Automates d'arbres

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Logique monadique ~~faible~~ du second ordre à deux successeurs

~~Weak~~ Second Order Monadic Logic with two successors

(S2S)

Second Order Monadic Logic with two successors (S2S)

- expresses properties of infinite binary trees.
- equivalent in expressiveness to Rabin automata.
  [Rabin theorem]

Infinite Terms as structures

- Σ is a finite alphabet ;
- $\mathcal{T}^\omega(\Sigma)$ is the set of infinite binary trees labeled with Σ.

S2S : same language as WS2S.

$$\mathcal{L}_\Sigma := \{=, <, S_1, S_2, L_a \mid a \in \Sigma\}.$$

to $t \in \mathcal{T}^\omega(\Sigma)$, we associate a structure $\underline{t}$ over $\mathcal{L}_\Sigma$

$$\underline{t} := \left\langle \{1,2\}^*, =, <, S_1, S_2, L_a^{\underline{t}}, L_b^{\underline{t}}, \cdots \right\rangle$$

where

- domain $= \{1,2\}^* =$ set of positions in infinite binary terms,
- $=$: equality over $\{1,2\}^*$,
- $<$: prefix ordering over $\{1,2\}^*$,
- $S_i = \left\{\langle p, p \cdot i\rangle \mid p \in \{1,2\}^*\right\}$, ($i^{\text{th}}$ successor position),
- $L_a^{\underline{t}} = \left\{p \in \{1,2\}^* \mid t(p) = a\right\}$.

S2S : syntax

- first order variables $x, y \ldots$
- second order variables $X, Y \ldots$
- $$\begin{array}{lll} \text{form} & ::= & x = y \mid x < y \mid x \in X \\ & & \mid S_1(x,y) \mid S_2(x,y) \mid L_a(x) \quad a \in \Sigma \\ & & \mid \text{form} \wedge \text{form} \mid \text{form} \vee \text{form} \mid \neg\text{form} \\ & & \mid \exists x \text{ form} \mid \exists X \text{ form} \mid \forall x \text{ form} \mid \forall X \text{ form} \end{array}$$

Notation : $\phi(x_1, \ldots, x_m, X_1, \ldots, X_n)$,
where $x_1, \ldots, x_m$, $X_1, \ldots, X_n$ are the free variables of ϕ.

S2S : semantics

- $t \in \mathcal{T}^\omega(\Sigma)$,
- valuation σ of first order variables into $\{1,2\}^*$,
- valuation δ of second order variables into subsets of $\{1,2\}^*$ (interpretation and quantification over infinite sets),
- $\underline{t}, \sigma, \delta \models x = y$ iff $\sigma(x) = \sigma(y)$,
- $\underline{t}, \sigma, \delta \models x < y$ iff $\sigma(x) <_{\mathit{prefix}} \sigma(y)$,
- $\underline{t}, \sigma, \delta \models x \in X$ iff $\sigma(x) \in \delta(X)$,
- $\underline{t}, \sigma, \delta \models S_i(x, y)$ iff $\sigma(y) = \sigma(x) \cdot i$,
- $\underline{t}, \sigma, \delta \models L_a(x)$ iff $t(\sigma(x)) = a$ i.e. $\sigma(x) \in L_a^t$,
- $\underline{t}, \sigma, \delta \models \phi_1 \wedge \phi_2$ iff $\underline{t}, \sigma, \delta \models \phi_1$ and $\underline{t}, \sigma, \delta \models \phi_2$,
- $\underline{t}, \sigma, \delta \models \phi_1 \vee \phi_2$ iff $\underline{t}, \sigma, \delta \models \phi_1$ or $\underline{t}, \sigma, \delta \models \phi_2$,
- $\underline{t}, \sigma, \delta \models \neg\phi$ iff $\underline{t}, \sigma, \delta \not\models \phi$,

S2S : semantics (quantifiers)

- $\underline{t}, \sigma, \delta \models \exists x\ \phi$ iff $x \notin dom(\sigma)$, x free in ϕ and exists $p \in \{1, 2\}^*$ s.t. $\underline{t}, \sigma \cup \{x \mapsto p\}, \delta \models \phi$,
- $\underline{t}, \sigma, \delta \models \forall x\ \phi$ iff $x \notin dom(\sigma)$, x free in ϕ and for all $p \in \{1, 2\}^*$, $\underline{t}, \sigma \cup \{x \mapsto p\}, \delta \models \phi$,
- $\underline{t}, \sigma, \delta \models \exists X\ \phi$ iff $X \notin dom(\delta)$, X free in ϕ and exists $P \subseteq \{1, 2\}^*$ s.t. $\underline{t}, \sigma, \delta \cup \{X \mapsto P\} \models \phi$,
- $\underline{t}, \sigma, \delta \models \forall X\ \phi$ iff $X \notin dom(\delta)$, X free in ϕ and for all $P \subseteq \{1, 2\}^*$, $\underline{t}, \sigma, \delta \cup \{X \mapsto P\} \models \phi$.

S2S : languages

Definition : S2S-definability

For $\phi \in$ S2S without free variables, $L(\phi) := \left\{ t \in \mathcal{T}^{\omega}(\Sigma) \;\middle|\; \underline{t} \models \phi \right\}$.

S2S : languages

- $T_0 \subseteq T^\omega(\{a,b\})$: $\exists$ path with infinite number of a.

  $$\phi_0 \equiv \exists X \left(\text{path}(X) \wedge \forall x\, (x \in X \Rightarrow \exists y\, (y \in X \wedge x < y \wedge L_a(y)))\right)$$

- path : maximal chain

  $$\text{path}(X) \equiv \text{chain}(X) \wedge \neg \exists Y \left(\text{chain}(Y) \wedge Y \neq X \wedge X \subseteq Y\right)$$

- chain : totalement ordered set

  $$\text{chain}(X) \equiv \forall x\, \forall y\, (x \in X \wedge y \in X \Rightarrow x < y \vee y < x \vee x = y)$$

- $T_1 \subseteq T^\omega(\{a,b\})$: all paths have a finite number of a.

  $$\phi_1 \equiv \neg \phi_0 \equiv$$
  $$\forall X \left(\text{path}(X) \Rightarrow \exists x\, (x \in X \wedge \forall y\, (y \in X \wedge x < y \Rightarrow \neg L_a(y)))\right)$$

Rabin's Theorem

Theorem : Rabin

Languages of S2S formula = Rabin Automata languages.

S2S formulae ↔ automata for infinite trees

infinite Tree Automata

Büchi Automata

Definition : Büchi tree Automata

A Büchi tree automaton over the alphabet Σ is a tuple $\mathcal{A} = (\Sigma, Q, q^{\mathsf{i}}, F, \Delta)$ where Q is a finite set of states, $q^{\mathsf{i}} \in Q$ is the initial state, $F \subseteq Q$ is the subset of final states and Δ is a set of top-down transition rules of the form : $q \to d(q_1, q_2)$ with $d \in \Sigma$ and $q, q_1, q_2 \in Q$.

A run of $\mathcal{A}$ on a term $t \in T^\omega(\Sigma)$ is a term $r \in \mathcal{T}^\omega(Q)$ compatible with Δ : for all position $p \in \{1,2\}^*$ of r,
$r(p) \to t(p)(r(p \cdot 1), r(p \cdot 2)) \in \Delta$.

The run r is successful (or accepting) iff for all (infinite) path π starting from the root of r, $\mathit{Inf}(\pi) \cap F \neq \emptyset$

$$\mathit{Inf}(\pi) := \left\{ q \in Q \;\middle|\; \exists^\infty i, \pi(i) = q \right\}$$

Büchi Languages

$\mathcal{A}$, Büchi automaton.

- $L(\mathcal{A})$ is the set of $t \in \mathcal{T}^{\omega}(\Sigma)$ on which there exists a successful run of $\mathcal{A}$.
- a set $L \subseteq \mathcal{T}^{\omega}(\Sigma)$ is Büchi-recognizable = Büchi language if there exists a Büchi automaton $\mathcal{A}$ s.t. $L(\mathcal{A}) = L$.

Büchi Languages : examples

- $T'_0 \subseteq T^\omega(\{a,b\})$: all path have infinite number of a.

$$\begin{array}{rcl@{\qquad}rcl}
q^{\mathsf{i}} & \to & a(q_a,q_a), & q^{\mathsf{i}} & \to & b(q_b,q_b), \\
q_a & \to & a(q_a,q_a), & q_a & \to & b(q_b,q_b), \\
q_b & \to & a(q_a,q_a), & q_b & \to & b(q_b,q_b), \\
F & = & \{q_a\} & & &
\end{array}$$

- $T_0 \subseteq T^\omega(\{a,b\})$: $\exists$ path with infinite number of a.

$$\begin{array}{rcl@{\qquad}rcl}
q^{\mathsf{i}}|q_a|q_b & \to & a(q_a,q_0), & q^{\mathsf{i}}|q_a|q_b & \to & a(q_0,q_a), \\
q^{\mathsf{i}}|q_a|q_b & \to & b(q_b,q_0), & q^{\mathsf{i}}|q_a|q_b & \to & b(q_0,q_b), \\
q_0 & \to & a(q_0,q_0), & q_0 & \to & b(q_0,q_0), \\
F & = & \{q_a,q_0\} & & &
\end{array}$$

- $T_1 \subseteq T^\omega(\{a,b\})$: all path have finite number of a.

  not Büchi !

Rabin Automata

Definition : Rabin tree automata

A Rabin automaton over the alphabet Σ is a tuple $\mathcal{A} = (\Sigma, Q, q^{\mathsf{i}}, \Omega, \Delta)$ where Q is a finite set of states, $q^{\mathsf{i}} \in Q$ is the initial state, $\Omega = \big\{(L_1, U_1), \ldots, (L_n, U_n)\big\}$ ($\forall i \leq n$, $L_i, U_i \subseteq Q$) and Δ is a set of top-down transition rules of the form : $q \longrightarrow d(q_1, q_2)$ avec $d \in \Sigma$ et $q, q_1, q_2 \in Q$.

A run r of $\mathcal{A}$ on t is successful (or accepting) iff for all infinite path π starting from the root of r, there exists $i \leq n$ s.t. $\mathit{Inf}(\pi) \cap L_i = \emptyset$ and $\mathit{Inf}(\pi) \cap U_i \neq \emptyset$.

Rabin vs Büchi

- $T_1 \subseteq T^\omega(\{a,b\})$: all path have finite number of a.

  is not Büchi !

  is Rabin !

$$\begin{array}{rclrcl} q^{\mathsf{i}} & \rightarrow & a(q_a, q_a), & q^{\mathsf{i}} & \rightarrow & b(q^{\mathsf{i}}, q^{\mathsf{i}}), \\ q_a & \rightarrow & a(q_a, q_a), & q_a & \rightarrow & b(q^{\mathsf{i}}, q^{\mathsf{i}}). \\ \Omega & = & \{(\{q_a\}, \{q^{\mathsf{i}}, q_a\})\} & & & \end{array}$$

Rabin's Complementation Theorem

Theorem : Rabin

For all Rabin automata $\mathcal{A}$ over Σ, there exists a Rabin automaton recognizing $\mathcal{T}^\omega(\Sigma) \setminus L(\mathcal{A})$.

Müller, Streett, Parity Automata

Classes equivalent to Rabin automata.

- Müller automata : $\mathcal{A} = (Q, q^{\mathsf{i}}, \mathcal{F}, \Delta)$ with $\mathcal{F} := \{F_1, \ldots, F_n\}$, $\forall i \leq n$, $F_i \subseteq Q$,

  a run r is successful iff for all infinite path π of r, $\mathit{Inf}(\pi) \in \mathcal{F}$.
- Streett automata : $\mathcal{A} = (Q, q^{\mathsf{i}}, \Omega, \Delta)$ with $\Omega = \big\{(L_1, U_1), \ldots, (L_n, U_n)\big\}$,

  a run r is successful iff for all infinite path π of r,
  for all $i \leq n$, if $\mathit{Inf}(\pi) \cap U_i \neq \emptyset$ then $\mathit{Inf}(\pi) \cap L_i \neq \emptyset$.
- Parity automata : $\mathcal{A} = (Q, q^{\mathsf{i}}, \mathit{index}, \Delta)$ with $\mathit{index} : Q \to \{1..k\}$, $k \in \mathbb{N}$,

  a run r is successful iff for all infinite path π of r,
  $\min\{\mathit{index}(q) \mid q \in \mathit{Inf}(\pi)\}$ is even.

Rabin's Theorem : Automata → Logic

S2S : Automata $\to$ Logic

Theorem :

For all Rabin automaton $\mathcal{A}$ over Σ,
there exists $\phi_{\mathcal{A}} \in$ S2S such that $L(\phi_A) = L(\mathcal{A})$.

$\mathcal{A} = (Q, q^{\mathsf{i}}, \Omega, \Delta)$ Rabin Automaton over Σ,
with $Q = \{0, \dots, m\}$, $q^{\mathsf{i}} = 0$.

Existence of an accepting run of $\mathcal{A}$ on $t \in \mathcal{T}^{\omega}\big(\{0,1\}^n\big)$ in S2S :

$$\phi_{\mathcal{A}} \equiv \exists Y_0 \dots \exists Y_m \; \phi_{\mathsf{run}}(\overline{Y}) \wedge \phi_{\mathsf{init}}(\overline{Y}) \wedge \phi_{\mathsf{tr}}(\overline{Y}) \wedge \phi_{\Omega}(\overline{Y})$$

S2S : Automata → Logic

$$\phi_{\mathcal{A}} \equiv \exists Y_0 \ldots \exists Y_m \; \phi_{\mathsf{run}}(\overline{Y}) \wedge \phi_{\mathsf{init}}(\overline{Y}) \wedge \phi_{\mathsf{tr}}(\overline{Y}) \wedge \phi_{\Omega}(\overline{Y})$$

$$\begin{array}{rcl}
\phi_{\mathsf{run}}(\overline{Y}) & \equiv & Y_0, \ldots, Y_m \text{ partition } \{1,2\}^* \\
\phi_{\mathsf{init}}(\overline{Y}) & \equiv & \forall x_0 \; \mathsf{root}(x_0) \Rightarrow x_0 \in Y_0 \\
\phi_{\mathsf{tr}}(\overline{Y}) & \equiv & \forall x, x_1, x_2 \displaystyle\bigvee_{i \to a(j_1,j_2) \in \Delta} \left(\begin{array}{c} x \in Y_i \wedge L_a(x) \\ \wedge\, S_1(x,x_1) \wedge S_2(x,x_2) \\ \wedge\, x_1 \in Y_{j_1} \wedge x_2 \in Y_{j_2} \end{array} \right) \\
\phi_{\Omega}(\overline{Y}) & \equiv & \forall Z \; \mathsf{path}(Z) \Rightarrow \\
 & & \displaystyle\bigvee_{(L,U) \in \Omega} \; \bigwedge_{j \in L} \exists x \; x \in Z \wedge \forall y \; (y \in Z \wedge x < y \Rightarrow y \notin Y_j) \\
 & & \displaystyle\bigvee_{j' \in U} \forall x \; x \in Z \Rightarrow \exists y \; (y \in Z \wedge x < y \wedge y \in Y_{j'})
\end{array}$$

Rabin's Theorem : Logic $\longrightarrow$ Automata

S2S : Logic $\rightarrow$ Automata

Theorem :

Every S2S language is a Rabin automata language.

For all formula $\phi \in$ S2S over Σ (without free variables) there exists a Rabin automaton $\mathcal{A}_\phi$ over Σ, such that $L(\mathcal{A}_\phi) = L(\phi)$.

Corollary :

S2S is decidable.

pr.: reduction to emptiness decision for $\mathcal{A}_\phi$.

Rabin Theorem

$\mathcal{A}_\phi$ is effectively constructed form ϕ, by induction.

✓ for free second order variables :

$$\begin{array}{c} t \in \mathcal{T}^\omega(\Sigma) \\ \delta : \{X_1, \ldots, X_n\} \to \mathcal{P}\big(\{1,2\}^*\big) \end{array} \quad \mapsto \quad t \times \delta \in \mathcal{T}^\omega(\Sigma \times \{0,1\}^n)$$

for all $p \in \{1,2\}^*$, $(t \times \delta)(p) = \langle t(p), b_1, \ldots, b_n \rangle$ where

for all $i \leq n$,

- $b_i = 1$ if $p \in \delta(X_i)$,
- $b_i = 0$ otherwise.

✓ free first order variables are interpreted as singletons.

S2S$_0$

simplified language S2S$_0$.

- no first order variables,
- only second order variables $X, Y \ldots$,
- form ::= $X \subseteq Y \mid Y = X \cdot 1 \mid Y = X \cdot 2$
  $\mid X \subseteq L_a \quad a \in \Sigma$
  $\mid$ form $\wedge$ form $\mid$ form $\vee$ form $\mid \neg$form
  $\mid \exists X$ form $\mid \forall X$ form

$Y = X \cdot i$: $X = \{x\}$, $Y = \{y\}$ and $y = x \cdot i$ ($i = 1, 2$).

Lemma :

Every S2S formula can be transformed into an equivalent S2S$_0$ formula.

pr.: analoguous to WSkS and WSkS$_0$.

Compilation : $\mathsf{S2S}_0 \longrightarrow$ Rabin Automata

notation : $\Sigma_{[m]} := \Sigma \times \{0,1\}^m$.

For all $\phi(X_1, \ldots, X_n) \in \mathsf{S2S}_0$ and $m \geq n$,
we construct a Rabin automaton $[\![\phi]\!]_m$ over $\Sigma_{[m]}$ recognizing

$$\big\{ t \times \delta \mid \delta : \{X_1, \ldots, X_m\} \to \mathcal{P}\big(\{1,2\}^*\big),\ \underline{t}, \delta \models \phi(X_1, \ldots, X_n) \big\}$$

induction on the $\mathsf{S2S}_0$ formula :

- atoms $X_1 \subseteq X_2$, $X_2 = X_1 \cdot 1$, $X_2 = X_1 \cdot 2$, $X_1 \subseteq L_a$,

  $\Rightarrow$ automata for formula with free variables.
- Boolean closures for Boolean connectors.
- $\exists$ quantifier : projection.

Atom : $X_1 \subseteq X_2$

Automaton $[\![X_1 \subseteq X_2]\!]_2$:

- signature $\Sigma_{[2]} = \Sigma \times \{0,1\}^2$.
- states : q_0
- initial states : q_0
- transitions : $$\begin{array}{lll} q_0 & \longrightarrow & \langle a,0,0\rangle(q_0,q_0) \quad \forall a \in \Sigma \\ q_0 & \longrightarrow & \langle a,0,1\rangle(q_0,q_0) \\ q_0 & \longrightarrow & \langle a,1,1\rangle(q_0,q_0) \\ & & (\langle a,1,0\rangle \text{ is forbidden}) \end{array}$$
- $\Omega = \big\{(\emptyset, \{q_0\})\big\}$

For $m \geq 2$,

$$[\![X_1 \subseteq X_2]\!]_m := cyl_{2,m}\big([\![X_1 \subseteq X_2]\!]_2\big)$$

Atom : $X_2 = X_1 \cdot 1$

Automaton $[\![X_2 = X_1 \cdot 1]\!]_2$:

- signature $\Sigma_{[2]} = \Sigma \times \{0,1\}^2$.
- states : q_i, q_0, q_1
- initial states : q_i
- transitions : $$\begin{array}{rcl} q_\mathsf{i} & \rightarrow & \langle a,0,0\rangle(q_\mathsf{i}, q_0) \quad \forall a \in \Sigma \\ q_\mathsf{i} & \rightarrow & \langle a,0,0\rangle(q_0, q_\mathsf{i}) \\ q_0 & \rightarrow & \langle a,0,0\rangle(q_0, q_0) \\ q_\mathsf{i} & \rightarrow & \langle a,1,0\rangle(q_1, q_0) \\ q_1 & \rightarrow & \langle a,0,1\rangle(q_0, q_0) \end{array}$$
- $\Omega = \big\{(\{q_\mathsf{i}\}, \{q_0\})\big\}$

For $m \geq 2$,

$$[\![X_2 = X_1 \cdot 1]\!]_m := cyl_{2,m}\big([\![X_2 = X_1 \cdot 1]\!]_2\big)$$

Atom : $X_1 \subseteq L_a$

Automaton $[\![X_1 \subseteq L_a]\!]_1$:

- signature $\Sigma_{[1]} = \Sigma \times \{0,1\}$.
- states : q_0
- initial states : q_0
- transitions : $$\begin{array}{lll} q_0 & \rightarrow & \langle a,0\rangle(q_0,q_0) \\ q_0 & \rightarrow & \langle a,1\rangle(q_0,q_0) \\ q_0 & \rightarrow & \langle b,0\rangle(q_0,q_0) \quad (b \neq a) \\ & & (\langle b,1\rangle \text{ is forbidden}) \end{array}$$
- $\Omega = \big\{(\emptyset, \{q_0\})\big\}$

For $m \geq 1$,

$$[\![X_1 \subseteq L_a]\!]_m := cyl_{1,m}\big([\![X_1 \subseteq L_a]\!]_1\big)$$

Induction step : Boolean connectors, quantifiers

- $[\![\phi(X_1,\ldots,X_n) \vee \phi(X_1,\ldots,X_{n'})]\!]_m :=$
  $[\![\phi(X_1,\ldots,X_n)]\!]_m \cup [\![\phi(X_1,\ldots,X_{n'})]\!]_m$
  with $m \geq \max(n,n')$
- $[\![\phi(X_1,\ldots,X_n) \wedge \phi(X_1,\ldots,X_{n'})]\!]_m :=$
  $[\![\phi(X_1,\ldots,X_n)]\!]_m \cap [\![\phi(X_1,\ldots,X_{n'})]\!]_m$
  with $m \geq \max(n,n')$
- $[\![\neg\phi(X_1,\ldots,X_n)]\!]_m := \mathcal{T}^\omega(\Sigma_{[m]}) \setminus [\![\phi(X_1,\ldots,X_n)]\!]_m$
  for $m \geq n$.
- $[\![\exists X_{n+1}\,\phi(X_1,\ldots,X_{n+1})]\!]_n := \mathit{proj}_n\big([\![\phi(X_1,\ldots,X_{n+1})]\!]_{n+1}\big)$
- $[\![\exists X_{n+1}\,\phi(X_1,\ldots,X_{n+1})]\!]_m :=$
  $\mathit{cyl}_{n,m}\big([\![\exists X_{n+1}\,\phi(X_1,\ldots,X_{n+1})]\!]_n\big)$ for $m \geq n$.
- $\forall = \neg\exists\neg$

Rabin Theorem

Theorem :

For all formula $\phi \in \mathsf{S2S}_0$ over Σ without free variables, there exists a Rabin automaton $\mathcal{A}_\phi$ over Σ, such that $L(\mathcal{A}_\phi) = L(\phi)$.

$\mathcal{A}_\phi = [\![\phi]\!]_0$ can be computed explicitely !
Size $\mathcal{A}_\phi =$ exponential tower of height $=$ quantifier depth of ϕ.

Corollary :

For all formula $\phi \in \mathsf{S2S}$ over Σ without free variables there exists a Rabin automaton $\mathcal{A}_\phi$ over Σ, such that $L(\mathcal{A}_\phi) = L(\phi)$.

using translation of $\mathsf{S2S}$ into $\mathsf{S2S}_0$ first.

S2S and WS2S

Proposition :

WS2S $\subset$ S2S

It is possible to express that X is finite in S2S.

- $x <_{lex} y \equiv x \leq y \vee \exists z \, (z \cdot 1 \leq x \wedge z \cdot 2 \leq y)$
- finite$(X) \equiv$
  $$\forall Y \big(Y \subseteq X \wedge Y \neq \emptyset \Rightarrow$$
  $$(\exists y \, y \, \text{minimal for} <_{lex} \text{in} \, Y \wedge y \, \text{maximal for} <_{lex} \text{in} \, Y)\big)$$

S2S and WS2S

Proposition :

Un sous-ensemble de $\mathcal{T}^\omega(\Sigma)$ est un langage de Büchi ssi il est défini par une formule de S2S de la forme $\exists Y_1 \ldots \exists Y_m \, \phi(Y_1, \ldots, Y_m)$ avec $\phi \in$ WS2S.

Corollary :

S2S $\not\subset$ WS2S

car les automates (d'arbres) de Büchi ne sont pas clos par complément.

Proposition :

Un sous-ensemble $L \subseteq \mathcal{T}^\omega(\Sigma)$ est défini en WS2S ssi L et $\mathcal{T}^\omega(\Sigma) \setminus L$ sont des langages de Büchi.

Theorem :

Toute formule de S2S de la forme $\phi(x_1, \ldots, x_m)$ se traduit en WS2S.

Extensions

- SkS
- SωS
- théorie du second ordre monadique de fonctions d'arité 1 (sur un domaine quelconque dénombrable).
- théorie monadique faible de la grille $\langle \mathbb{N} \times \mathbb{N}, \rightarrow, \uparrow \rangle$ indécidable
- WS2S $+\ 1 \cdot \_$ (concaténation à gauche) indécidable
- WS2S $+$ prédicat de *longueur égale* indécidable.
  en effet pour $x, y \in 1^*2^*$,

$$\begin{aligned}
&y = 1 \cdot x \equiv \\
&(x \in 1^* \wedge S_1(x, y)) \vee \\
&(\exists z\, \exists z'\, z \in 1^* \wedge x \in z \cdot 2^* \wedge z' \in z \cdot 1 \cdot 2^* \wedge |z| = |z'| \wedge S_2(z', y))
\end{aligned}$$