{a \in M}, m \rangle$:
  - each R_a is an equivalence relation,
  - for every $w \in W$, there is $\preceq \in lin(\mathcal{L})$ such that for all $a, b \in M_0$, if $a \preceq b$, then $R_a(w) \subseteq R_b(w)$.
- DALLA$'$ is an LA-logic with lin(DALLA$'$) being the set of all linear orderings over M_0.
- Existence of a logarithmic space reduction from DALLA to DALLA$'$ (with renaming technique).

SIM [Konikowska97] formulae

- Countably infinite set $\mathrm{PROP} = \{p_1, p_2, \ldots\}$ of propositional variables.

- Countably infinite set $\mathrm{NOM} = \{\mathrm{x}_1, \mathrm{x}_2, \ldots\}$ of object nominals.

- The set P of parameter expressions is the smallest set containing
  - a countably infinite set $\mathrm{PNOM} = \{\mathrm{E}_1, \mathrm{E}_2, \ldots\}$ of parameter nominals and
  - a countably infinite set $\mathrm{PARVAR} = \{\mathrm{C}_1, \mathrm{C}_2, \ldots\}$ of parameter variables,

  and that is closed under the Boolean operators $\cap, \cup, -$.

- Formulae: $\phi ::= p \mid \mathrm{x} \mid \neg\phi \mid \phi \wedge \phi \mid [\mathrm{A}]\phi$ ($\mathrm{A} \in \mathrm{P}$).
  Example: $[\mathrm{E}_2 \cap -\mathrm{E}_2]\mathrm{x} \Rightarrow [\mathrm{E}_1 \cup \mathrm{C}_1](\mathrm{x} \vee p)$.

P-*interpretation*

- A P-interpretation m is a map $m : \mathrm{P} \to \mathcal{P}(PAR)$ where PAR is a non-empty set and for all $\mathrm{A}_1, \mathrm{A}_2 \in \mathrm{P}$,
  - if $\mathrm{A}_1, \mathrm{A}_2 \in \mathrm{PNOM}$ and $\mathrm{A}_1 \neq \mathrm{A}_2$, then $m(\mathrm{A}_1) \neq m(\mathrm{A}_2)$,
  - if $\mathrm{A}_1 \in \mathrm{PNOM}$, then $m(\mathrm{A}_1)$ is a singleton,
  - $m(\mathrm{A}_1 \cap \mathrm{A}_2) = m(\mathrm{A}_1) \cap m(\mathrm{A}_2)$,
  - $m(\mathrm{A}_1 \cup \mathrm{A}_2) = m(\mathrm{A}_1) \cup m(\mathrm{A}_2)$,
  - $m(-\mathrm{A}_1) = PAR \setminus m(\mathrm{A}_1)$.
- $\mathrm{A} \equiv \mathrm{B}$ [resp. $\mathrm{A} \sqsubseteq \mathrm{B}$] $\overset{\text{def}}{\Leftrightarrow}$ for every P-interpretation m, we have $m(\mathrm{A}) = m(\mathrm{B})$ [resp. $m(\mathrm{A}) \subseteq m(\mathrm{B})$].

SIM-model

A SIM-model $\mathcal{M}$ is a structure $\mathcal{M} = \langle W, (\mathcal{R}_P)_{P \subseteq PAR}, m \rangle$, where

($\star$) W and PAR are non-empty sets,

($\star\star$) $(\mathcal{R}_P)_{P \subseteq PAR}$ is a family of binary relations on W,

(uni) $\mathcal{R}_\emptyset$ is the cartesian product $W \times W$,

(refl) $\mathcal{R}_P$ is reflexive for every $P \subseteq PAR$,

(sym) $\mathcal{R}_P$ is symmetric for every $P \subseteq PAR$,

(inter) $\mathcal{R}_{P \cup Q} = \mathcal{R}_P \cap \mathcal{R}_Q$ for all $P, Q \subseteq PAR$.

($\star\star\star$) $m : \mathrm{NOM} \cup \mathrm{PROP} \cup \mathrm{P} \rightarrow \mathcal{P}(W) \cup \mathcal{P}(PAR)$ is such that $m(p) \subseteq W$ for every $p \in \mathrm{PROP}$, $m(\mathrm{x}) = \{w\}$, where $w \in W$ for every $\mathrm{x} \in \mathrm{NOM}$, and the restriction of m to P is a P-interpretation.

Properties

- **Theorem.** [Vakarelov87] The class of information frames $\langle OB, (sim_A)_{A \subseteq AT} \rangle$ derived from information systems is precisely the class of SIM-frames.
- The parameter expressions are interpreted within the Boolean algebra $\mathcal{B} = \langle \mathcal{P}(PAR), \cup, \cap, -, 1, 0 \rangle$ for some non-empty set PAR.
- Conditions on $(\mathcal{R}_P)_{P \subseteq PAR}$ induce a semi-lattice structure of $\mathcal{L} = \langle \{\mathcal{R}_P : P \in \mathcal{B}\}, \cap \rangle$ with zero element $W \times W$.
- Condition **(inter)** allows SIM to capture intersection on relations. $R_{m(\mathrm{A} \cup \mathrm{B})} = R_{m(\mathrm{A})} \cap R_{m(\mathrm{B})}$.
- SIM contains universal modality since $R_{m(\mathrm{A} \cap - \mathrm{A})} = W \times W$.

Satisfaction relation

- $\mathcal{M}, w \models p$ iff $w \in m(p)$ for $p \in \mathrm{PROP} \cup \mathrm{NOM}$,
- $\mathcal{M}, w \models \neg\phi$ iff not $\mathcal{M}, w \models \phi$,
- $\mathcal{M}, w \models \phi \wedge \psi$ iff $\mathcal{M}, w \models \phi$ and $\mathcal{M}, w \models \psi$,
- $\mathcal{M}, w \models [\mathrm{A}]\phi$ iff for every $w' \in W$, if $\langle w, w' \rangle \in \mathcal{R}_{m(\mathrm{A})}$, then $\mathcal{M}, w' \models \phi$.

Some other information logics

- Logics with knowledge operators [Orłowska89].
- Relative versions of NIL, IL, $\ldots$
- Variants of SIM (IND, FORIN, $\ldots$).
- Logic of indiscerniblity relations [Orłowska93], relative variant of DAL.
- Information logics with relative frames of level > 1 [Balbiani&Orłowksa99].

Deciding NIL by filtration

- A NIL formula ϕ has a model iff it has a model of size at most $2^{\mathcal{O}(|\phi|)}$.
- Proved by a filtration construction [Vakarelov87].
- **Corollary.** NIL satisfiability is in NEXPTIME.
- NEXPTIME upper bound for IL can be shown with an even more sophisticated filtration construction [Vakarelov96].

Deciding NIL by translation

- Reduction to satisfiability for Propositional Dynamic Logic with Converse (CPDL) known to be EXPTIME-complete.

- Logarithmic space reduction:
  - $f(p) = p$,
  - f is homomorphic wrt Boolean connectives,
  - $f([\sigma]\phi) = [(c_2^{-1})^*; (c_1 \cup c_1^{-1} \cup id); c_2^*]f(\phi)$,
  - $f([\leq]\phi) = [c_2^*]f(\phi)$,
  - $f([\geq]\phi) = [(c_2^{-1})^*]f(\phi)$.

- ϕ is NIL satisfiable iff $f(\phi)$ is CPDL satisfiable.

- NIL is a regular grammar logic in the sense of [Demri&DeNivelle05].

Algorithm à la Ladner for NIL

- In [Ladner77], PSPACE algorithm is designed for modal logics K and S4.
- Extension to tense S4 (NIL without $[\sigma]$) in [Spaan93].
- Principle: construction of a finite tree with nodes labeled by sets of formulae from which a model of the formula ϕ can be built.
- The algorithm can be viewed as a strategy for building proofs in some sequent/tableaux calculus.

Closure

Let X be a set of NIL-formulae. Let $\mathrm{cl}(X)$ be the smallest set of formulae such that:

- $X \subseteq \mathrm{cl}(X)$,
- if $\neg\phi \in \mathrm{cl}(X)$, then $\phi \in \mathrm{cl}(X)$,
- if $\phi_1 \wedge \phi_2 \in \mathrm{cl}(X)$, then $\phi_1, \phi_2 \in \mathrm{cl}(X)$,
- if $[\leq]\phi \in \mathrm{cl}(X)$, then $\phi \in \mathrm{cl}(X)$,
- if $[\geq]\phi \in \mathrm{cl}(X)$, then $\phi \in \mathrm{cl}(X)$,
- if $[\sigma]\phi \in \mathrm{cl}(X)$, then $[\geq]\phi \in \mathrm{cl}(X)$,
- if $[\sigma]\phi \in \mathrm{cl}(X)$ and ϕ is not a $[\leq]$-formula, then $[\sigma][\leq]\phi \in \mathrm{cl}(X)$,
- if $[\sigma][\leq]\phi \in \mathrm{cl}(X)$, then $[\sigma]\phi \in \mathrm{cl}(X)$.

Directed closure

- $\mathrm{card}(\mathrm{cl}(\{\phi\})) < 5 \times |\phi|$.

- $s \in \{sim, fin, bin\}^*$, let $\mathrm{cl}(s, \phi)$ be the smallest set such that:
  - $\mathrm{cl}(\lambda, \phi) = \mathrm{cl}(\{\phi\})$, $\mathrm{cl}(s, \phi)$ is closed,
  - if $[\sigma][\leq]\psi \in \mathrm{cl}(s, \phi)$, then $[\leq]\psi \in \mathrm{cl}(s \cdot sim, \phi)$,
  - if $[\leq]\psi \in \mathrm{cl}(s, \phi)$, then $[\leq]\psi \in \mathrm{cl}(s \cdot fin, \phi)$,
  - if $[\geq]\psi \in \mathrm{cl}(s, \phi)$, then $[\geq]\psi \in \mathrm{cl}(s \cdot bin, \phi)$,
  - if $[\sigma][\leq]\psi \in \mathrm{cl}(s, \phi)$, then $[\sigma][\leq]\psi \in \mathrm{cl}(s \cdot bin, \phi)$.

- **Lemma.** Let ϕ be a formula and $s \in \{sim, fin, bin\}^*$ be such that neither $bin \cdot bin$ nor $fin \cdot fin$ is a substring of s and $|s| \geq 3 \times |\phi|$. Then $\mathrm{cl}(s, \phi) = \emptyset$.

Syntactic relations

- The binary relation $\approx$ on sets of NIL-formulae is defined as follows: $X \approx Y \stackrel{\text{def}}{\Leftrightarrow}$
  - for every $[\sigma]\psi \in X$, $\psi \in Y$,
  - for every $[\sigma]\psi \in Y$, $\psi \in X$.

- The binary relation $\preceq$ is defined as follows: $X \preceq Y \stackrel{\text{def}}{\Leftrightarrow}$
  - for every $[\leq]\psi \in X$, $[\leq]\psi, \psi \in Y$,
  - for every $[\geq]\psi \in Y$, $[\geq]\psi, \psi \in X$,
  - for every $[\sigma]\psi \in Y$, $[\sigma]\psi \in X$.

Consistency

- X be a subset of $\mathrm{cl}(s,\phi)$ for some $s \in \{sim, bin, fin\}^*$ and for some formula ϕ. X is s-consistent $\overset{\text{def}}{\Leftrightarrow}$ for every $\psi \in \mathrm{cl}(s,\phi)$:
  - if $\psi = \neg\varphi$, then $\varphi \in X$ iff not $\psi \in X$,
  - if $\psi = \varphi_1 \wedge \varphi_2$, then $\{\varphi_1, \varphi_2\} \subseteq X$ iff $\psi \in X$,
  - if $\psi = [\alpha]\varphi$ for some $\alpha \in \{\sigma, \leq, \geq\}$ and $\psi \in X$, then $\varphi \in X$,
  - if $\psi = [\sigma]\varphi$, $\varphi \neq [\leq]\varphi'$ and $\psi \in X$, then $[\sigma][\leq]\varphi \in X$,
  - if $\psi = [\sigma][\leq]\varphi$ and $\psi \in X$, then $[\sigma]\varphi \in X$,
  - if $\psi = [\sigma]\varphi$ and $\psi \in X$, then $[\geq]\varphi \in X$.

- **Lemma.** Let $\mathcal{M} = \langle W, R_{\leq}, R_{\geq}, R_{\sigma}, m\rangle$ be a NIL model, $w \in W$, $s \in \{sim, fin, bin\}^*$, ϕ be a NIL formula. Then, the set $\{\psi \in \mathrm{cl}(s,\phi) : \mathcal{M}, w \models \psi\}$ is s-consistent.

Principle of the algorithm

- Construction of a finite tree with nodes labeled by sets of formulae from which a model of the formula ϕ can be built.
- $\texttt{NIL-WORLD}(\Sigma, s, \phi)$ returns a Boolean: Σ is a nonempty finite sequence of subsets of $\mathrm{cl}(\{\phi\})$ and $s \in \{sim, fin, bin\}^*$.
- For any $X \subseteq \mathrm{cl}(\{\phi\})$ and for any call $\texttt{NIL-WORLD}(\Sigma, s, \phi)$ in $\texttt{NIL-WORLD}(X, \lambda, \phi)$ (at any recursion depth), we have $last(\Sigma) \subseteq \mathrm{cl}(s, \phi)$.
- Cycle detection because of the S4 modalities $[\leq]$ and $[\geq]$.

Algorithm

`function` `NIL-WORLD`(Σ, s, ϕ)

if $last(\Sigma)$ is not s-consistent, then return false;

for $[\sigma]\psi \in \mathrm{cl}(s, \phi) \setminus last(\Sigma)$ do

for each $X_\psi \subseteq \mathrm{cl}(s \cdot sim, \phi) \setminus \{\psi\}$ such that $last(\Sigma) \approx X_\psi$, call `NIL-WORLD`$(X_\psi, s \cdot sim, \phi)$. If all these calls return false, then return false;

for $[\leq]\psi \in \mathrm{cl}(s, \phi) \setminus last(\Sigma)$ do

if there is no $X \in \Sigma$ such that $\psi \notin X$, $last(\Sigma) \preceq X$, and $last(s) = fin$, then for each $X_\psi \subseteq \mathrm{cl}(s \cdot fin, \phi) \setminus \{\psi\}$ such that $last(\Sigma) \preceq X_\psi$, if $last(s) = fin$, then call `NIL-WORLD`$(\Sigma \cdot X_\psi, s, \phi)$, otherwise call `NIL-WORLD`$(X_\psi, s \cdot fin, \phi)$. If all these calls return false, then return false;

+ similar instructions for $[\geq]$ ...

Return true.

Correction and complexity

- **Lemma.** ϕ is NIL satisfiable iff there is $X \subseteq \mathrm{cl}(\{\phi\})$ such that $\phi \in X$ and `NIL-WORLD`(X, λ, ϕ) returns true.

- Let $X \subseteq \mathrm{cl}(\{\phi\})$.
  - `NIL-WORLD`(X, λ, ϕ) terminates and requires space in $\mathcal{O}(|\phi|^4)$.
  - Let `NIL-WORLD`(Σ, s, ϕ) be a call in the computation of `NIL-WORLD`(X, λ, ϕ). Then, $|\Sigma| \leq 25 \times |\phi|^2$ and $|s| \leq 3 \times |\phi|$.

- **Theorem.** NIL satisfiability is in PSPACE.

PSPACE-complete LA-logics

- Ladner-like algorithms for “nice” LA-logics.
- PSPACE-hardness can be shown by reducing QBF.
- **Corollary.** The logics below are PSPACE-complete:
  - DALLA$'$,
  - DALLA,
  - Nakamura’s logic of graded modalities [Nakamura93].

Tree automata and SIM

- Numerous reductions to the emptiness problem for tree automata (PDL, modal μ-calculus, etc.).
  ϕ is satisfiable iff $\mathrm{L}(\mathcal{A}_\phi)$ is non-empty.
- Tree model property: for every satisfiable formula there is a (possibly infinite) tree from which can be built easily a model.
- For sake of presentation, we consider SIM without parameter nominals.
- Only Büchi tree automata are needed.

Global information for SIM-models

- Guessing a global information for a given formula ϕ will correspond to the primary non-deterministic choice in the automata built for ϕ.
- A global information G for ϕ is a structure $\langle UF, EF, EQ, NOM, \mathrm{R_N} \rangle$ such that
  - UF and EF are subsets of $\{\varphi \in sub(\phi) : \varphi = [\mathrm{A}]\psi\}$,
  - $EQ \subseteq \mathrm{NOM}(\phi)^2$,
  - NOM is a map $NOM : \mathrm{NOM}(\phi) \rightarrow \mathcal{P}(sub(\phi))$,
  - $\mathrm{R_N} \subseteq \mathrm{NOM}(\phi)^2 \times \mathrm{P}(\phi)$.
- Definition of SIM-consistency for G, i.e. EQ is an equivalence relation or $[\mathrm{A}]\psi \in EF \cup UF$ implies $m(\mathrm{A}) = \emptyset$ for every m.

Consistency and syntactic relation

- X subset of $sub(\phi)$. X is locally SIM-consistent $\stackrel{\text{def}}{\Leftrightarrow}$ for every $\psi \in sub(\phi)$,
  - if $\psi = \neg\varphi$, then $\varphi \in X$ iff $\psi \notin X$,
  - if $\psi = \varphi_1 \wedge \varphi_2$, then $\{\varphi_1, \varphi_2\} \subseteq X$ iff $\psi \in X$,
  - if $\psi = [\mathrm{A}]\varphi$ and $\psi \in X$, then $\varphi \in X$.

- Let G be a SIM-consistent global information. Given two locally SIM-consistent sets X and Y and a parameter expression A occurring in ϕ, we write $X \sim_{G,\mathrm{A}} Y$ to denote that,
  - for every $[\mathrm{B}]\psi \in X$, if $\mathrm{B} \sqsubseteq \mathrm{A}$, then $\psi \in Y$, and
  - for every $[\mathrm{B}]\psi \in Y$, if $\mathrm{B} \sqsubseteq \mathrm{A}$, then $\psi \in X$.

Symbolic states

A symbolic state for ϕ is either $\perp$ or a triple $q = \langle \mathrm{A}, X, T \rangle$ such that

- $\mathrm{A} \in \mathrm{P}(\phi)$. A refers to the relation $\mathcal{R}_{m(\mathrm{A})}$ which relates q's (unique) predecessor to q.
- $X \in \mathcal{P}(sub(\phi))$. X is the set of formulae satisfied in q.
- $T \subseteq \mathrm{P}(\phi) \times \mathrm{NOM}(\phi)$. T is a *table* such that, for every $\langle \mathrm{B}, \mathrm{x} \rangle \in T$, $\langle q, w \rangle \in \mathcal{R}_{m(\mathrm{A})}$ for $m(\mathrm{x}) = \{w\}$.
- The “dummy” value $\perp$ is used for those nodes in a tree not representing objects.

Consistency wrt to G

- G SIM-consistent global information. A symbolic state $q = \langle \mathrm{A}, X, T \rangle$ is locally SIM-consistent with respect to $G \overset{\text{def}}{\Leftrightarrow} q$ is dummy or if it satisfies
  - X is locally SIM-consistent,
  - for every $\mathrm{x} \in \mathrm{NOM}(\phi)$, $\mathrm{x} \in q$ implies $X = NOM(\mathrm{x})$ and $T = \{\langle \mathrm{B}, \mathrm{y} \rangle \mid \langle \mathrm{x}, \mathrm{y}, \mathrm{B} \rangle \in \mathrm{R_N}\}$,
  - for every $\langle \mathrm{A}, \mathrm{x} \rangle \in T$, $X \sim_{G,\mathrm{A}} NOM(\mathrm{x})$,
  - for all $\langle \mathrm{A}_1, \mathrm{x}_1 \rangle, \ldots, \langle \mathrm{A}_n, \mathrm{x}_n \rangle \in T$ with $n \geq 1$, if $\mathrm{x}_1 = \ldots = \mathrm{x}_n$ then, for every $\mathrm{A} \in \mathrm{P}(\phi)$ with $\mathrm{A} \sqsubseteq \mathrm{A}_1 \cup \ldots \cup \mathrm{A}_n$, we have $\langle \mathrm{A}, \mathrm{x}_1 \rangle \in T$,
  - for every $\mathrm{B} \in \mathrm{P}(\phi)$ such that $\mathrm{B} \equiv \emptyset$, for every $\mathrm{x} \in \mathrm{NOM}(\phi)$, $\langle \mathrm{B}, \mathrm{x} \rangle \in T$,
  - $UF \subseteq X$ and $EF \cap X = \emptyset$.
- $\mathrm{SYMB}(\phi)$: set of symbolic states of ϕ, and $\mathrm{SYMB}_G(\phi)$: set of symbolic states of ϕ that are locally SIM-consistent wrt G.

Hintikka trees (I)

- Given $K \geq 1$ and a finite alphabet Σ, an infinite Σ, K-tree $\mathcal{T}$ is a mapping $\mathcal{T} : \{1, \ldots, K\}^* \rightarrow \Sigma$.
- Let ϕ be a SIM-formula with $K = |\phi|$.
- A $\mathrm{SYMB}(\phi), K$-tree $\mathcal{T}$ is a Hintikka tree for $\phi \overset{\text{def}}{\Leftrightarrow}$ there exists a SIM-consistent global information $G = \langle UF, EF, EQ, NOM, \mathrm{R_N} \rangle$ for ϕ such that
  - $\mathcal{T}(\epsilon)$ is dummy,
  - there is $i \in \{1, \ldots, K\}$ such that $\phi \in \mathcal{T}(i)$,
  - for every $\mathrm{x} \in \mathrm{NOM}(\phi)$, there is a unique $i \in \{1, \ldots, K\}$ such that $\mathrm{x} \in \mathcal{T}(i)$ (this i is then written i_{x}),

  and each $s \in \{1, \ldots, K\}^+$ satisfies the following conditions:

Hintikka trees (II)

and each $s \in \{1, \ldots, K\}^+$ satisfies the following conditions:

- $\mathcal{T}(s)$ is locally SIM-consistent with respect to G,
- if $\mathcal{T}(s)$ is dummy, then $\mathcal{T}(s \cdot 1), \ldots, \mathcal{T}(s \cdot K)$ are also dummy,
- if s is of length at least 2, then $\mathcal{T}(s)$ is not a named symbolic state,
- if $\mathcal{T}(s) = \langle \mathrm{A}, X, T \rangle$ is not dummy and $[\mathrm{B}]\psi \in sub(\phi) \setminus X$, then
  1. either there is $i \in \{1, \ldots, K\}$ with $\mathcal{T}(s \cdot i) = \langle \mathrm{B}, X', T' \rangle$, $\mathcal{T}(s \cdot i)$ is not dummy, and $\psi \not\in X'$ or
  2. there is $\mathrm{x} \in \mathrm{NOM}(\phi)$ such that $\langle \mathrm{B}, \mathrm{x} \rangle \in T$ and $\psi \not\in \mathcal{T}(i_{\mathrm{x}})$;
- for every $i \in \{1, \ldots, K\}$, if both $\mathcal{T}(s) = \langle \mathrm{A}, X, T \rangle$ and $\mathcal{T}(s \cdot i) = \langle \mathrm{B}, X', T' \rangle$ are not dummy, then $X \sim_{G,\mathrm{B}} X'$.

Complexity of SIM

- **Lemma.** For every SIM-formula ϕ, ϕ is SIM-satisfiable iff ϕ has a Hintikka tree.
- The class of Hintikka tree for ϕ can be defined as the language recognized by a Büchi tree automaton.
- SIM can be decided in EXPTIME by using the complexity of the translation combined by that of checking emptiness for Büchi tree automata [Demri&Sattler02].
- SIM is EXPTIME-hard as a consequence of [Hemaspaandra96].

Other proof techniques

- Filtration, see e.g. [Vakarelov97].
  For numerous information logics only decidability is known.

- Translation of information logics characterized by relative frames into standard modal logics.

- Optimal complexity upper bounds sometimes obtained via the renaming technique.

- Submodel construction to show NP upper bound of the logic of indiscernibility and complementarity.

Concluding remarks

- How to deal with natural extensions of known logics?
  For instance, the indiscernibility variant of SIM is not known to be decidable.

- How to characterize first-order specifications on information systems that lead to decidable information logics?

- How to distinguish the information logics that are the most useful in applications? i.e. to make some order in the jungle of information logics.

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