Logics with concrete domains: an introduction

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Concrete domains in TCS

- Constraint satisfaction problems (CSP).
- Satisfiability Modulo Theory (SMT) solvers.
  String theories, arithmetical theories, array theories, etc.
  See e.g. [Barrett & Tinelli, Handbook 2018]
- Description logics with concrete domains.
  [Baader & Hanschke, IJCAI'91, Lutz, PhD 2002]
- Temporal logics with arithmetical constraints.
  See e.g. [Bouajjani et al., LICS 95; Comon & Cortier, CSL'00]
- Verification of database-driven systems.
  [Deutsch & Hull & Vianu, SIGMOD 2014]

Concrete domains and constraints

- Concrete domain $\mathcal{D} = (\mathbb{D}, R_1, R_2, \dots)$: fixed non-empty domain with a family of relations.
- $(\mathbb{Z}, +, <, =, 0, 1)$, $(\mathbb{N}, <, +1)$, $(\mathbb{R}, <, =)$, $(\mathbb{D}, \equiv)$.
- Terms are built from variables x and expressions $\mathsf{X}^i x$.
- Constraint C: Boolean combination of atomic constraints of the form $R(\mathtt{t}_1, \dots, \mathtt{t}_d)$.

$$(\mathsf{X}x_1 = x_2 + \mathsf{XXX}x_3) \vee (x_1 > \mathsf{X}x_4)$$

- Constraints are interpreted on valuations $\mathfrak{v}$ that assign elements from $\mathbb{D}$ to the terms and

$$\mathfrak{v} \models R(\mathtt{t}_1, \dots, \mathtt{t}_d) \text{ iff } (\mathfrak{v}(\mathtt{t}_1), \dots, \mathfrak{v}(\mathtt{t}_d)) \in R^{\mathcal{D}}.$$

- A constraint C over $\mathcal{D}$ is satisfiable $\overset{\text{def}}{\Leftrightarrow}$ there is a valuation $\mathfrak{v}$ such that $\mathfrak{v} \models C$.

More examples

- $(\mathbb{Q}, <, =)$, $(\mathbb{R}, <, =)$, $(\mathbb{Z}, <, =)$, $(\mathbb{N}, <, =)$.
- $(\{0,1\}^*, \preceq_{pre})$ with binary strings.
- Temporal concrete domain $\mathcal{D}_A = (I_{\mathbb{Q}}; (R_i)_{i \in [1,13]})$ with
  - $I_{\mathbb{Q}}$: set of closed intervals $[r, r'] \subseteq \mathbb{Q}$
  - $(R_i)_{i \in [1,13]}$ is the family of 13 Allen's relations.

[Allen83; CACM 1983]

- Concrete domain RCC8 with space regions in $\mathbb{R}^2$ contains topological relations between spatial regions.

See e.g. [Wolter & Zakharyaschev, KR'00]

Symbolic models – the linear case

- AC_k: set of atomic constraints built over $\{x_1, \ldots, x_k\}$ and $\{\mathsf{X}x_1, \ldots, \mathsf{X}x_k\}$. ('$\mathsf{X}x$' refers to the next value of x.)

- Symbolic model $\mathtt{w} : \mathbb{N} \to \mathcal{P}(\mathrm{AC}_k)$. ($\omega$-sequence)

$$\begin{tikzcd}
x_1 & \bullet \arrow[r, "="'] \arrow[d, "="'] & \bullet \arrow[r, "<"'] \arrow[d, "="] & \bullet \arrow[r, "<"'] & \bullet \arrow[r, "<"'] & \bullet & \cdots \\
x_2 & \bullet \arrow[r, "="] \arrow[d, "="'] & \bullet \arrow[r, "<"] \arrow[d, "="] & \bullet \arrow[r, "="] & \bullet \arrow[r, "<"] & \bullet & \cdots \\
x_3 & \bullet \arrow[r, "="] & \bullet \arrow[r, "="] & \bullet \arrow[r, "="] & \bullet \arrow[r, "="] & \bullet & \cdots \\
1 & \bullet \arrow[r, "="] \arrow[u, "<"] & \bullet \arrow[r, "="] \arrow[u, "="] & \bullet \arrow[r, "="] \arrow[u, "<"] & \bullet \arrow[r, "="] \arrow[u, "="] & \bullet & \cdots \\
0 & \bullet \arrow[r, "="] \arrow[u, "<"] & \bullet \arrow[r, "="] \arrow[u, "<"] & \bullet \arrow[r, "="] \arrow[u, "<"] & \bullet \arrow[r, "="] \arrow[u, "<"] & \bullet \arrow[u, "<"] & \cdots
\end{tikzcd}$$

- $\mathtt{w}$ is $\mathcal{D}$-satisfiable $\overset{\text{def}}{\Leftrightarrow}$ there is $\mathfrak{v} : \mathbb{N} \times \{x_1, \ldots, x_k\} \to \mathbb{D}$ such that for all i, $\{c \in \mathrm{AC}_k \mid \mathfrak{v}, i \models c\} = \mathtt{w}(i)$.

- $\mathfrak{v}, i \models x = \mathsf{X}y$ iff $\mathfrak{v}(i, x) = \mathfrak{v}(i+1, y)$.

A selection of problems

- Given a concrete domain $\mathcal{D}$, how to characterise the class of $\mathcal{D}$-satisfiable symbolic models?

  ($\{x > \mathsf{X}x\}^\omega$ not $\mathbb{N}$-satisfiable)

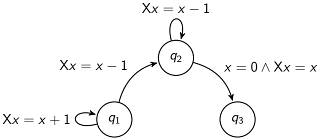
- Given a formalism to define symbolic models (logics, automata, etc.), how to determine whether a recognized $\mathcal{D}$-satisfiable symbolic model exists?

- Can the class of $\mathcal{D}$-satisfiable symbolic models be expressed by a given formalism?

  (ω-regularity/Büchi automata?)

- In this talk:
  - Concrete domains: $(\mathbb{Q}, <, =)$, $(\mathbb{N}, <, =)$.
  - Formalisms: constrained automata, constrained LTL, description logics, MSO-like logics.

Constrained automata



- $\mathcal{D}$-automaton $\mathbb{A} = (S, \delta, I, F)$ with k variables:
  - S is a non-empty finite set of control states,
  - Set $I \subseteq S$ of initial states; set $F \subseteq S$ of final states,
  - δ is a finite subset of $S \times \mathrm{C_k} \times \mathrm{S}$, where $\mathrm{C_k}$ is the set of $\mathcal{D}$-constraints built over $\{x_1, \ldots, x_k\} \cup \{\mathrm{X}x_1, \ldots, \mathrm{X}x_k\}$.

[Revesz, Book 2002]

- $\mathfrak{v}_0\mathfrak{v}_1 \cdots \in \mathrm{L}(\mathbb{A}) \overset{\text{def}}{\Leftrightarrow}$ there is $q_0 \xrightarrow{C_0} q_1 \xrightarrow{C_1} \cdots$ such that
  - $q_0 \in I$ and $q \in F$ occurs infinitely often in $q_0 q_1 q_2 \cdots$.
  - for all $i \in \mathbb{N}$,

$$q_i \xrightarrow{C_i} q_{i+1} \in \delta \text{ and } \mathfrak{v}_i, \mathfrak{v}_{i+1} \models C_i.$$

Non-emptiness problem

- Non-emptiness problem for $\mathcal{D}$-automata takes as input a $\mathcal{D}$-automaton $\mathbb{A}$ and asks whether $\mathrm{L}(\mathbb{A}) \neq \emptyset$.

- $\mathrm{L}(\mathbb{A}) \neq \emptyset$ iff for some symbolic model $\mathtt{w} : \mathbb{N} \to \mathcal{P}(\mathrm{AC}_k)$,
    - there is an infinite run $q_0 \xrightarrow{C_0} q_1 \xrightarrow{C_1} \cdots$ such that for all $i \in \mathbb{N}$, validity of

$$(\bigwedge_{c \in \mathtt{w}(i)} c) \wedge (\bigwedge_{c \in (\mathrm{AC}_k \setminus \mathtt{w}(i))} \neg c) \Rightarrow C_i$$

    - $\mathtt{w}$ is $\mathcal{D}$-satisfiable,

$\mathrm{LTL}(\mathcal{D})$: LTL with concrete domain $\mathcal{D}$

$$\phi ::= R(\mathtt{t}_1, \ldots, \mathtt{t}_d) \mid \phi \wedge \phi \mid \neg\phi \mid \mathsf{X}\phi \mid \phi \mathsf{U} \phi$$

(the $\mathtt{t}_i$'s are terms of the form $\mathsf{X}^j x$)

- $\mathrm{LTL}(\mathcal{D})$ model $\mathfrak{v} : \mathbb{N} \times \mathrm{VAR} \to \mathbb{D}$.

Satisfaction relation

- $\mathfrak{v}, i \models R(\mathtt{t}_1, \ldots, \mathtt{t}_d) \overset{\text{def}}{\Leftrightarrow} (\mathfrak{v}(i, \mathtt{t}_1), \ldots, \mathfrak{v}(i, \mathtt{t}_d)) \in R^{\mathcal{D}}$
- $\mathfrak{v}, i \models \mathsf{X}\phi \overset{\text{def}}{\Leftrightarrow} \mathfrak{v}, i+1 \models \phi$

x_1	0	$\frac{3}{8}$	$\frac{1}{9}$	3	$\ldots$
x_2	$\frac{1}{2}$	$\mathbf{0}$	$\frac{3}{4}$	2	$\ldots$
x_3	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\mathbf{1}$	$\ldots$
x_4	1	2	3	4	$\ldots$

$\models \mathsf{F}(x_2 < \mathsf{X}^2 x_3)$

- Automata-based approach for temporal logics applies!

[Vardi & Wolper, IC 1994]

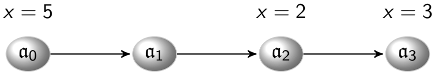
Branching-time temporal logics

- $\mathcal{D}$-decorated Kripke structure $\mathcal{K}$ is a structure of the form $(\mathcal{D}, \mathcal{W}, \mathcal{R}, l, \mathfrak{v})$ such that
  - Concrete domain $\mathcal{D} = (\mathbb{D}, \sigma)$, Kripke structure $(\mathcal{W}, \mathcal{R}, l)$
  - $\mathfrak{v} : \mathcal{W} \times \mathrm{VAR} \to \mathbb{D}$ is a valuation function.

- $\mathrm{CTL}^*(\mathcal{D})$ formulae

$$\phi := \neg\phi \mid \phi \wedge \phi \mid \mathsf{E}\Phi \qquad \Phi := \phi \mid R(\mathtt{t}_1, \ldots, \mathtt{t}_d) \mid \neg\Phi \mid \Phi \wedge \Phi \mid \mathsf{X}\Phi \mid \Phi \mathsf{U} \Phi$$

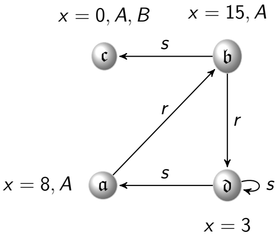
- Satisfaction relation
  - $\mathcal{K}, w \models \mathsf{E}\Phi$ iff there is an infinite path π starting from w such that $\mathcal{K}, \pi \models \Phi$,
  - $\mathcal{K}, \pi \models R(\mathtt{t}_1, \ldots, \mathtt{t}_d)$ iff $(\mathfrak{v}(\pi(0), \mathtt{t}_1), \ldots, \mathfrak{v}(\pi(0), \mathtt{t}_d)) \in R^{\mathcal{D}}$ &
    $\mathfrak{v}(\pi(0), \mathsf{X}^j x) \stackrel{\text{def}}{=} \mathfrak{v}(\pi(j), x)$



$\mathfrak{a}_0 \models \mathsf{E}(x = \mathsf{X}^2 x + \mathsf{X}^3 x)$

Description logics with concrete domains

- Description logics are well-known logical formalisms for knowledge representation. [Baader et al., Book 2017]
- Concrete domains in DLs to refer to concrete objects and built-in predicates on these objects for designing concepts. [Baader & Hanschke, IJCAI'91, Lutz, PhD 2002]
- Role names $\mathsf{N_R} = \{r, s, \ldots\}$ and role path $P = r_1 \cdots r_n$.



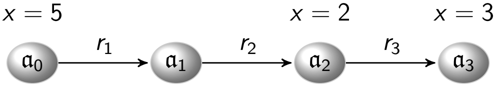
$\mathcal{D}$-decorated interpretations $(\mathcal{D}, \mathcal{W}, (\mathcal{R}_r)_{r \in \mathsf{N_R}}, I, \mathfrak{v})$ with $\mathfrak{v} : \mathcal{W} \times \mathrm{VAR} \to \mathbb{D}$. (often partial in the literature)

$\mathrm{ALC}^{\ell}(\mathcal{D})$ (with "linear-path constraints")

- $\mathrm{ALC}^{\ell}(\mathcal{D})$-formulae (unorthodox pre

  $\phi ::= p \mid \mathsf{E} r_1 \cdots r_n\ R(\mathtt{t}_1, \ldots, \mathtt{t}_d) \mid \phi \wedge \phi \mid \neg \phi \mid \mathsf{EX}_r \phi$

- $\mathcal{K}, w \models \mathsf{EX}_r \phi \overset{\text{def}}{\Leftrightarrow}$ there is $w' \in \mathcal{R}_r(w)$ s.t. $\mathcal{K}, w' \models \phi$,



$\mathfrak{a}_0 \models \mathsf{E} r_1 r_2 r_3\ (x = \mathsf{X}^2 x + \mathsf{X}^3 x)$

- Logics of the form $\mathrm{ALC}^{\ell}(\mathcal{D})$ considered in

  [Carapelle & Turhan, ECAI'16; Labai & Ortiz & Simkus, KR'20]

- Conditions on $\mathcal{D}$ for decidability/low complexity studied in [Lutz & Milićic, JAR 2007; Baader & Rydval, IJCAR'20]

  ... but this excludes domains such as $(\mathbb{N}, <, =)$.

What's next?

1. Characterisations of satisfiable symbolic models.
   - Characterisation for $(\mathbb{R}, <, =)$ (and $(\mathbb{Q}, <, =)$).
   - Characterisation of $\mathcal{D}$-satisfiable symbolic models for $\mathcal{D} = (\mathbb{N}, <, =)$.

2. 3 methods for handling $\mathbb{N}$-satisfiable symbolic models.
   - EHD approach with BMW.
   - Nonemptiness problem for $\mathbb{N}$-automata.
   - Approximating condition $\mathcal{UPM}_{\mathbb{N}}$ for $(\mathbb{N}, <, =)$ with ultimately periodic symbolic models.

Easy case with $(\mathbb{R}, <, =)$ and $(\mathbb{Q}, <, =)$

- Symbolic model $\mathtt{w}$ is $\mathbb{Q}$-satisfiable iff for all $i \in \mathbb{N}$,

$\mathcal{C}_{\mathbb{Q}}(1)$: $\mathtt{w}(i)$ and $\mathtt{w}(i+1)$ are satisfiable,
$\mathcal{C}_{\mathbb{Q}}(2)$: $\{\mathrm{X}x_1, \ldots, \mathrm{X}x_k\}$ in $\mathtt{w}(i)$ and $\{x_1, \ldots, x_k\}$ in $\mathtt{w}(i+1)$ are related in the same way.

- The set of $\mathbb{Q}$-satisfiable symbolic models is ω-regular.

(good new

- Sat. problem for $\mathrm{LTL}(\mathbb{Q}, <, =)$ is $\mathrm{PS{\scriptsize PACE}}$-complete.

[Balbiani & Condotta, FroCoS'02]

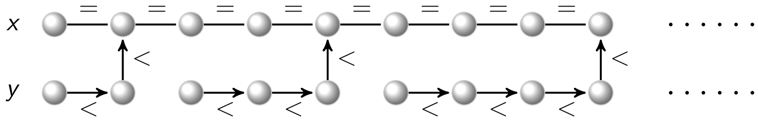
- $\mathrm{LTL}(\mathcal{D}_A)$ $\mathrm{PS{\scriptsize PACE}}$-complete too with the temporal concrete domain $\mathcal{D}_A = (I_{\mathbb{Q}}; (R_i)_{i \in [1,13]})$.

[Balbiani & Condotta, FroCoS'02]

Characterisation for $(\mathbb{N}, <, =)$

- Symbolic model $\mathtt{w} : \mathbb{N} \to \mathcal{P}(\mathrm{AC}_k)$ understood as an infinite labelled graph on $\{x_1, \ldots, x_k\} \times \mathbb{N}$.

- A simple non $\mathbb{N}$-satisfiable symbolic model.



- Strict length of the path π:

$$\mathrm{slen}(\pi) \stackrel{\text{def}}{=} \text{number of edges labelled by } <.$$

- Strict length of (x, i):

$$\mathrm{slen}((x,i)) \stackrel{\text{def}}{=} sup\ \{\mathrm{slen}(\pi) : \text{ path } \pi \text{ from } (x', i') \text{ to } (x, i)\}$$

$\mathbb{N}$-satisfiable symbolic models

- Symbolic model $\mathtt{w}$ is $\mathbb{N}$-satisfiable iff

$\mathcal{C}_{\mathbb{N}}(1)$: local consistency between two consecutive positions and, $(\mathcal{C}_{\mathbb{Q}}(1) \wedge \mathcal{C}_{\mathbb{Q}}(2))$

$\mathcal{C}_{\mathbb{N}}(2)$: any node has a finite strict length.

[Cerans, ICALP'94; Demri & D'Souza, IC 07;Carapelle & Kartzow & Lohrey, CONCUR'13; Exibard & Filiot & Khalimov, STACS'21]

- The set of $\mathbb{N}$-satisfiable symbolic models is not ω-regular.

The EHD approach

- The set of $\mathbb{N}$-satisfiable symbolic models is not ω-regular but can it be captured by decidable extensions of MSO?

  (MSO = monadic 2nd logic $\approx$ Büchi automata)

- Starting point of the EHD approach with the bounding quantifier B. [Carapelle & Kartzow & Lohrey, CONCUR'13]

- Bounding quantifier B: $\mathsf{B}X.\phi(X)$ expresses that there is a finite bound on the size of the sets that satisfy $\phi(X)$.
  [Bojańczyk, CSL'04]

- B fits well to express the condition $\mathcal{C}_{\mathbb{N}}(2)$.

Decidability status

- Satisfiability $\mathrm{MSO{+}B}$ is undecidable over ω-words.
  [Bojańczyk & Parys & Toruńczyk, STACS'16]

- Boolean combinations of MSO and $\mathrm{WMSO{+}B}$ (BMW) is decidable over infinite trees of finite branching degree.
  [Carapelle & Kartzow & Lohrey, CONCUR'13]

- Negation-closed $\mathcal{D}$ with EHD(BMW)-property.
  Satisfiability problem for $\mathrm{CTL}^*(\mathcal{D})$ is decidable.
  [Carapelle & Kartzow & Lohrey, JCSS 2016]
  (tree model property + decidability of BMW)

EHD approach: two conditions

- $\mathcal{D}$ negation-closed if complements of relations definable by positive existential first-order formulae over $\mathcal{D}$.
  $(\neg(x = n) \Leftrightarrow \exists\ y\ (y = n) \wedge ((x < y) \vee (y < x)))$

- EHD(BMW) property for symbolic models.
  There is ϕ_{SAT} in BMW for ω-words such that
  $$\texttt{w} \text{ is } \mathbb{N}\text{-satisfiable iff } \texttt{w} \models \phi_{\mathrm{SAT}}.$$

- EHD(BMW) property (complete version).
  For every finite subsignature τ, one can compute ϕ_τ such that for every countable τ-structure $\mathcal{S}$,
  $$\underbrace{\text{there is an homomorphism from } \mathcal{S} \text{ to } \mathcal{D}}_{\approx\ \mathcal{D}\text{-satisfiability}} \text{ iff } \mathcal{S} \models \phi_\tau.$$

- EHD = "the Existence of a Homomorphism is Definable".

New decidability results

- $(\mathbb{Z}, <, =, (=_n)_{n\in\mathbb{Z}})$ has the EHD(BMW)-property.

- The satisfability problem for $\mathrm{CTL}^*(\mathbb{Z}, <, =, (=_n)_{n\in\mathbb{Z}})$ is decidable. [Carapelle & Kartzow & Lohrey, JCSS 2016]

- Satisfiability w.r.t. TBoxes for $\mathrm{ALC}^\ell(\mathbb{Z}, <, =, (=_n)_{n\in\mathbb{Z}})$ is decidable [Carapelle & Turhan, ECAI'16]

$\mathbb{N}$-automata

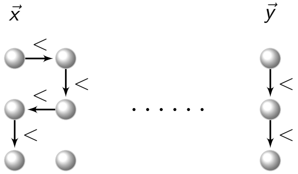
- EHD good for decidability, unsatisfactory for complexity!
- Concrete domains $\mathcal{D} = (\mathbb{D}, <, P_1, \ldots, P_l, =_{\mathfrak{d}_1}, \ldots, =_{\mathfrak{d}_m})$, where $(\mathbb{D}, <)$ is a linear ordering and the P_i's are unary relations. [Segoufin & Toruńczyk, STACS'11]
- Existence of accepting runs characterised by existence of extensible lassos.

$\mathbb{N}$-automata: extensible lassos

$\mathbb{A}$ has an accepting run iff there are finite runs π, λ s.t.

1. $\pi = (q_I, \vec{x}_0) \xrightarrow{*} (q_F, \vec{x})$ and $\lambda = (q_F, \vec{x}) \xrightarrow{+} (q_F, \vec{y})$
2. "$\mathrm{type}(\vec{x}) = \mathrm{type}(\vec{y})$", $\vec{x} \leq \vec{y}$ and $\mathrm{dv}(\vec{x}) \leq \mathrm{dv}(\vec{y})$.

$$0 \overset{7}{\frown} 7 \overset{2}{\frown} 9 \overset{6}{\frown} 15 \qquad \mathrm{dv}\left(\begin{pmatrix} 15 \\ 9 \\ 7 \end{pmatrix}\right) = \begin{pmatrix} 7 \\ 2 \\ 6 \end{pmatrix}$$



Conditions (2) and (3) allow us to repeat infinitely λ.

3. For all $j \in [1, k]$ such that $\vec{x}[j] = \vec{y}[j]$, there is no j' such that $\vec{x}[j'] < \vec{y}[j']$ and $\vec{x}[j'] < \vec{x}[j]$.

$\mathbb{N}$-automata: lasso detection in PSpace

- Existence of finite runs π, λ can be checked in $\textsc{PSpace}$.
- The non-emptiness problem for $(\mathbb{N}, <)$-automata is $\textsc{PSpace}$-complete. [Segoufin & Toruńczyk, STACS'11]
- A similar method used in [Kartzow & Weidner, CoRR 2015].
- $\textsc{PSpace}$-completeness for the concrete domains
  - $\mathcal{D}_{\mathbb{Q}^*} = (\mathbb{Q}^*; \preceq_{\mathsf{pre}}, \preceq_{\mathsf{lex}}, =_{\mathfrak{d}_1}, \ldots, =_{\mathfrak{d}_m})$.
  - $\mathcal{D}_{[1,\alpha]^*} = ([1,\alpha]^*; \preceq_{\mathsf{pre}}, \preceq_{\mathsf{lex}}, =_{\mathfrak{d}_1}, \ldots, =_{\mathfrak{d}_m})$, $\alpha \geq 2$.

[Kartzow & Weidner, CoRR 2015]

Condition $\mathcal{UPM}_{\mathbb{N}}$: ultimately periodic models

- $\mathbb{N}$-automata and $\mathrm{LTL}(\mathbb{N}, <, =)$ define ω-regular classes of symbolic models with uninterpreted constraints.

- A symbolic model $\mathtt{w}$ is ultimately periodic iff $\mathtt{w}$ of the form

$$\mathtt{w}(0) \cdots \mathtt{w}(I-1) \cdot \Big(\mathtt{w}(I) \cdots \mathtt{w}(I+J)\Big)^{\omega}$$

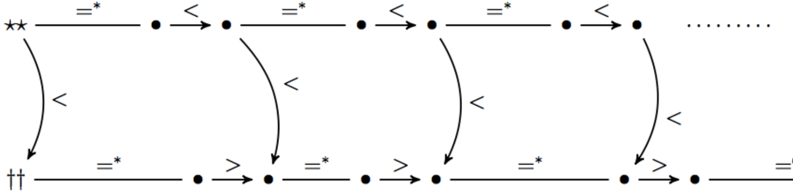
- Characterisation for $\mathbb{N}$-satisfiable ultimately periodic models might be simpler than the general case.

Condition $\mathcal{UPM}_{\mathbb{N}}$: definition

Symbolic model $\mathtt{w}$ satisfies the condition $\mathcal{UPM}_{\mathbb{N}}$ iff

1. Local consistency btw. two consecutive positions holds.
2. There is no infinite $(z_1, j_1) \xrightarrow{a_1} (z_2, j_2) \xrightarrow{a_2} (z_3, j_3) \cdots$ s.t. $\{a_1, a_2, \ldots\} \subseteq \{=, >\}$ and an infinite amount of a_j's are equal to $>$.
3. There do not exist nodes $\star\star$ and $\dagger\dagger$ such that

(infinite amount of $<$)



(finite amount of $>$)

$$\mathcal{C}_{\mathbb{N}}(1) \wedge \mathcal{C}_{\mathbb{N}}(2) \Rightarrow \mathcal{UPM}_{\mathbb{N}}$$

Condition $\mathcal{UPM}_{\mathbb{N}}$: properties

- Ultimately periodic symbolic model $\mathtt{w}$. Equivalence btw.
  - $\mathtt{w}$ is $\mathbb{N}$-satisfiable.
  - $\mathtt{w}$ satisfies the condition $\mathcal{UPM}_{\mathbb{N}}$.

[Demri & D'Souza, IC 2007; Exibard & Filiot & Reynier, STACS'21]

- The class of symbolic models having $\mathcal{UPM}_{\mathbb{N}}$ is ω-regular.
- By-products:
  - Non-emptiness problem for $\mathbb{N}$-automata is in $\mathrm{PS{\scriptsize PACE}}$.
  - Satisfiability problem for $\mathrm{LTL}(\mathbb{N}, <, =)$ is in $\mathrm{PS{\scriptsize PACE}}$.
- Remarkable generalisation to description logics:
  - $\mathcal{UPM}_{\mathbb{N}}$ for regular tree symbolic models and regularity via Rabin tree automata.
  - Satisfiability problem w.r.t. TBoxes in $\mathrm{ALC}^{\ell}(\mathbb{N}, <, =)$ is in $\mathrm{E{\scriptsize XP}T{\scriptsize IME}}$. [Labai & Ortiz & Šimkus, KR'20]
- Results apply to $(\mathbb{Z}, <, =)$ with adequate adaptations.

Concluding remarks

- Presentation of three methods for handling $\mathbb{N}$-satisfiable symbolic models.
  1. MSO-like logics (EHD approach),
  2. $\mathcal{D}$-automata (for linear domains or strings)
  3. overapproximation (condition $\mathcal{UPM}_{\mathbb{N}}$)

- A selection of open problems.
  - Decidability status for $\mathrm{LTL}(\{0,1\}^*, \preceq_{pre}, \preceq_{suf})$.
  - Satisfiability w.r.t. TBoxes for $\mathrm{ALC}^{\ell}(\mathbb{Z}; <, =, (=_n)_{n \in \mathbb{Z}})$ in $\textsc{ExpTime}$ with integers in binary.