An introduction to timed automata

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General books on model-checking: [CGP99,SBB+01,AILS08,BK08]

1 Introduction to timed systems: why and how?

Context: verification of embedded systems

– The time naturally appears in real systems.

Example 1.
- Autonomous systems
- GPS
- Communication protocols
- Processors
- Schedulers

– The time naturally appears in properties, for instance bounded response time properties.

Example 2. After having called the lift, we want it to arrive no longer than 3 minutes later.
Stupid controller: would keep moving and stopping at each floor. ⌟

We thus need models and specification languages which integrate timing aspects.

1.1 How can we add time information to behaviours?

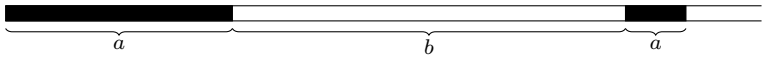
– An untimed behaviour: a sequence of events

$$(ab)^n \text{ or } (ab)^\omega \text{ with } a, b \in \Sigma \text{ (finite alphabet)}$$

– Time integrated as explicit delays: an alternating sequence of events and delays

$$1.5\ a\ 2\ b\ 3.6\ a \cdots \text{ with } a, b, c \in \Sigma$$

– Time integrated as durations of events/atomic propositions:



– Time integrated as dates of events

$$(a, 1.5)(b, 3.6)(a, 4.8)$$

We will choose this last formalism.

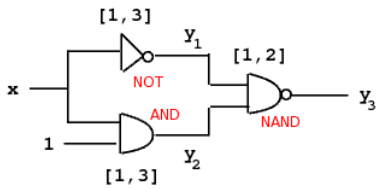
There are several possible semantics for the time:

– the time domain is discrete: $\mathbb{N}$
– the time domain is dense: $\mathbb{Q}_+$ or $\mathbb{R}_+$

1.2 Why we will consider dense time [Alu91]

Dense-time is more general than discrete time. But can we not always discretize? We will discuss this issue in the context of asynchronous circuits [BS91].

Example 3. We consider the following circuit:



Intervals labelling gates represent set of possible delays before stabilization of the gate. For instance, if $x = 0$ and $y_1 = 1$ (which is a stable configuration), when x is set to 1, gate y_1 will be updated to 0, but this will take a delay from 1 up to 3 time units.

We start with input $x = 0$. The corresponding stable configuration is $y = [101]$. The input x changes to 1. The corresponding stable configuration is $y = [011]$. One of the possible behaviours to reach that stable configuration is (values represent absolut time):

$$[\underline{1}01] \xrightarrow[1.2]{y_2} [\underline{1}\underline{1}1] \xrightarrow[2.5]{y_3} [\underline{1}10] \xrightarrow[2.8]{y_1} [01\underline{0}] \xrightarrow[4.5]{y_3} [011]$$

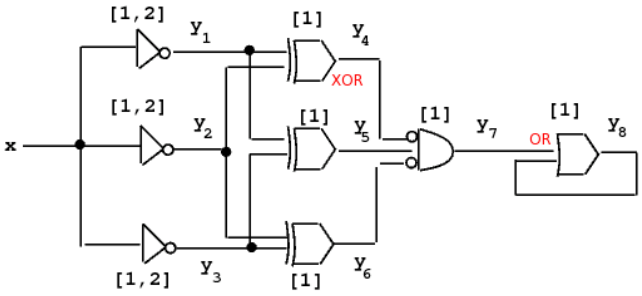
(an underlined value is not correct)

In this circuit, the set of reachable configurations is

$$\{[101], [111], [110], [010], [011], [001]\}$$

⌟

Example 4. Consider now the following circuit:



We will see that this circuit **is not** 1-discretizable.

We assume that initially, $x = 0$, and $y = [11100000]$ (stable configuration). The input x is set to 1.

– In dense time, the following behaviour is possible:

$[\underline{111}00000] \xrightarrow[1]{y_1} [0\underline{1100}000] \xrightarrow[1.5]{y_2} [00\underline{1}0\underline{00}00] \xrightarrow[2]{y_3,y_5} [0000\underline{1}0\underline{0}0] \xrightarrow[3]{y_5,y_7} [00000\underline{10}] \xrightarrow[4]{y_7,y_8} [00000001]$

– In discrete time (granularity 1), the possible behaviours are (up to permutation):
  - $[\underline{111}00000] \xrightarrow[1]{y_1,y_2,y_3} [00000000]$
  - $[\underline{111}00000] \xrightarrow[1]{y_1} [0\underline{1111}000] \xrightarrow[2]{y_2,y_3,y_4,y_5} [00000000]$
  - $[\underline{111}00000] \xrightarrow[1]{y_1,y_2} [00\underline{1}0\underline{00}00] \xrightarrow[2]{y_3,y_5,y_6} [000\underline{1}\underline{1}00] \xrightarrow[3]{y_5,y_6} [00000000]$
  - $[\underline{111}00000] \xrightarrow[1]{y_1,y_3} [0\underline{1}0\underline{1}0\underline{1}00] \xrightarrow[2]{y_2,y_4,y_6} [00000000]$

  Thus, the state $[00000001]$ is not reachable in discrete time.

  We will admit the following theorem.

Theorem 1 ([BS91]). *For every $k \geq 1$, there exists a digital circuit such that the reachability set of states in dense-time is strictly larger than the one in discrete time with granularity $\frac{1}{k}$.*

Claim. Finding a correct granularity is as difficult as analysing the system in dense time.

1.3 Classical timed models

They most of the time extend classical models:

– timed circuits,
– time(d) Petri nets,
– timed automata,
– *etc.*

2 The timed automaton model

2.1 Preliminary notations

If Z is a set, let Z^* be the set of *finite* sequences of elements in Z. We consider as time domain $\mathbb{T}$ the set $\mathbb{Q}_+$ of non-negative rationals or the set $\mathbb{R}_+$ of non-negative reals and Σ as a finite set of *actions*. A *time sequence* over $\mathbb{T}$ is a finite non-decreasing sequence $\tau = (t_i)_{1 \leq i \leq p} \in \mathbb{T}^*$. A *timed word* $w = (a_i, t_i)_{1 \leq i \leq p}$ is an element of $(\Sigma \times \mathbb{T})^*$, also written as a pair $w = (\sigma, \tau)$, where $\sigma = (a_i)_{1 \leq i \leq p}$ is a word in Σ^* and $\tau = (t_i)_{1 \leq i \leq p}$ a time sequence in $\mathbb{T}^*$ of same length. In the following, we denote by $\mathsf{Untime}(w)$ the finite word σ over the alphabet Σ.

Clock valuations, operations on clocks We consider a finite set X of variables, called *clocks*. A *(clock) valuation* over X is a mapping $v : X \to \mathbb{T}$ which assigns to each clock a time value. The set of all clock valuations over X is denoted $\mathbb{T}^X$, and $\mathbf{0}_X$ denotes the valuation assigning 0 to every clock $x \in X$.

Let $v \in \mathbb{T}^X$ be a valuation and $t \in \mathbb{T}$, the valuation $v+t$ is defined by $(v+t)(x) = v(x)+t$ for every $x \in X$. For a subset Y of X, we denote by $[Y \leftarrow 0]v$ the valuation such that for every $x \in Y$, $([Y \leftarrow 0]v)(x) = 0$ and for every $x \in X \setminus Y$, $([Y \leftarrow 0]v)(x) = v(x)$.

Clock constraints Given a finite set of clocks X, we introduce two sets of *clock constraints over* X. The most general one, denoted $\mathcal{C}(X)$, is defined by the grammar:

$$g ::= x \bowtie c \mid x - y \bowtie c \mid g \wedge g \mid \mathit{true}$$
$$\text{where } x, y \in X,\ c \in \mathbb{Z} \text{ and } \bowtie \in \{<, \leq, =, \geq, >\}.$$

Remark 1. We could allow rational constants in clock constraints (*i.e.*, have $c \in \mathbb{Q}$ in the above grammar), everything that follows would still hold, but developments would then be a bit more technical.

A clock constraint of the form $x - y \bowtie c$ is said *diagonal*. Next we also use the proper subset of *diagonal-free* clock constraints where the diagonal constraints are not allowed. This set, denoted $\mathcal{C}_{df}(X)$, is defined by the grammar:

$$g ::= x \bowtie c \mid g \wedge g \mid \mathit{true}$$
$$\text{where } x \in X,\ c \in \mathbb{Z} \text{ and } \bowtie \in \{<, \leq, =, \geq, >\}.$$

A *k-bounded* clock constraint is a clock constraint which involves only (integral) constants c between $-k$ and $+k$. The set of k-bounded (resp. k-bounded diagonal-free) clock constraints is denoted $\mathcal{C}^k(X)$ (resp. $\mathcal{C}^k_{df}(X)$).

If $v \in \mathbb{T}^X$ is a clock valuation, we write $v \models g$ when v satisfies the clock constraint g, and we say that v satisfies $x \bowtie c$ (resp. $x - y \bowtie c$) whenever $v(x) \bowtie c$ (resp. $v(x) - v(y) \bowtie c$). If g is a clock constraint, we write $[\![g]\!]_X$ the set of clock valuations $\{v \in \mathbb{T}^X \mid v \models g\}$.

Example 5. The valuation v over $\{x, y\}$ such that $v(x) = 4.1$ and $v(y) = 0$ satisfies the constraint $(x \leq 5) \wedge (x - y > 3)$. ⌟

2.2 The model of timed automata

A *timed automaton* over $\mathbb{T}$ is a tuple $\mathcal{A} = (L, L_0, L_F, X, \Sigma, T)$, where L is a finite set of locations, $L_0 \subseteq L$ is the set of initial locations, $L_F \subseteq L$ is the set of final locations, X is a finite set of clocks, Σ is a finite alphabet of actions, and $T \subseteq L \times \mathcal{C}(X) \times \Sigma \times 2^X \times L$ is a finite set of transitions.[1] If all constraints appearing in $\mathcal{A}$ are diagonal-free (*i.e.* are in $\mathcal{C}_{df}(X)$), we say that $\mathcal{A}$ is a *diagonal-free* timed automaton.

For modelling purpose some definitions assume invariants in the model. Invariants are clock constraints assigned to locations which have to be satisfied while the system is in the location: they restrict time elapsing and may enforce the system be live. A *timed automaton with invariants* is a tuple $\mathcal{A} = (L, L_0, L_F, X, \Sigma, T, \mathsf{Inv})$ where $(L, L_0, L_F, X, \Sigma, T)$ is a timed automaton in the previous sense, and $\mathsf{Inv}\colon L \to \mathcal{C}_{df}(X)$, the invariant, assigns a clock constraint to every location. A timed automaton as defined initially is a special case of a timed automaton with invariants where the invariant assigns true to every location. In the

[1] For more readability, a transition will often be written as $\ell \xrightarrow{g,a,Y} \ell'$ or even as $\ell \xrightarrow{g,a,Y:=0} \ell'$ instead of simply the tuple (ℓ, g, a, Y, ℓ').

sequel when we speak of timed automata we will equivalently mean timed automata with or without invariants.

Several semantics can be given to timed automata. We first give an operational semantics to timed automata, and then give a language-based semantics.

Semantics as a timed transition system. We first give an operational semantics as a timed transition system. Let $\mathcal{A} = (L, L_0, L_F, X, \Sigma, T, \mathsf{Inv})$ be a timed automaton. Its operational semantics is given as the *timed transition system* $\mathcal{T}_\mathcal{A} = (S, S_0, S_F, \to)$ over alphabet Σ, where the set of states is $S = \{(\ell, v) \in L \times \mathbb{T}^X \mid v \models \mathsf{Inv}(\ell)\}$, the set of initial states is $S_0 = \{(\ell_0, \mathbf{0}_X) \in S \mid \ell_0 \in L_0\}$, the set of final states is $S_F = S \cap (L_F \times \mathbb{T}^X)$,[2] and $\to \subseteq S \times (\mathbb{T} \cup \Sigma) \times S$ is the set of moves defined as follows:

- *delay moves*: if $(\ell, v) \in S$ and $d \in \mathbb{T}$, there is a move $(\ell, v) \xrightarrow{d} (\ell, v + d)$ whenever for every $0 \leq d' \leq d$, $(\ell, v + d') \in S$ (*i.e.* $v + d' \models \mathsf{Inv}(\ell)$);
- *action moves*: if $(\ell \xrightarrow{g,a,Y} \ell') \in T$ and $(\ell, v) \in S$, there is a move $(\ell, v) \xrightarrow{a} (\ell', [Y \leftarrow 0]v)$ whenever $v \models g$ and $([Y \leftarrow 0]v) \models \mathsf{Inv}(\ell')$. In that case we say that the action move is associated with the transition $\ell \xrightarrow{g,a,Y} \ell'$.

Next it will be more convenient to consider *mixed moves*, and hence to consider the timed transition system $\mathcal{T}^m_\mathcal{A} = (S, S_0, \to)$ where S and S_0 are defined as in $\mathcal{T}_\mathcal{A}$, and $\to \subseteq S \times (\mathbb{T} \times \Sigma) \times S$. In that case a move is composed of a delay move directly followed by an action move: $(\ell, v) \xrightarrow{d,a} (\ell', v')$ is a mixed move if $(\ell, v) \xrightarrow{d} (\ell, v + d)$ is a delay move and $(\ell, v + d) \xrightarrow{a} (\ell', v')$ is an action move.

Remark 2. We will later use this semantics for defining behavioural equivalences (and preorders).

Semantics as a timed language. We now give a language-based semantics to a timed automaton. Let $\mathcal{A} = (L, L_0, L_F, X, \Sigma, T, \mathsf{Inv})$ be a timed automaton. A *path* in $\mathcal{A}$ is a finite sequence of consecutive transitions:

$$P = \ell_0 \xrightarrow{g_1,a_1,Y_1} \ell_1 \xrightarrow{g_2,a_2,Y_2} \ \dots \ \ell_{p-1} \xrightarrow{g_p,a_p,Y_p} \ell_p$$

where $\ell_{i-1} \xrightarrow{g_i,a_i,Y_i} \ell_i \in T$ for every $1 \leq i \leq p$. The path is said to be *accepting* if it starts in an initial location ($\ell_0 \in L_0$) and ends in a final location ($\ell_p \in L_F$). A *run* of the automaton along the path P is a sequence of the form:

$$\varrho = (\ell_0, v_0) \xrightarrow{d_1,a_1} (\ell_1, v_1) \xrightarrow{d_2,a_2} \ \dots \ (\ell_{p-1}, v_{p-1}) \xrightarrow{d_p,a_p} (\ell_p, v_p)$$

where $v_0 \models \mathsf{Inv}(\ell_0)$, and for each $1 \leq i \leq p$, $(\ell_{i-1}, v_{i-1}) \xrightarrow{d_i,a_i} (\ell_i, v_i)$ is a mixed move of the timed transition system $\mathcal{T}^m_\mathcal{A}$ (see Section 2.2) associated with the transition $\ell_{i-1} \xrightarrow{g_i,a_i,Y_i} \ell_i$. The run is *accepting* if the underlying path is accepting and if $v_0 = \mathbf{0}_X$.

[2] Note that S_F is optional, but it will be useful to have it later.

The label of the run ϱ is the timed word $w = (a_1, t_1) \dots (a_p, t_p)$ where for each $1 \leq i \leq p$, $t_i = \sum_{j=1}^{i} d_j$ is the absolute time at which the i-th action a_i occurs (we also say that run ϱ *reads* the timed word w). If the run ϱ is accepting then the timed word w is said to be accepted by $\mathcal{A}$. The set of all timed words accepted by $\mathcal{A}$ is denoted $L(\mathcal{A})$ and is the *timed language accepted (or equivalently recognized) by* $\mathcal{A}$.

Remark 3. In these notes, we mostly consider finite paths, finite runs and finite timed words, but as for classical finite automata, one could consider infinite timed words and ω-regular accepting conditions (Büchi, Muller, etc...), see [AD94].

2.3 An example of a timed automaton

We consider the (diagonal-free) timed automaton in Figure 1 with two clocks x and y and over the alphabet $\{\texttt{problem}, \texttt{delayed}, \texttt{repair}, \texttt{done}\}$. It has four locations ('safe', 'alarm', 'failsafe', 'repairing'), and location 'safe' is both initial and final. Transitions are depicted in a standard way as labelled edges between locations. The transition between 'alarm' and 'repairing' has constraint $x < 15$, action label `repair` and resets clock y. A state of this

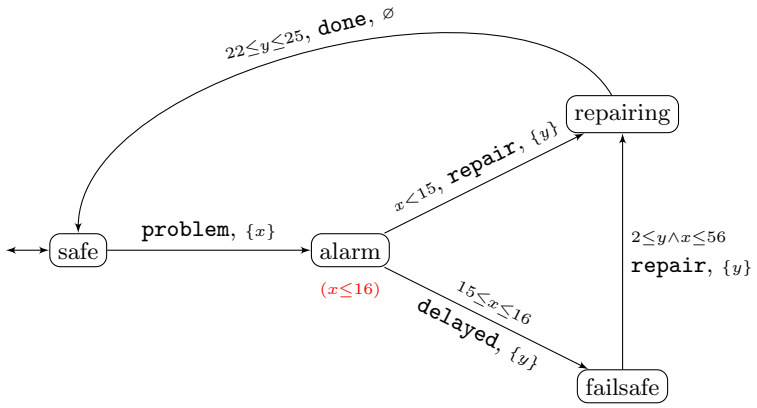


Fig. 1: *An example of a timed automaton*

automaton is a location, and a valuation for the two clocks x and y. The following is a sequence of delay and action moves in the timed transition system of the above timed automaton (we write a state vertically, with the name of the location on top, then the

value of clock x and the value of clock y at the bottom):

	safe	$\xrightarrow{23}$	safe	$\xrightarrow{\texttt{problem}}$	alarm	$\xrightarrow{15.6}$	alarm	$\xrightarrow{\texttt{delayed}}$	failsafe	
x	0		23		0		15.6		15.6	$\cdots$
y	0		23		23		38.6		0	

	failsafe	$\xrightarrow{2.3}$	failsafe	$\xrightarrow{\texttt{repair}}$	repairing	$\xrightarrow{22.1}$	repairing	$\xrightarrow{\texttt{done}}$	safe
$\cdots$	15.6		17.9		17.9		40		40
	0		2.3		0		22.1		22.1

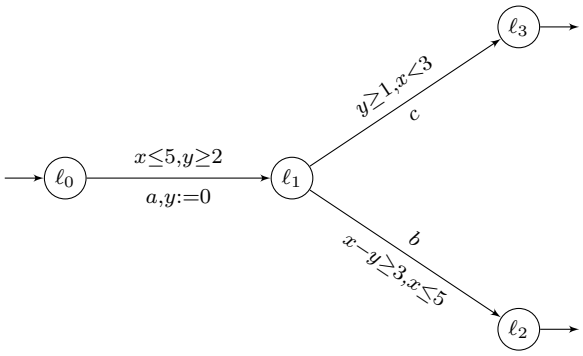
It corresponds to the following accepting run of the timed automaton (with the same convention for the representation of states):

	safe	$\xrightarrow{23,\texttt{problem}}$	alarm	$\xrightarrow{15.6,\texttt{delayed}}$	failsafe	$\xrightarrow{2.3,\texttt{repair}}$	repairing	$\xrightarrow{22.1,\texttt{done}}$	safe
x	0		0		15.6		17.9		40
y	0		23		0		0		22.1

This run reads (and accepts) the timed word

$$(\texttt{problem}, 23)(\texttt{delayed}, 38.6)(\texttt{repair}, 40.9), (\texttt{done}, 63)$$

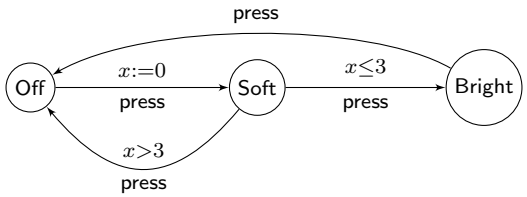
Exercise 1. Consider the following timed automata $\mathcal{A}$:



Do the timed words $w = (a, 4.7)(b, 4.9)$ and $w' = (a, 2.1)(c, 3.1)$ belong to $L(\mathcal{A})$? Justify your answer. ⌟

Exercise 2. Give a model for the intelligent light controller: If press is issued twice quickly (say, in no more than 3 time units) then the light will get brighter. Otherwise the light is switched off.

A possible solution:



⌟

2.4 Parallel composition of timed systems

Modelling a system with a unique timed automaton is not always convenient. A solution is to define small components of the system, and show how they communicate and interact. This is the role of the parallel composition.

Let $(\mathcal{A}_i)_{1\leq i\leq n}$ be n timed automata, with $\mathcal{A}_i = (L_i, L_{i,0}, L_{i,F}, X_i, \Sigma_i, T_i, \mathsf{Inv}_i)$. We assume that all X_i's are disjoint. If Σ is a new alphabet, given a (partial) synchronization function $f : \prod_{i=1}^n (\Sigma_i \cup \{\bullet\}) \setminus \{\bullet, \ldots, \bullet\} \to \Sigma$, the *synchronized product* (or *parallel composition*) $(\mathcal{A}_1 \parallel \cdots \parallel \mathcal{A}_n)_f$ is the timed automaton $\mathcal{A} = (L, L_0, L_F, X, \Sigma, T, \mathsf{Inv})$ where $L = L_1 \times \ldots L_n$, $L_0 = L_{1,0} \times \ldots L_{n,0}$, $L_F = L_{1,F} \times \ldots L_{n,F}$, for every $(\ell_1, \ldots, \ell_n) \in L_1 \times \ldots L_n$, $\mathsf{Inv}((\ell_1, \ldots, \ell_n)) = \bigwedge_{i=1}^{n} \mathsf{Inv}_i(\ell_i)$, $X = \bigcup_{i=1}^n X_i$, and the set T is composed of the transitions $(\ell_1, \ldots, \ell_n) \xrightarrow{g,a,Y} (\ell'_1, \ldots, \ell'_n)$ such that there exists $(a_1, \ldots, a_n) \in f^{-1}(a)$ where:

- $a_i = \bullet$ implies $\ell_i = \ell'_i$;
- $a_i \neq \bullet$ implies that there exists $\ell_i \xrightarrow{g_i,a_i,Y_i} \ell'_i$ in T_i;
- $g = \bigwedge\{g_i \mid a_i \neq \bullet\}$ and $Y = \bigcup\{Y_i \mid a_i \neq \bullet\}$;

Remark 4. For modelling purpose we may use finitely-valued variables, and use shared variables. We however do not enter into more details here.

Exercise 3 (The train crossing example). A gate is controlled by a controller, which sends signals (GoDown and GoUp) for closing or opening the barrier of the gate. Once such a signal is received by the gate, 10 time units are required for processing the action. The trains communicate with the controller. A train sends a signal Approach to the controller when it is approaching the gate and Exit when it is exiting the gate. After having sent signal Approach, the train will be on the gate within 20 and 30 time units. It will exit the gate within 10 and 20 time units later.

The controller receives the Approach and Exit signals from the train, and sends the GoUp and GoDown instructions to the gate. After having received signal Approach the controller sends quickly (in no more than 10 time units) the signal GoDown, and whenever it receives the signal Exit it sends instruction GoUp to the gate precisely 20 time units later.

Give a model of the whole system, first with a single train, and then with two trains. The two trains should be similar.

Here are some properties that we could check on that model:

- is the gate closed when the train crosses the road? (safety)
- is the gate always closed for less than 50 time units? (liveness)

Exercise 4. Build a model for a 3-floor lift as a product of several small automata. It will be made of the cabin, one door at each floor, one button at each floor (or two directional buttons), 3 buttons in the cabin, and a controller. ⌟

3 Reachability analysis, why and how?

For verification purposes, the most fundamental properties that one should be able to verify on a model are reachability properties: basic safety properties (like 'avoid some bad states of the system') can be expressed as (non-)reachability properties. Usually a class of models is said *decidable* whenever checking reachability properties in this class is decidable. Otherwise this class is said *undecidable*. For timed automata we will focus on (location) reachability properties of the form: "Is the set of final locations of a timed automaton $\mathcal{A}$ reachable from an initial state?" *I.e.*, "is there a run starting in an initial state leading to some final location ℓ?" There is no requirement as for the values of the clocks when reaching location ℓ (note however that any clock constraint could be added as a requirement). This problem is equivalent to the so-called *emptiness problem* (from a language-theoretical perspective), where the question is whether the language accepted by a timed automaton is empty or not.

Several classes of systems are known to be decidable, and the simplest such class of systems is that of finite automata, whose emptiness problem is known to be NLOGSPACE-complete [HU79]. The case of timed automata is much harder as there are potentially infinitely many states (recall that a state is a pair (ℓ, v) where ℓ is a location and v is a valuation). Techniques used for finite automata can thus not be applied to timed automata, and specific symbolic techniques and abstractions have to be developed.

In the next section we present the basic technics for solving the decidability of the emptiness problem in timed automata. The idea will be to build a finite abstraction of the timed automaton (actually a finite automaton), which preserves time-abstract properties (and in particular reachability properties).

4 The region abstraction: A key for decidability

The region automaton construction is due to Rajeev Alur and David Dill [AD90,AD94]. It is an abstraction of a timed automaton into a finite automaton which preserves many interesting properties of the system. In this section we present a more abstract version of this construction, which has been introduced in a wider context in [BDFP04]. This approach decouples the main arguments from the technicalities. As we focus on reachability properties we assume w.l.o.g. that timed automata have no invariants (they can be transferred to constraints on incoming and outgoing transitions while preserving reachability properties).

4.1 A tool: time-abstract bisimulation

Viewing the semantics of timed automata as timed transition systems allows to define behavioural equivalence relations (or pre-orders).

Standard bisimulation on finite transition systems. We first recall the notion of bisimulation. Let $\mathcal{T} = (S, S_0, S_F, \to)$ and $\mathcal{T}' = (S', S'_0, S'_F, \to')$ be two finite transition systems (corresponding *eg* to finite automata) over the same alphabet Σ (meaning that

$\to \subseteq S \times \Sigma \times S$ and $\to' \subseteq S' \times \Sigma \times S'$). Let $\mathfrak{R} \subseteq S \times S'$ be a relation. It is a *bisimulation relation* whenever the following condition holds:

- if $s_1 \xrightarrow{a} s_2$ and $(s_1, s_1') \in \mathfrak{R}$, then there exists $s_2' \in S'$ such that $s_1' \xrightarrow{a}{}' s_2'$ and $(s_2, s_2') \in \mathfrak{R}$;
- if $s_1' \xrightarrow{a}{}' s_2'$ and $(s_1, s_1') \in \mathfrak{R}$, then there exists $s_2 \in S$ such that $s_1 \xrightarrow{a} s_2$ and $(s_2, s_2') \in \mathfrak{R}$.

We can schematize the first condition as follows:

$$\begin{array}{cccccccc} & s_1 & \xrightarrow{a} & s_2 & & & s_1 & \xrightarrow{a} & s_2 \\ \forall & \mathfrak{R} & & & \Rightarrow & \exists & \mathfrak{R} & & \mathfrak{R} \\ & s_1' & & & & & s_1' & \xrightarrow{a} & s_2' \end{array}$$

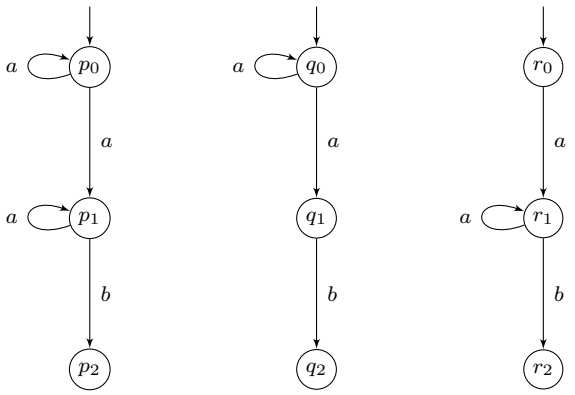
Such a relation *respects the final states* whenever $(s, s') \in \mathfrak{R}$ implies ($s \in S_F$ iff $s' \in S_F'$).

We say that the two transition systems $\mathcal{T}$ and $\mathcal{T}'$ are *bisimilar* whenever there exists a bisimulation relation $\mathfrak{R}$ such that for every $s_0 \in S_0$ there exists $s_0' \in S_0'$ with $(s_0, s_0') \in \mathfrak{R}$, and vice-versa.

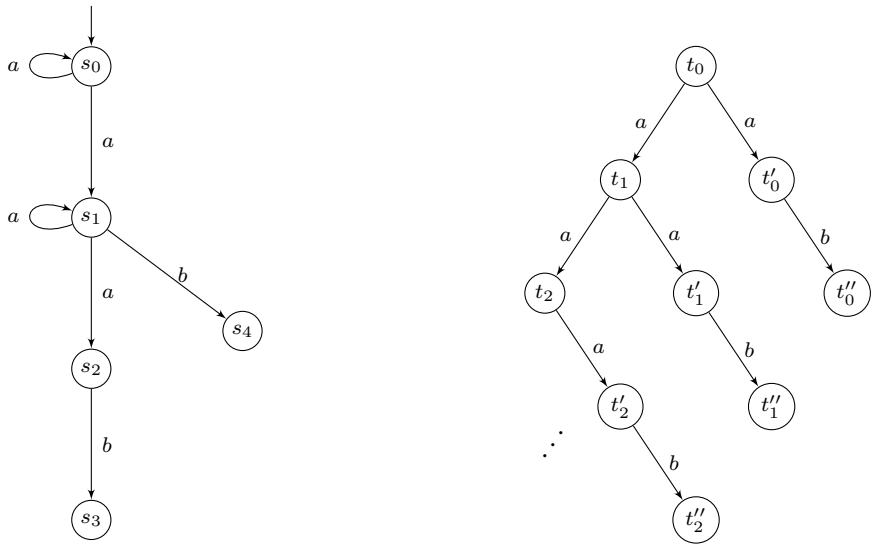
Lemma 1. *If $\mathfrak{R}$ is a bisimulation relation which respects the final states, such that for every $s_0 \in S_0$, there exists $s_0' \in S_0'$ with $(s_0, s_0') \in \mathfrak{R}$ and vice-versa, then $L(\mathcal{T}) = L(\mathcal{T}')$.*

Exercise 5. Write the proof of this lemma. ⌟

Exercise 6 (On bisimulation relations). Consider the three following automata. Are they bisimilar? Justify.



How do the following transitions relate (in a bisimulation sense) with the previous automata?



⌟

Exercise 7. How do the determinized version and the minimal automaton relate to the original finite automaton?

Strong timed bisimulation. We extend the previous notion in an obvious way to timed transition systems. We choose to define it with mixed moves. Let $\mathcal{T}_1^m = (S_1, S_{1,0}, S_{1,F}, \to_1)$ and $\mathcal{T}_2^m = (S_2, S_{2,0}, S_{2,F}, \to_2)$ be two timed transition systems with mixed moves over the same alphabet Σ. A relation $\mathfrak{R} \subseteq S_1 \times S_2$ is a *strong timed bisimulation* whenever the following properties hold:

- if $s_1 \mathfrak{R} s_2$ and $s_1 \xrightarrow{d,a}_1 s'_1$ for some $d \in \mathbb{T}$ and some $a \in \Sigma$, then $s_2 \xrightarrow{d,a}_2 s'_2$ and $s'_1 \mathfrak{R} s'_2$;
- *vice-versa* (swap mixed moves in $\mathcal{T}_1^m$ and in $\mathcal{T}_2^m$).

This equivalence is really strong and will seldom be used.

Time-abstract bisimulation. Let $\mathcal{T}_1^m = (S_1, S_{1,0}, \to_1)$ and $\mathcal{T}_2^m = (S_2, S_{2,0}, \to_2)$ be two timed transition systems with mixed moves over the same alphabet Σ. A relation $\mathfrak{R} \subseteq S_1 \times S_2$ is a *time-abstract bisimulation* whenever the following properties hold:

- if $s_1 \mathfrak{R} s_2$ and $s_1 \xrightarrow{d_1,a}_1 s'_1$ for some $d_1 \in \mathbb{T}$ and some $a \in \Sigma$, then there exists $d_2 \in \mathbb{T}$ such that $s_2 \xrightarrow{d_2,a}_2 s'_2$ and $s'_1 \mathfrak{R} s'_2$;
- *vice-versa* (swap mixed moves in $\mathcal{T}_1^m$ and in $\mathcal{T}_2^m$).

This relation is called time-abstract because it existentially quantifies over possible delays.

Unlike bisimulation, time-abstract bisimulation does not preserve languages, but it does preserve emptiness of languages.

Lemma 2. *If $\mathfrak{R}$ is a time-abstract bisimulation relation which respects final states, such that for every $s_0 \in S_0$, there exists $s'_0 \in S'_0$ with $(s_0, s'_0) \in \mathfrak{R}$ and vice-versa, then $L(\mathcal{T}) = \varnothing$ iff $L(\mathcal{T}') = \varnothing$.*

4.2 The region automaton construction

The aim of this construction is to finitely abstract behaviours of timed automata, so that checking a reachability property in a timed automaton reduces to checking a reachability property in a finite automaton. The construction relies on the definition of an equivalence relation over the set of valuations, which has finite index, and is called the *region equivalence*.

The region equivalence. We fix a finite set of clocks X, and we let $\mathcal{C}$ be a finite set of constraints over X. Let $\mathcal{R}$ be a partition of the set of (clock) valuations $\mathbb{T}^X$. We define three compatibility conditions as follows:

① $\mathcal{R}$ is *compatible with the set of constraints* $\mathcal{C}$ if for every constraint g in $\mathcal{C}$, for every R in $\mathcal{R}$, either $R \subseteq [\![g]\!]_X$ or $[\![g]\!]_X \cap R = \varnothing$;

② $\mathcal{R}$ is *compatible with the elapsing of time* if for all R and R' in $\mathcal{R}$, if there exists some $v \in R$ and $t \in \mathbb{T}$ such that $v + t \in R'$, then for every $v' \in R$, there exists some $t' \in \mathbb{T}$ such that $v' + t' \in R'$;

③ $\mathcal{R}$ is *compatible with the resets* whenever for all R and R' in $\mathcal{R}$, for every subset $Y \subseteq X$, if $([Y \leftarrow 0]R) \cap R' \neq \varnothing$, then $([Y \leftarrow 0]R) \subseteq R'$.[3]

If $\mathcal{R}$ is finite and satisfies these three conditions, we say that $\mathcal{R}$ is a *set of regions* for the set of clocks X and the set of constraints $\mathcal{C}$ or simply a set of regions (if X and $\mathcal{C}$ are clear in the context). An element $R \in \mathcal{R}$ is then called a *region*. The set $\mathcal{R}$ defines in a natural way an equivalence relation $\equiv_{\mathcal{R}}$ over valuations:

$$v \equiv_{\mathcal{R}} v' \text{ iff (for each region } R \text{ of } \mathcal{R},\ v \in R \Leftrightarrow v' \in R)$$

If v is a valuation we note $[v]_{\mathcal{R}}$ the (unique) region to which v belongs.

The intuition behind these conditions is the following: we want to finitely abstract behaviours of timed automata. To that aim, we finitely abstract the (infinite) set of valuations (or equivalently the state space of the system): a valuation v will be abstracted into the region $[v]_{\mathcal{R}}$ (and a state (ℓ, v) will be abstracted into the pair $(\ell, [v]_{\mathcal{R}})$). In order for the abstraction to preserve (at least) reachability properties, it must be the case that if two states are equivalent, then future behaviours from those states should be more or less the same. The three conditions above express this property: condition ① says that two equivalent valuations satisfy the same clock constraints, condition ② says that time elapsing does not distinguish between two equivalent valuations, whereas condition ③ says that resetting clocks does not distinguish between two equivalent valuations. Note that condition ② is a *time-abstract* property: the precise values of delays needs not be the same from two equivalent valuations. This 'abstraction of time' is the main reason which will make the region equivalence finite index for timed automata.

Example 6. Let us consider the partition of $\mathbb{R}_+^{\{x,y\}}$ made of five pieces

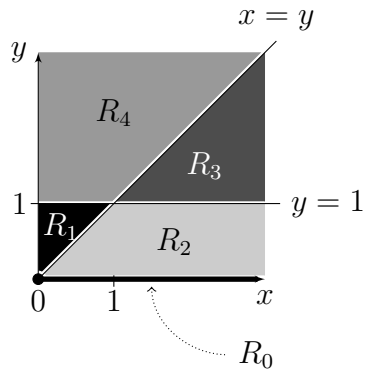
$$\mathcal{R} = \{R_0, R_1, R_2, R_3, R_4\}$$

[3] $[Y \leftarrow 0]R$ is the set of valuations $\{[Y \leftarrow 0]v \mid v \in R\}$.

which are defined by the following constraints:

$$\underset{\begin{pmatrix} x \geq 0 \\ y = 0 \end{pmatrix}}{R_0} \quad \underset{\begin{pmatrix} 0 \leq x < 1 \\ 0 \leq y \leq 1 \\ x < y \end{pmatrix}}{R_1} \quad \underset{\begin{pmatrix} x \geq 0 \\ 0 < y \leq 1 \\ x \geq y \end{pmatrix}}{R_2} \quad \underset{\begin{pmatrix} x > 1 \\ y > 1 \\ x \geq y \end{pmatrix}}{R_3} \quad \underset{\begin{pmatrix} x \geq 0 \\ y > 1 \\ x < y \end{pmatrix}}{R_4}$$

This partition is represented below:



It is easy to verify that $\mathcal{R}$ is a set of regions for the set of clocks $\{x, y\}$ and for the set of constraints $\{y \leq 1, y > 1, x \geq y, x < y\}$. ⌟

The main property of the region equivalence is the following:

Proposition 1. *Let $\mathcal{A} = (L, L_0, L_F, X, \Sigma, T)$ be a timed automaton with set of constraints $\mathcal{C}$. Assume that $\mathcal{R}$ is a set of regions for X and $\mathcal{C}$. Then the equivalence relation $\cong_{\mathcal{R}}$ defined on configurations of $\mathcal{A}$ by:*

$$(\ell, v) \cong_{\mathcal{R}} (\ell', v') \text{ iff } \ell = \ell' \text{ and } v \equiv_{\mathcal{R}} v'$$

is a time-abstract bisimulation, which is called the region equivalence.

Exercise 8. Write the proof of this proposition. ⌟

This property of the region equivalence will be used to construct a finite automaton, which will somehow be the quotient of the original timed automaton with the region equivalence. We start by constructing the region graph, which represents the (abstract) evolution of time and clock valuations. Then we construct the region automaton, which is a finite automaton, representing (in an abstract way) the behaviours of the timed automaton.

The region graph. From a set of regions $\mathcal{R}$ as characterized in the previous paragraph, one can define the so-called *region graph*, which represents the possible time evolutions of the system: the region graph is a finite automaton whose set of states is $\mathcal{R}$ and whose transitions are:

$$\begin{cases} R \xrightarrow{\varepsilon} R' \text{ if } R' \text{ is a time successor of } R, \text{[4]} \\ R \xrightarrow{Y} R' \text{ if } [Y \leftarrow 0]R \subseteq R'. \end{cases}$$

[3] R' is a time successor of R whenever there is some $v \in R$ and some $t \in \mathbb{T}$ such that $v + t \in R'$. Note that, due to the compatibility condition ②, if R' is a time successor of R, then for every $v \in R$, there is some $t \in \mathbb{T}$ such that $v + t \in R'$.

Intuitively, the region graph records possible evolutions of time in the system: there is a transition $R \xrightarrow{\varepsilon} R'$ if, from some (or equivalently every, by condition ②) valuation of region R, it is possible to let some time elapse and reach region R'. There is a transition $R \xrightarrow{Y} R'$ if, from region R, the region R' can be reached by resetting clocks in Y. Note that this graph is closed by reflexivity and transitivity for ε-transitions, *i.e.*, $R \xrightarrow{\varepsilon} R$, and if $R \xrightarrow{\varepsilon} R'$ and $R' \xrightarrow{\varepsilon} R''$, then $R \xrightarrow{\varepsilon} R''$.

Example 7. The region graph associated with the set of regions $\mathcal{R}$ mentioned in Example 6 is represented on Figure 2. Plain edges are $\xrightarrow{\varepsilon}$ transitions of the region graph, whereas dashed (resp. dotted) edges are $\xrightarrow{\{x\}}$ (resp. $\xrightarrow{\{y\}}$) transitions. We have omitted transitions $\xrightarrow{\{x,y\}}$ (that reset both clocks) for readability reasons: there should be such a transition from any state to region R_0. ⌟

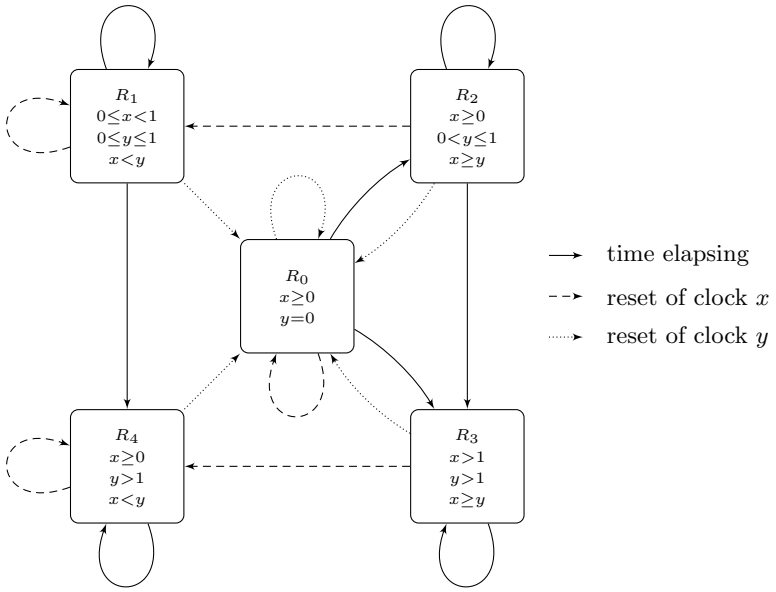


Fig. 2: *A simple example of a region graph*

The region automaton. Let $\mathcal{A} = (L, L_0, L_F, X, \Sigma, T)$ be a timed automaton, and assume that the set of constraints occurring in $\mathcal{A}$ is $\mathcal{C}$. Let $\mathcal{R}$ be a finite set of regions for X and $\mathcal{C}$ (*i.e.*, a partition of $\mathbb{T}^X$ satisfying conditions ①, ② and ③). The *region automaton* $\Gamma_{\mathcal{R}}(\mathcal{A})$ is the finite automaton $(Q, Q_0, Q_F, \Sigma, T')$ where $Q = L \times \mathcal{R}$ is the set of states, $Q_0 = L_0 \times \{[\mathbf{0}_X]_{\mathcal{R}}\}$ is the set of initial states, $Q_F = L_F \times \mathcal{R}$ is the set of final states, Σ is the same alphabet as that of $\mathcal{A}$, and T' is the set of transitions defined as follows: there is a transition $(\ell, R) \xrightarrow{a} (\ell', R')$ in T' whenever there exists some region $R'' \in \mathcal{R}$ and some

transition $\ell \xrightarrow{g,a,Y} \ell'$ in $\mathcal{A}$ such that

$$\begin{cases} R \xrightarrow{\varepsilon} R'' \text{ is a transition in the region graph,} \\ R'' \subseteq [\![g]\!]_X, \\ R'' \xrightarrow{Y} R' \text{ is a transition in the region graph.} \end{cases}$$

A transition of the region automaton abstracts a delay followed by an action in the original timed automaton. More precisely, whenever in $\mathcal{A}$ we can delay some time and make an a, then in $\Gamma_{\mathcal{R}}(\mathcal{A})$, we can make some a, and *vice-versa*. This is formalized by the following lemma, which can be derived from the definition of $\Gamma_{\mathcal{R}}(\mathcal{A})$ and from Proposition 1:

Lemma 3. *The three following sentences are equivalent:*

(*a*) *there is a transition* $(\ell, R) \xrightarrow{a} (\ell', R')$ *in* $\Gamma_{\mathcal{R}}(\mathcal{A})$*;*

(*b*) *there exists* $v \in R$*, there exists* $d \in \mathbb{T}$ *and there exists* $v' \in R'$ *such that* $(\ell, v) \xrightarrow{d,a} (\ell', v')$ *is a mixed move in* $\mathcal{A}$*;*

(*c*) *for every* $v \in R$*, there exists* $d \in \mathbb{T}$ *and there exists* $v' \in R'$ *such that* $(\ell, v) \xrightarrow{d,a} (\ell', v')$ *is a mixed move in* $\mathcal{A}$*.*

We are now ready to state the main language-related property of the region automaton construction.

Proposition 2. *Let* $\mathcal{A}$ *be a timed automaton with set of clocks* X *and set of constraints* $\mathcal{C}$*. We assume we can construct a set of regions* $\mathcal{R}$ *for* X *and* $\mathcal{C}$*. Then,*

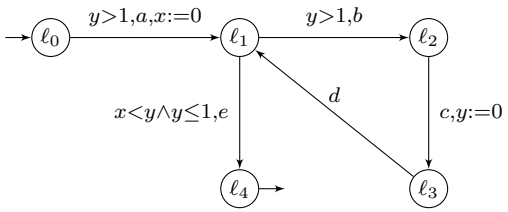
$$\mathsf{Untime}(L(\mathcal{A})) = L(\Gamma_{\mathcal{R}}(\mathcal{A}))$$

where $L(\Gamma_{\mathcal{R}}(\mathcal{A}))$ *is the (untimed) language accepted by the finite automaton* $\Gamma_{\mathcal{R}}(\mathcal{A})$ *and* $\mathsf{Untime}((a_1, t_1) \dots (a_p, t_p)) = a_1 \dots a_p$ *assigns to a timed word (and by extension to a timed language) a finite word (and by extension a classical language).*

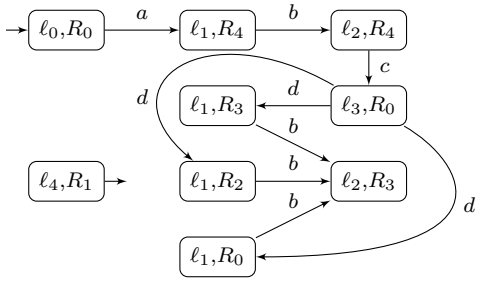
Proof. A standard induction on the length of runs allows to build runs in $\mathcal{A}$ and paths in $\Gamma_{\mathcal{R}}(\mathcal{A})$ which coincide in the following sense: there is a run $(\ell_0, v_0) \xrightarrow{d_1,a_1} \dots \xrightarrow{d_p,a_p} (\ell_p, v_p)$ in $\mathcal{A}$ iff there is a path $(\ell_0, [v_0]_{\mathcal{R}}) \xrightarrow{a_1} \dots \xrightarrow{a_p} (\ell_p, [v_p]_{\mathcal{R}})$. This implies the expected result. ⌟

Remark 5. In terms of behavioural equivalence, the finite automaton $\Gamma_{\mathcal{R}}(\mathcal{A})$ is the quotient of the timed transition system $\mathcal{T}_{\mathcal{A}}^{m}$ w.r.t. the time-abstract bisimulation $\cong_{\mathcal{R}}$.

Example 8. We consider the following small timed automaton, and we would like to know whether location ℓ_4 is reachable from the initial state $(\ell_0, \mathbf{0}_{\{x,y\}})$.



A possible set of regions for that automaton has been given in Example 6, and the corresponding region automaton is the following finite automaton.



Applying Proposition 2, we have that location ℓ_4 is reachable in the original timed automaton iff the state (ℓ_4, R_1) is reachable in the region automaton (actually it should be iff any of the (ℓ_4, R_i) is reachable, but due to the constraint labelling the transition from ℓ_1 to ℓ_4, we can focus on (ℓ_4, R_1)). We see that it is not the case, and hence that there is no run which starts in the initial state $(\ell_0, \mathbf{0}_{\{x,y\}})$ and ends in ℓ_4. ⌟

For every timed automaton $\mathcal{A}$ for which we can *effectively* construct a finite set of regions $\mathcal{R}$ (satisfying conditions ①, ② and ③), we can transfer the checking of reachability properties in $\mathcal{A}$ to the finite automaton $\Gamma_{\mathcal{R}}(\mathcal{A})$. It remains to see how we can effectively build sets of regions for timed automata.

4.3 Effective construction of sets of regions

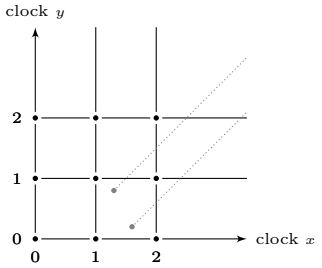
In the previous section, we have presented an abstract construction which allows to reduce for instance the model-checking of reachability properties in timed automata to the model-checking of reachability properties in finite automata (under the condition that there is a finite set of regions for the set of constraints used in the timed automaton). However, we did not explain how we construct a set of regions for timed automata, which is the basis to the whole construction.

In this section, we fix a finite set of clocks X.

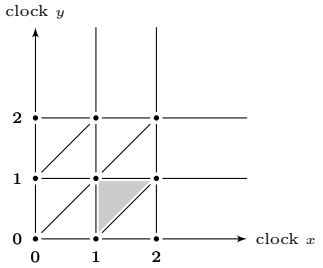
Regions for sets of diagonal-free constraints. Let $M \in \mathbb{N}$ be an integer. We define a set of regions for X and the set of M-bounded diagonal-free clock constraints $\mathcal{C}_{df}^{M}(X)$. A natural partition would be to take the partition induced by the set of constraints itself, see Figure 3(a) for an illustration with two clocks. But this is actually not fine enough because compatibility condition ③ is not satisfied (as illustrated on the figure by the two gray valuations). A correct partition is then the one of Figure 3(b).

We formalize this idea and we define the equivalence relation $\equiv_{df}^{X,M}$ over $\mathbb{T}^X$ as follows. Let v and v' be two valuations of $\mathbb{T}^X$, we say that $v \equiv_{df}^{X,M} v'$ if all three following conditions are satisfied:

(*a*) $v(x) > M$ iff $v'(x) > M$ for each $x \in X$,



(a) Partition compatible with the 2-bounded clock constraints and the resets (conditions ① and ③), but not with time elapsing (condition ②): the two gray points are not equivalent



(b) Partition $\mathcal{R}^2_{df}(\{x,y\})$ satisfying all the compatibility constraints ①, ② and ③ for the set of constraints $\mathcal{C}^2_{df}(\{x,y\})$

Fig. 3: *Region construction for set of diagonal-free constraints* $\mathcal{C}^2_{df}(\{x,y\})$

(*b*) if $v(x) \leq M$, then $\lfloor v(x) \rfloor = \lfloor v'(x) \rfloor$ and $\Big(\{v(x)\} = 0 \text{ iff } \{v'(x)\} = 0\Big)$ for each $x \in X$,[5] and

(*c*) if $v(x) \leq M$ and $v(y) \leq M$, then $\{v(x)\} \leq \{v(y)\}$ iff $\{v'(x)\} \leq \{v'(y)\}$ for all $x,\ y \in X$.

The relation $\equiv^{X,M}_{df}$ is an equivalence relation of finite index, and it naturally induces a finite partition $\mathcal{R}^M_{df}(X)$ of $\mathbb{T}^X$ (defined as the set of equivalence classes of $\mathbb{T}^X{}_{/\equiv^{X,M}_{df}}$). The construction for two clocks is precisely that illustrated on Figure 3(b).

The following lemma states the correctness of the above partition.

Proposition 3. *The partition* $\mathcal{R}^M_{df}(X)$ *is a finite set of regions for clocks* X *and* M*-bounded diagonal-free clock constraints* $\mathcal{C}^M_{df}(X)$.

Proof. The complete proof of this lemma is rather technical and is often ommitted. However we think this is worth giving some hints here. Conditions ① and ③ are rather easy to check, we thus omit the details. The case of condition ② requires more careful developments. Assume that $v_1 \in R_1$ for some $R_1 \in \mathcal{R}^M_{df}(X)$, and that $v_2 = v_1 + t \in R_2$ for some $t \in \mathbb{T}$ and some $R_2 \in \mathcal{R}^M_{df}(X)$. Take $v'_1 \in R_1$. We will exhibit some $t' \in \mathbb{T}$ such that $v'_2 = v'_1 + t' \in R_2$. We distinguish between several cases.

- First assume that all clocks $x \in X$ are such that $v_2(x) > M$. Then we can pick $t' > M$ and set $v'_2 = v'_1 + t'$. It is then obvious that $v'_2 \equiv^{M,X}_{df} v_2$ because for every clock $x \in X$, $v'_2(x) > M$.
- Assume next that there is some clock $x \in X$ such that $v_2(x) \leq M$ and $v_2(x)$ is an integer that we denote α. Then, $t = \alpha - v_1(x)$, and we define t' as $\alpha - v'_1(x)$. First note that $t' \in \mathbb{T}$ (as $0 \leq \alpha \leq M$, and v_1 and v'_1 are region equivalent, $\alpha - v_1(x) \geq 0$ implies $\alpha - v'_1(x) \geq 0$). Then we prove that $v'_2 \in R_2$ (*i.e.*, $v'_2 \equiv^{X,M}_{df} v_2$).

[5] $\lfloor \cdot \rfloor$ (resp. $\{\cdot\}$) represents the integral (resp. fractional) part.

Pick some clock $y \in X$. If $v_1(y), v'_1(y) > M$, then obviously $v_2(y), v'_2(y) > M$, and conditions (a) and (b) are satisfied. Assume now that $v_1(y), v'_1(y) \leq M$. Then:

$$\begin{aligned} v'_2(y) &= v'_1(y) + t' \\ &= \lfloor v'_1(y) \rfloor + \{v'_1(y)\} + \alpha - \lfloor v'_1(x) \rfloor - \{v'_1(x)\} \\ &= \big(\lfloor v_1(y) \rfloor + \alpha - \lfloor v_1(x) \rfloor\big) + \big(\{v'_1(y)\} - \{v'_1(x)\}\big) \\ &= \alpha_{x,y} + \big(\{v'_1(y)\} - \{v'_1(x)\}\big) \end{aligned}$$

for some non-negative integer $\alpha_{x,y}$. We have a similar expression for $v_2(y)$: $v_2(y) = \alpha_{x,y} + \big(\{v_1(y)\} - \{v_1(x)\}\big)$. As $v_1 \equiv_{df}^{X,M} v'_1$, both differences $\big(\{v'_1(y)\} - \{v'_1(x)\}\big)$ and $\big(\{v_1(y)\} - \{v_1(x)\}\big)$ lie in the same interval $(-1, 0)$, $[0, 0]$, or $(0, 1)$. In the first case:

$$\begin{cases} \lfloor v_2(y) \rfloor = \alpha_{x,y} - 1 & \text{and} \quad \{v_2(y)\} = 1 + \big(\{v_1(y)\} - \{v_1(x)\}\big) \\ \lfloor v'_2(y) \rfloor = \alpha_{x,y} - 1 & \text{and} \quad \{v'_2(y)\} = 1 + \big(\{v'_1(y)\} - \{v'_1(x)\}\big) \end{cases}$$

while in the two last cases:

$$\begin{cases} \lfloor v_2(y) \rfloor = \alpha_{x,y} & \text{and} \quad \{v_2(y)\} = \big(\{v_1(y)\} - \{v_1(x)\}\big) \\ \lfloor v'_2(y) \rfloor = \alpha_{x,y} & \text{and} \quad \{v'_2(y)\} = \big(\{v'_1(y)\} - \{v'_1(x)\}\big) \end{cases}$$

In particular, $\lfloor v_2(y) \rfloor = \lfloor v'_2(y) \rfloor$, and $\{v_2(y)\} = 0$ iff $\{v'_2(y)\} = 0$. Conditions (a) and (b) of the definition for a region are thus satisfied.
Assume now that y and z are two clocks such that $v'_2(y) \leq M$ and $v'_2(z) \leq M$. From what precedes, we get that $v_2(y) \leq M$ and $v_2(z) \leq M$ as well. Moreover,

$$\begin{cases} v_2(y) = \alpha_{x,y} + \big(\{v_1(y)\} - \{v_1(x)\}\big) \\ v_2(z) = \alpha_{x,z} + \big(\{v_1(z)\} - \{v_1(x)\}\big) \\ v'_2(y) = \alpha_{x,y} + \big(\{v'_1(y)\} - \{v'_1(x)\}\big) \\ v'_2(z) = \alpha_{x,z} + \big(\{v'_1(z)\} - \{v'_1(x)\}\big) \end{cases}$$

for some non-negative integers $\alpha_{x,y}$ and $\alpha_{x,z}$. We have similar expressions as above for integral and fractional parts of $v_2(y)$, $v'_2(y)$, $v_2(z)$ and $v'_2(z)$, and by examining all possible combinations, we get that $\{v_2(y)\} \leq \{v_2(z)\}$ iff $\{v'_2(y)\} \leq \{v'_2(z)\}$. Condition (c) for a region is thus satisfied.

– Finally assume that there is some clock $x \in X$ such that $v_2(x) \leq M$ (in particular $t \leq M$), but for every such clock $x \in X$, $v_2(x)$ is not an integer. We then assume that z_{extra} is an extra clock which does not belong to X, and we define $\widetilde{X} = X \cup \{z_{extra}\}$. We extend valuation v_1 to set of clocks $\widetilde{X}$ by defining the valuation $\widetilde{v}_1$ as $\widetilde{v}_1(x) = v_1(x)$ if $x \in X$ and $\widetilde{v}_1(z_{extra}) = M - t$ (this is a well-defined valuation as $t \leq M$). The valuation $\widetilde{v}_2 = \widetilde{v}_1 + t$ extends valuation v_2 to set of clocks $\widetilde{X}$, and $\widetilde{v}_2(z_{extra}) = M$ is an integer.
The idea is now to define a valuation $\widetilde{v}'_1$ over $\widetilde{X}$ which is region equivalent (for equivalence $\equiv_{df}^{\widetilde{X},M}$) to $\widetilde{v}_1$ and extends valuation v'_1 to set $\widetilde{X}$. Then we apply the construction of the first item (because $\widetilde{v}_2(z_{extra}) = M$ is an integer), and we find $t' \in \mathbb{T}$ such that

defining $\widetilde{v}'_2 = \widetilde{v}'_1 + t'$, we get that $\widetilde{v}_2 \equiv_{df}^{\widetilde{X},M} \widetilde{v}'_2$. We obtain v'_2 as the projection of $\widetilde{v}'_2$ onto X, and we get that $v'_2 = v'_1 + t'$ and $v'_2 \equiv_{df}^{X,M} v_2$.

It remains to explain how we define a valuation $\widetilde{v}'_1$ over $\widetilde{X}$ which is region equivalent (for equivalence $\equiv_{df}^{\widetilde{X},M}$) to $\widetilde{v}_1$ and extends valuation v'_1 to set $\widetilde{X}$. To do so, we distinguish between several cases:

- if $\widetilde{v}_1(z_{extra})$ is an integer (equal to $M - t \leq M$), we set $\widetilde{v}'_1(z_{extra}) = \widetilde{v}_1(z_{extra})$;
- if there is some clock $x \in X$ such that $v_1(x) \leq M$ and $\{v_1(x)\} = \{\widetilde{v}_1(z_{extra})\}$, then we set $\widetilde{v}'_1(z_{extra}) = \lfloor \{\widetilde{v}_1(z_{extra})\} \rfloor + \{v'_1(x)\}$;
- otherwise, if there is some clock $x \in X$ such that $\{v_1(x)\} < \{\widetilde{v}_1(z_{extra})\}$ then we choose clock $x_1 \in X$ satisfying this condition while maximizing $\{v_1(x)\}$. If there is some clock $x \in X$ such that $\{\widetilde{v}_1(z_{extra})\} < \{v_1(x)\}$ then we choose clock $x_2 \in X$ satisfying this condition while minimizing $\{v_1(x)\}$. We then define the two values γ_1 and γ_2 as follows:

$$\gamma_1 = \begin{cases} v'_1(x_1) \text{ if } x_1 \text{ defined} \\ 0 \qquad \text{otherwise} \end{cases} \qquad \gamma_2 = \begin{cases} v'_1(x_2) \text{ if } x_2 \text{ defined} \\ 1 \qquad \text{otherwise} \end{cases}$$

  As $v_1 \equiv_{df}^{X,M} v'_1$, it is the case that $\gamma_1 < \gamma_2$, and we then pick γ in the open interval (γ_1, γ_2) and set $\widetilde{v}'_1(z_{extra}) = \gamma$.

We define now $\widetilde{v}'_1(x) = v'_1(x)$ if $x \in X$. The valuation $\widetilde{v}'_1$ obviously extends valuation v'_1 to the set of clocks $\widetilde{X}$. And the careful reader can check that $\widetilde{v}_1 \equiv_{df}^{X,M} \widetilde{v}'_1$, which was precisely what was missing in the proof.

Hence, condition ② is satisfied by the partition $\mathcal{R}_{df}^M(X)$. ⌟

We now give another description of the regions in $\mathcal{R}_{df}^M(X)$, which makes it easier to give an upper bound on the number of regions in $\mathcal{R}_{df}^M(X)$. An interval of $\mathbb{T}$ with integral bounds is said *M-simple* if it is of one of the following forms: $(c, c+1)$ with $0 \leq c < M$, or $[c, c]$ with $0 \leq c \leq M$, or $(M, +\infty)$. It is said bounded if it is one of the two first forms, and singular in the second form. Each region of $\mathcal{R}_{df}^M(X)$ can then be characterized uniquely by:

- an M-simple interval I_x for every clock $x \in X$, and
- a preorder $\prec$ on the set of clocks

$$Z_{(I_x)_{x \in X}} = \{x \in X \mid I_x \text{ bounded and non-singular}\}.$$

Intuitively the interval I_x is the interval to which x belongs, and the preorder is given by the preorder on the fractional parts of all clocks bounded by M with non-integral values. For instance, the lightgray region depicted in Figure 3(b) is characterized by:

$$\begin{cases} x \in (1,2) \\ y \in (0,1) \\ x \prec y,\ y \not\prec x \text{ (meaning } \{x\} < \{y\}) \end{cases}$$

Exercise 9. Prove the correctness of this characterization. ⌟

Lemma 4. *The number of regions in $\mathcal{R}^M_{df}(X)$ is bounded by $(2M+2)^{|X|}.|X|!.2^{|X|}$ where $|X|$ is the cardinal of X.*

Proof. We use the previous characterization of regions to get the expected upper bound. The factor $(2M+2)^{|X|}$ is for the number of possible assignments of M-simple intervals to clocks. Now, to understand the rest of the formula, we define $X_0 = \{x \in X \mid I_x \text{ is singular}\}$ and $X_\infty = \{x \in X \mid I_x = (M, +\infty)\}$. Finally if $X \setminus (X_0 \cup X_\infty)$ is non-empty, we define non-empty and disjoint sets of clocks $(X_i)_{1 \leq i \leq p}$ that form a partition of $X \setminus (X_0 \cup X_\infty)$ satisfying furthermore the property that $(x \in X_i,\ y \in X_j$ and $i \leq j \Leftrightarrow x \prec y)$. We can then write down a list of the clocks in X_0 (in any order) followed by a list of the clocks in X_1, *etc*, followed by a list of the clocks in X_p, and finally followed by a list of the clocks in X_∞. There are at most $|X|!$ such lists that can be written. The region is finally completely characterized by designing any first element of each group of clocks, hence the last factor $2^{|X|}$. ⌟

Note that a tighter bound can be obtained, but the computation is much more involved, see [Kop96].[6]

Regions for sets of general constraints. Let $M \in \mathbb{N}$ be an integer. The aim is to define a set of regions for X and the set of M-bounded clock constraints $\mathcal{C}^M(X)$. The partition we have defined in the previous paragraph is no more compatible with the set of constraints $\mathcal{C}^M(X)$, we thus need to refine it.

We define an equivalence relation $\equiv^{X,M}$ over $\mathbb{T}^X$ as follows: let v and v' be two valuations of $\mathbb{T}^X$, we say that $v \equiv^{X,M} v'$ if all two following conditions hold:

- $v \equiv^{X,M}_{df} v'$, and
- $v \models (x - y \sim c)$ iff $v' \models (x - y \sim c)$ for every $(x - y \sim c) \in \mathcal{C}^M(X)$.

This equivalence relation refines $\equiv^{X,M}_{df}$ in that two equivalent valuations satisfy in addition the same diagonal constraints (bounded by M). This new partition is denoted $\mathcal{R}^M(X)$ and is illustrated for two clocks on Figure 4.

We omit the proof of the following proposition, which states the correctness of the refinement.

Proposition 4. *The partition $\mathcal{R}^M(X)$ is a set of regions for X and the set of M-bounded clock constraints $\mathcal{C}^M(X)$.*

As previously, each region of $\mathcal{R}^M(X)$ can be characterized by:

- an M-simple interval I_x for every clock $x \in X$, and
- an M-simple interval, or minus an M-simple interval, $J_{x,y}$ for all clocks $x, y \in X$.

[6] Thanks to Luca Aceto who pointed out this reference.

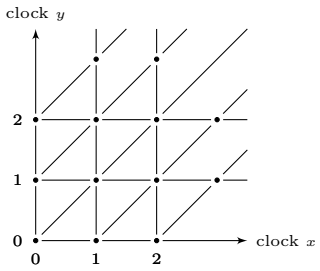


Fig. 4: *Set of regions $\mathcal{R}^2(X)$ for 2-bounded clock constraints with two clocks*

Intuitively the interval I_x gives the constraint on clock x, and $J_{x,y}$ gives the constraint on the difference $x - y$. The order between fractional parts of two clocks x and y can now be infered from the intervals I_x, I_y and $J_{x,y}$.

As a direct consequence of this characterization, we get the following upper bound on the number of regions.

Lemma 5. *The number of regions in $\mathcal{R}^M(X)$ is bounded by $(4M+3)^{(|X|+1)^2}$ where $|X|$ is the cardinal of X.*

Remark 6. Note that $\mathcal{R}^M(X)$ is also a set of regions for the set of constraints $\mathcal{C}^M_{df}(X)$.

Remark 7. Note that sets of regions we have described could be made smaller: there is no need to have the same maximal constant for all clocks, one maximal constant for each clock could be used. However, for the purpose of these notes, there is no need for such a refinement.

Exercise 10. We assume we can update clocks with more involve operations than resets to zero. For this exercise an update *up* is a function which assigns to each clock x an affine combination of all the clocks: for each clock x, $up(x)$ is of the form $\sum_y a_{x,y}y + b_x$, where $a_{x,y}$ and b_x are (integer) constants.

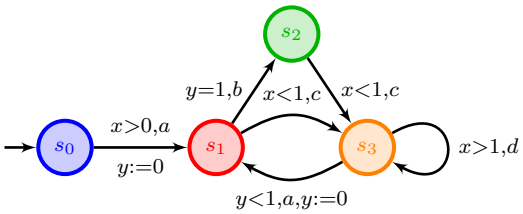
How should we strengthen conditions ①, ② and ③ in the construction of a finite set of regions to take into account those new updates?

Can you build a (finite) set of regions in the following cases? Explain.

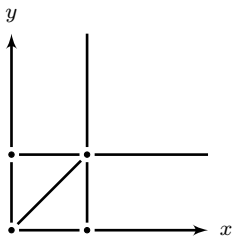
- diagonal-free clock constraints, resets to zero, and updates of the form $x := x + 1$;
- general clock constraints, resets to zero, and updates of the form $x := x + 1$;
- diagonal-free clock constraints, resets to zero, and updates of the form $x := x - 1$;
- general clock constraints, resets to zero, and updates of the form $x := x - 1$.

4.4 An example of region automaton [AD94]

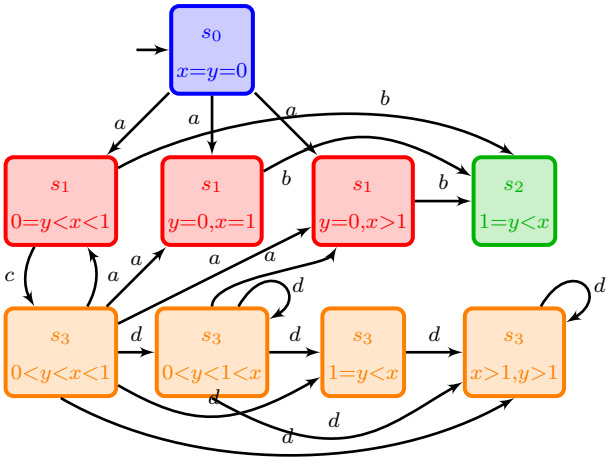
Consider the following timed automaton:



It has two clocks and maximal constant 1. The set of regions can be described as follows:



The corresponding region automaton is therefore:



5 Applications of the region automaton construction

5.1 Application to the reachability problem

Let $\mathcal{A}$ be a timed automaton with set of clocks X. Let M be the maximal constant involved in one of the constraints of $\mathcal{A}$, the set $\mathcal{R}^M(X)$ (or even $\mathcal{R}^M_{df}(X)$ in case $\mathcal{A}$ is a diagonal-free timed automaton) is a set of regions for $\mathcal{A}$. Hence the region abstraction and all the developments made in the previous section can be used. The following result, due to Alur and Dill [AD90,AD94], is the core of the verification of timed systems.

Theorem 2. *The reachability problem is decidable for timed automata. It is a* PSPACE*-complete problem (for both diagonal-free and general timed automata).*

Although this theorem has been first stated in [AD90,AD94], the proof we present here is taken from [AL02].

Proof. To prove PSPACE membership, we use the region automaton construction, and check reachability properties in this finite abstraction (see Proposition 2). Applying Lemmas 4 and 5, we know that the size of this finite automaton is exponential in the size of the original timed automaton (in both diagonal-free and general cases). Moreover, using the characterizations of the regions we have given we know that each state of the finite automaton can be stored in polynomial space, and, given a state, we can guess a successor in polynomial space. Hence, we apply the classical NLOGSPACE algorithm for checking reachability properties in finite automata, and get a PSPACE algorithm for checking reachability properties in the original timed automaton.

The PSPACE-hardness can be proved by reducing the termination of a (non-deterministic) linearly bounded Turing machine (LBTM for short) on some given input to the reachability problem in timed automata. We present the proof for general timed automata, and then explain informally how it can be extended to diagonal-free timed automata. The complete reduction can be found in the appendices of [AL02].

Let $\mathcal{M}$ be a LBTM and w_0 an input for $\mathcal{M}$. We write N for a (linear) bound on the length of the tape which is used when $\mathcal{M}$ executes on w_0. We encode the behaviour of $\mathcal{M}$ on w_0 as the behaviour of a timed automaton. The encoding is as follows: assuming the alphabet of $\mathcal{M}$ is $\{a, b\}$ (this can be done w.l.o.g.), and writing # for the blank symbol, the content of cell C_i of the tape of the LBTM is encoded by a constraint on two clocks x_i and y_i. Cell C_i contains a symbol a when the constraint $x_i = y_j$ holds, and cell C_j contains a symbol b when the constraint $x_j < y_j$ holds. If the cell is empty (or equivalently contains the blank symbol #), then the constraint $x_j > y_j$ holds. Note that these three constraints are invariant by time elapsing. This is illustrated on Figure 5.

We assume that the set of states of $\mathcal{M}$ is Q, its initial state is q_0, and its halting state is q_F. We construct a timed automaton $\mathcal{A} = (L, L_0, L_F, X, \Sigma, T)$ as follows:

- $L = \big(Q \times \{1, \ldots, N\}\big) \cup \{\mathsf{init}\}$;
- $L_0 = \{\mathsf{init}\}$;

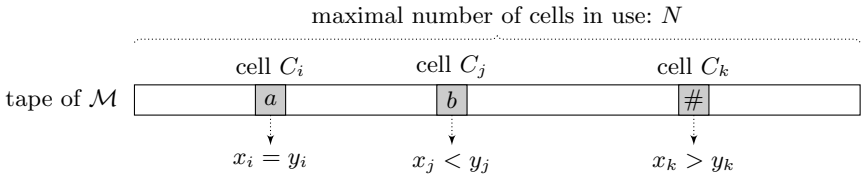


Fig. 5: *Encoding of the tape of the LBTM $\mathcal{M}$*

- $L_F = \{(q_F, i) \mid i \in \{1, \ldots, N\}\}$;
- $X = \{x_i, y_i \mid i \in \{1, \ldots, N\}\} \cup \{u\}$;
- $\Sigma = \{a\}$;
- For every rule of $\mathcal{M}$, we will have several transitions in T. We define the constraint $g_{a,i}$ by $(u > 0 \wedge x_i < y_i)$, the constraint $g_{b,i}$ by $(u > 0 \wedge x_i = y_i)$, and the constraint $g_{\#,i}$ by $(u > 0 \wedge x_i > y_i)$. Those constraints express that cell i contains an a, respectively a b and a blank character. We define the resetting sets $Y_{a,i} = \{u, x_i\}$ and $Y_{b,i} = \{u, x_i, y_i\}$. Those are sets of clocks to be reset for expressing the fact that we write an a, respectively a b, in cell i.
  - Consider a rule $(q, \mathsf{Read}_\alpha, \mathsf{Write}_\beta, \mathsf{Right}, q')$ [7] in $\mathcal{M}$. For every $i \in \{1, \ldots, N-1\}$, there is a transition $(q, i) \xrightarrow{g_{\alpha,i}, a, Y_{\beta,i}} (q', i+1)$ in T.
  - Consider a move $(q, \mathsf{Read}_\alpha, \mathsf{Write}_\beta, \mathsf{Left}, q')$ in $\mathcal{M}$. For every $i \in \{2, \ldots, N\}$, there is a transition $(q, i) \xrightarrow{g_{\alpha,i}, a, Y_{\beta,i}} (q', i-1)$ in T.

  There is an extra transition from state init to initialize the input word w_0 on the tape: $\mathsf{init} \xrightarrow{u>0, a, Y_{w_0}} (q_0, 1)$ where $Y_{w_0} = \{u\} \cup \{x_i \mid w_0(i) = a\} \cup \{x_i, y_i \mid w_0(i) = b\} \cup \{y_i \mid i > |w_0|\}$ ($w_0(i)$ denotes the i-th letter of w_0).

We claim that there is a halting computation in $\mathcal{M}$ iff $L(\mathcal{A}) \neq \varnothing$, and we let the careful reader get convinced of this equivalence. As the halting problem for LBTMs is PSPACE-hard, we get the expected lower bound.

See next exercise for the case of diagonal-free clock constraints. ⌟

Remark 8. In [CY92], a proof of PSPACE-hardness is given for diagonal-free timed automata with only three clocks, it is rather technical, hence we have chosen not to present it here.

Exercise 11. Think of a PSPACE lower bound reduction for diagonal-free timed automata. ⌟

Solution: The proof for diagonal-free timed automata is slightly more complicated, as it is no more possible to use time-elapsing invariant constraints to encode the content of the cells of the tape. In that case, the content of a cell C_j will be encoded using a single clock x_j. Cell C_j contains an a whenever $x_j \leq 1$, whereas cell C_j contains a b whenever $x_j > 1$. We will ensure with an extra clock u that the encoding of each transition takes exactly one

[7] This rule reads as follows: from state q, if we read an α in the current cell of the tape, then we write a β onto the current cell, move the head of the tape to the right and go to state q'.

time unit. Hence, reading an a in cell C_j will consist in checking that $x_j = 1$ (because x_i will have been reset one time unit earlier), and reading a b in cell C_j will consist in checking that $x_j > 1$. Writing an a in cell C_j will consist in resetting clock x_j, whereas writing a b in cell C_j will consist in not resetting clock x_j (that way, at the next transition, the value of x_j will be at least 2, hence greater than 1). It remains to update all clocks x_i with $i \neq j$, this is done using a 0-delay sequence of transitions which updates one-by-one every clock x_i with $i \neq j$ following the rule 'if the value of x_i is 1, then reset x_i'. This rule preserves the encoding we have described before. We do not formalize the construction here, but better refer to [AL02].

5.2 The case of 'simply-timed' timed automata

The previous complexity result holds for timed automata with three clocks or more (the proof of three clocks is rather involved, though). For some simpler systems, for instance for systems with a single clock, this result can be improved. Of course, the same set of regions cannot be used, because even though the number of clocks is one, the number of regions given in Lemma 4 remains exponential, due to the binary encoding of constants in the timed automaton. However we can choose a smaller and rougher set of regions which yields the following result, due to [LMS04].

Proposition 5. *The reachability problem for single-clock timed automata is* NLOGSPACE*-complete.*

Proof. NLOGSPACE-hardness follows from that of reachability in finite graphs [HU79].

The NLOGSPACE membership can be obtained using a rougher set of regions than that presented in Section 4.3. Given a finite set $\mathcal{C}$ of constraints over a single clock x, we define the set of constants $C = \{c \in \mathbb{N} \mid \exists (x \bowtie c) \in \mathcal{C}\} \cup \{0\}$, and we assume that this set is ordered: $\mathcal{C} = \{c_0 < c_1 < c_2 < \cdots < c_p\}$. We define the partition $\mathcal{R}_\mathcal{C}$ as the (finite) set of intervals of one of the forms: (i) $\{c_i\}$ with $0 \leq i \leq p$, or (ii) (c_i, c_{i+1}) with $0 \leq i < p$, or (iii) $(c_p, +\infty)$. This is not hard to prove that $\mathcal{R}_\mathcal{C}$ is a set of regions for $\{x\}$ and $\mathcal{C}$. The size of $\mathcal{R}_\mathcal{C}$ is polynomial in the size of $\mathcal{C}$, which yields a polynomial-size region automaton, hence the expected result. ⌟

The case of two-clocks timed automata has also been considered in the literature [LMS04,Nav06], but the precise complexity of the reachability problem for this class of systems is still an open (and difficult) problem. The best known lower bound is NP while the best known upper bound is PSPACE.

Exercise 12. Prove the above NP lower bound. ⌟

5.3 Further applications of the region automaton abstraction

The region automaton abstraction is sound for verifying reachability properties. This is due to the time-abstract bisimulation property which links the region automaton and the original timed automaton. As a consequence, the region automaton abstraction can be

used to verify all properties that are invariant by time-abstract bisimulation. This is for instance the case of safety properties, of ω-regular properties, or of untimed properties expressed in LTL [Pnu77] or in CTL [CE81]. However, this construction cannot be directly used to verify properties expressed in a timed temporal logic like TCTL [ACD90,ACD93] because a property like "reaching a state in exactly 5 units of time" is not invariant by time-abstract bisimulation. For these properties a refined construction is required, which uses an extra clock for the formula, and then construct a region automaton taking into account this additional clock. We do not develop this construction here but better refer to original articles on the subject [ACD90,ACD93].

Exercise 13. In the above we did not speak much about infinite behaviours. However the region automaton can be used to detect infinite behaviours satisfying (for instance) a Büchi condition. To any infinite behaviour of the automaton corresponds an infinite path in the region automaton (see the proof of Proposition 2). What can you say about this infinite path in case only Zeno infinite behaviours can be read?[8] ⌟

5.4 Can we extend timed automata and preserve the decidability?

Exercise 14. Prove that for any timed automaton $\mathcal{A}$, we can construct a diagonal-free timed automaton $\mathcal{B}$ that recognizes the same timed language. What is the size of $\mathcal{B}$? Do you think we can avoid that blowup? ⌟

Exercise 15. Prove that reachability is undecidable for timed automata which allow:

① incremental updates $x := x + 1$ and general clock constraints;
② decremental updates $x := x - 1$ and diagonal-free clock constraints. ⌟

Exercise 16. If we add linear constraints to the model, *i.e.*, constraints of the form $\sum_{x \in X} \alpha_x x \sim c$ with $\alpha_x \in \mathbb{Z}$ for every $x \in X$ and $c \in \mathbb{Z}$, prove that the reachability problem becomes undecidable. ⌟

Exercise 17. What happens if we consider the set of clocks $X = \{x, y\}$, and if we add clock constraints of the form $x + y \sim c$ to the model? Is the reachability problem still decidable? ⌟

6 The language-theoretic perspective

In the previous section we have presented the region automaton abstraction, which can be used to model-check several kinds of simple properties, like reachability properties. From a language perspective, this means that the emptiness problem is decidable for timed automata. In this section we study further language-theoretic properties of timed languages accepted by timed automata, and show in particular some negative results.

[8] An infinite behaviour is said Zeno whenever it is time-bounded (in particular infinitely many actions are made in bounded time).

6.1 Boolean operations

Closure under Boolean operations is a basic property which is interesting for modelling and verification reasons.

Proposition 6. *The class of timed languages accepted by timed automata is closed under finite union and finite intersection.*

Proof (Sketch of proof). Closure under finite union is rather straightforward by taking the disjoint union of all timed automata.

The closure under finite intersection follows the lines of the standard product construction used in the case of finite automata. Only clock constraints, invariants and resets of clocks need be carefully handled. We illustrate the general construction with the intersection of two timed automata $\mathcal{A}_1 = (L_1, L_{1,0}, L_{1,F}, X_1, \Sigma, T_1, \mathsf{Inv}_1)$ and $\mathcal{A}_2 = (L_2, L_{2,0}, L_{2,F}, X_2, \Sigma, T_2, \mathsf{Inv}_2)$ over a single alphabet Σ. We assume that the two sets of clocks X_1 and X_2 are disjoint (otherwise we rename clocks so that it is actually the case). Then we define the timed automaton $\mathcal{A} = (L, L_0, L_F, X, \Sigma, T, \mathsf{Inv})$ by:

- $L = L_1 \times L_2$, $L_0 = L_{1,0} \times L_{2,0}$, $L_F = L_{1,F} \times L_{2,F}$;
- $X = X_1 \cup X_2$ (disjoint union);
- the set T is composed of transitions of the form $(\ell_1, \ell_2) \xrightarrow{g,a,Y} (\ell'_1, \ell'_2)$ whenever there exist two transitions $\ell_1 \xrightarrow{g_1,a,Y_1} \ell'_1$ in T_1 and $\ell_2 \xrightarrow{g_2,a,Y_2} \ell'_2$ in T_2 such that:
  - $g = g_1 \wedge g_2$;
  - $Y = Y_1 \cup Y_2$;
- $\mathsf{Inv}((\ell_1, \ell_2)) = \mathsf{Inv}_1(\ell_1) \wedge \mathsf{Inv}_2(\ell_2)$.

This is straightforward to prove that a timed word is accepted by $\mathcal{A}$ iff it is both accepted by $\mathcal{A}_1$ and $\mathcal{A}_2$.

The following proposition is on the contrary rather bad news.

Proposition 7. *The class of timed languages accepted by timed automata is not closed under complementation.*

The most well-known timed automaton, already given in [AD94], which cannot be complemented is given in Figure 6. This automaton, over the single-letter alphabet $\{a\}$,

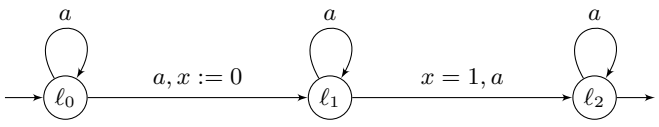


Fig. 6: *A non-complementable timed automaton*

recognizes the timed language:

$$\{(a,t_1)(a,t_2)\dots(a,t_n) \mid n \in \mathbb{N},\ n \geq 2 \text{ and there exist } 1 \leq i < j \leq n \text{ with } t_j - t_i = 1\}$$

Intuitively, to be recognized by a timed automaton, the complement of this timed language would require an unbounded number of clocks, because for any action a, we need to check that there is no a-action one time unit later, so a fresh clock is intuitively required. However the complete proof is rather technical and harassing [Bou98], and we do not provide it here.

An alternative and elegant proof of the above proposition has been proposed in [AM04], and this is the one we have decided to present here.

Proof. We consider the timed automaton of Figure 7. It accepts the following timed lan-

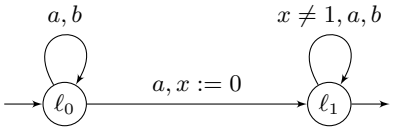


Fig. 7: *Another non-complementable timed automaton*

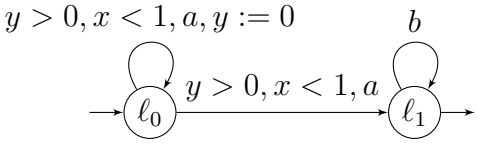
guage over the alphabet $\{a, b\}$:

$$\{(\alpha_1, t_1) \dots (\alpha_n, t_n) \mid n \in \mathbb{N},\ n \geq 1, \exists 1 \leq i \leq n \text{ s.t. } \alpha_i = a \text{ and } \forall i < j \leq n,\ t_j - t_i \neq 1\}$$

We assume towards a contradiction that $\overline{L}$ (the complement of L) can be recognized by a timed automaton. It is not hard to get convinced that the timed language over the alphabet $\{a, b\}$

$$L' = \{(a^+b^*, \tau) \mid \text{all } a's \text{ happen before 1 and no two } a's \text{ simultaneously}\}$$

is accepted by the timed automaton:



Hence by Proposition 6, $\overline{L} \cap L'$ is accepted by some timed automaton. The following lemma is just a matter of expanding and manipulating the definition of $\overline{L} \cap L'$.

Lemma 6. *The untiming of $\overline{L} \cap L'$ is the non-regular language*

$$\{a^n b^m \mid n \in \mathbb{N},\ n \geq 1 \text{ and } m \geq n\}.$$

This lemma yields a contradiction with the fact that $\overline{L} \cap L'$ is accepted by some timed automaton, say $\mathcal{B}$, because the untiming of $\overline{L} \cap L'$ should then be recognized by the region automaton of $\mathcal{B}$. Hence we conclude that the complement of L is not recognized by any timed automaton.

6.2 The universality and inclusion problems

The *universality problem* asks, given a timed automaton $\mathcal{A}$, whether $\mathcal{A}$ accepts all (finite) timed words. The *inclusion problem* asks, given two timed automata $\mathcal{A}$ and $\mathcal{B}$, whether all timed words accepted by $\mathcal{B}$ are also accepted by $\mathcal{A}$, that is whether $L(\mathcal{B}) \subseteq L(\mathcal{A})$. Note that the universality problem is a special instance of the inclusion problem, where $\mathcal{B}$ is universal, *i.e.* accepts all (finite) timed words. The following result is bad news in the verification context, as argued in Section 3.

Theorem 3 ([AD90,AD94]). *The universality problem is undecidable for timed automata.*

Proof. We encode the halting problem for a two-counter machine as a universality problem of a timed automaton.

Let $\mathcal{M}$ be a deterministic two-counter machine. We assume Q is the set of states of $\mathcal{M}$. A configuration of $\mathcal{M}$ is a triple (q, c, d) where $q \in Q$ is the current state, $c \in \mathbb{N}$ is the value of the first counter, and $d \in \mathbb{N}$ is the value of the second counter. We encode a finite execution $(q_0, c_0, d_0) \to (q_1, c_1, d_1) \to \ldots \to (q_n, c_n, d_n)$ of $\mathcal{M}$ as a finite timed word w over the alphabet $\Sigma = Q \cup \{c, d\}$ such that:

1. $\mathsf{Untime}(w) = q_0.c^{c_0}.d^{d_0}.q_1.c^{c_1}.d^{d_1} \ldots q_n.c^{c_n}.d^{d_n}$;
2. no two events happen at the same date, and we write

$$t_0 < \tau^1_{c,0} < \ldots < \tau^{c_0}_{c,0} < \tau^1_{d,0} < \ldots < \tau^{d_0}_{d,0} < t_1 < \ldots < t_n < \tau^1_{c,n} < \ldots < \tau^{c_n}_{c,n} < \tau^1_{d,n} < \ldots < \tau^{d_n}_{d,n}$$

   for the corresponding increasing sequence of dates;
3. event q_i happens at time i ($t_i = i$);
4. states faithfully follow instructions of $\mathcal{M}$:
   - if the instruction starting in $q_i = q$ is of the form
     “ q: `if` $c = 0$ `then goto` q' `else` $c := c - 1$; `goto` q'' ”
     - if $c_i = 0$, then $q_{i+1} = q'$
     - if $c_i > 0$, then $q_{i+1} = q''$
   - if the instruction starting in $q_i = q$ is of the form
     “ q: $c := c + 1$; `goto` q' ”
     then $q_{i+1} = q'$;
5. depending on the nature of the instruction starting in location q_i, we distinguish several rules:
   (i) if the instruction does not change the first counter (this is for instance the case when this is the second counter which is updated), then $c_{i+1} = c_i$ and for all $1 \leq j \leq c_i$, $\tau^j_{c,i+1} = 1 + \tau^j_{c,i}$
   (ii) if the instruction increments the first counter, then $c_{i+1} = c_i + 1$ and for all $1 \leq j \leq c_i$, $\tau^j_{c,i+1} = 1 + \tau^j_{c,i}$
   (iii) if the instruction checks that the value of the first counter is positive and then decrements it, then $c_{i+1} = c_i - 1$ and for all $1 \leq j \leq c_i - 1$, $\tau^j_{c,i+1} = 1 + \tau^j_{c,i}$

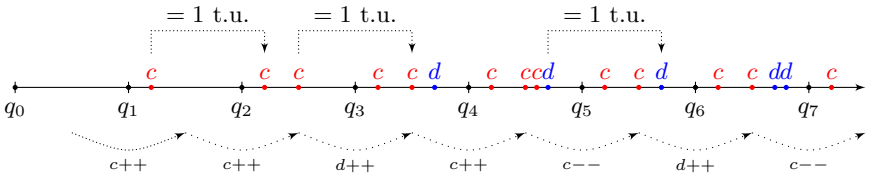


Fig. 8: *Encoding of the two-counter machine as a timed word*

Similarly for the second counter.

The constraints are illustrated on Figure 8. We use $c++$ (resp. $c--$) as macros to denote an incrementation (resp. decrementation) of counter c.

We construct a timed automaton which will accept all timed words over the alphabet Σ that **are not** encodings of (finite) halting executions of $\mathcal{M}$. This automaton will be some kind of test automaton [ABBL03] for those timed words. This automaton will be 'highly' non-deterministic and will deny one-by-one the conditions for a timed word to be an encoding of a halting execution of $\mathcal{M}$ (an execution as above is halting whenever it starts in the initial state and ends in the halting state of $\mathcal{M}$).

The first condition for being an encoding of an execution of $\mathcal{M}$, as well as the characteristics for being a halting execution, can be denied by a(n untimed) finite automaton. The second condition can be denied by a simple timed automaton which does two actions in 0-delay. The third and fourth conditions can also be denied by simple timed automata.

We now explain how we can deny condition 5.(i). A similar reasoning can be made for all the other conditions, we will omit it in these notes. Condition 5.(i) is made of two main constraints, and we deny each of them separately:

- It says that every c that happens within the interval $(i, i+1)$ must be followed one time unit later by another c. There are two ways of denying it:
  - either there is a c in $(i, i+1)$ which is not followed by any action one time unit later,
  - or there is a c in $(i, i+1)$ which is followed by some action one time unit later, but this is not a c as expected.

  A gadget to test for the above is given on Figure 9. Whenever the current instruction starts from state q, the automaton non-deterministically guesses a c (of the current configuration), such that either there is no action one time unit later (first branch), or there is an action, but it is not a c (second branch). On the picture, $\neg c$ means 'any action in Σ except c', and no action label on a transition means 'any action in Σ'.
- It says that every c that happens within $(i+1, i+2)$ must be preceded one time unit earlier by another c. As in the previous case, there are two ways of denying it:
  - either there is a c in $(i+1, i+2)$ which is not preceded by any action one time unit earlier,
  - or there is a c in $(i+1, i+2)$ which is preceded by an action one time unit earlier, but this is not a c.

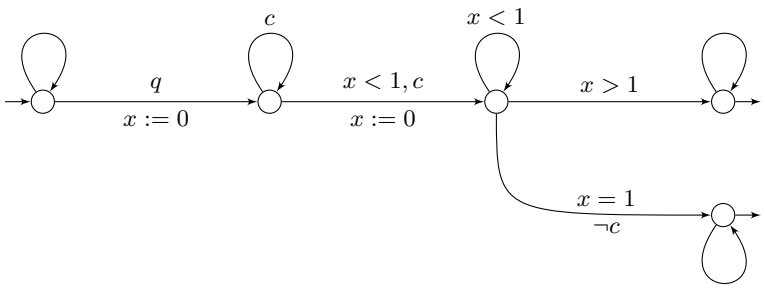


Fig. 9: *Denies that every c in* $(i, i+1)$ *is followed by a c* 1 *t.u. later*

The construction is illustrated on Figure 10 ($\neg Q$ means 'any action which does not belong to Q'). There are two possibilities: Either (first automaton on the picture) it non-deterministically guesses two consecutives letter within the interval $(i, i+1)$ (*i.e.*, in the part encoding the current configuration) that happen at dates say t and t' and checks that there is a c within the interval $(t+1, t'+1)$, hence not one time unit after another c (there is no action in the interval (t, t')). Or (second automaton on the picture) it non-deterministically guesses an action of the encoding of the first configuration which is not a c and checks that there is a c one time unit later.

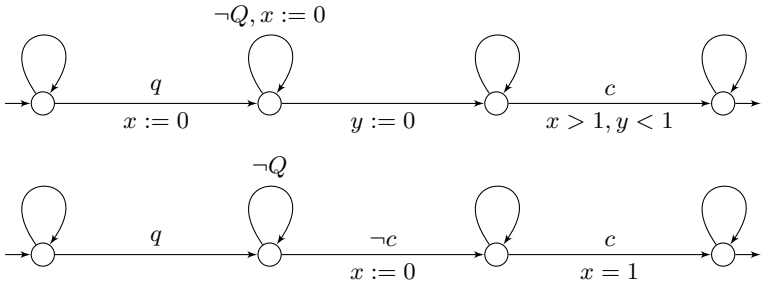


Fig. 10: *Denies that every c in* $(i+1, i+2)$ *is preceded* 1 *t.u. earlier by a c*

We omit all other cases, but the reader is invited to take one of the remaining conditions and to construct the corresponding gadget as an exercise.

The following is a straightforward corollary of the initial observation that the universality problem is a special instance of the inclusion problem.

Corollary 1. *The inclusion problem is undecidable for timed automata.*

It is interesting to notice that the reduction used in the above proof builds a timed automaton with two clocks. And actually, the universality problem (and also the inclusion problem) is decidable (but non-primitive recursive) for single-clock timed automata,

see [ADOW05]. Recent developments have considered alternating timed automata (a natural extension of timed automata with alternations) [LW05,OW05,LW08,OW07], but Theorem 3 implies that the emptiness problem is undecidable for alternating timed automata.

6.3 Timed automata and determinism

In the context of formal languages, determinism is a standard and central notion which expresses that for a word there is at most one execution which reads that word. For regular languages determinism does not restrict recognition of languages, but for context-free languages this is not the case [HMU01]. We discuss in this section the issue of determinism in the context of timed automata, which gives some explanation to the previous negative results.

The class of deterministic timed automata. We give a syntactical definition of determinism in timed automata (with no invariants, for simplicity). A timed automaton $\mathcal{A} = (L, L_0, L_F, X, \Sigma, T)$ is *deterministic* whenever L_0 is a singleton, and for every $\ell \in L$, for every $a \in \Sigma$, $(\ell, g_1, a, Y_1, \ell_1) \in T$ and $(\ell, g_2, a, Y_2, \ell_2) \in T$ imply $[\![g_1 \wedge g_2]\!]_X = \varnothing$. This notion extends in a natural way the standard notion of determinism in finite automata. In a deterministic timed automaton, for every timed word, there is at most one run that reads that timed word from a given configuration.

Example 9. The timed automaton of Figure 1 (see page 6) is deterministic. From location 'alarm', there are two outgoing transitions, but the constraints labelling those two transitions are disjoint. From the other locations, there is only one outgoing transition.

On the other hand, the timed automata of Figures 6 and 7 are not deterministic. In the first one, there is a non-deterministic choice from location ℓ_1, but it can be removed by strengthening the constraint on the self-loop (adding one self-loop with the constraint $x < 1$ and another one with the constraint $x > 1$). There is another non-determinisc choice from location ℓ_0, and this one cannot be removed (note that this is in general not obvious to see whether a non-deterministic choice can be removed or not!): it is not possible to predict when will be the occurrence of an a that will be followed one time unit later by another a.

Deterministic timed automata form a strict subclass of timed automata.[9] Using a product construction, as done in the proof of Proposition 6 for the intersection, it is easy to get convinced that this subclass is closed under finite union and finite intersection. On the other hand the two timed automata we have given to illustrate the non-closure under complementation of the class of standard timed automata (Proposition 7) are not deterministic. And actually it is not very hard to get convinced that the class of deterministic timed automata is closed under complementation: add a sink location, add transitions to that sink from every location, with constraints complementing the union of all the constraints

[9] The strictness is obvious at the syntactical level, and also holds at the semantical level, as will be argued later: there exists a timed automaton such that no deterministic timed automata accepts the same timed language.

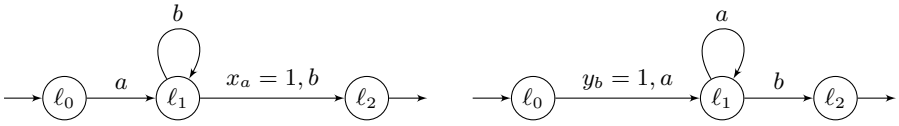
labelling the outgoing transitions from that location, and finally swap final and non-final locations. As a consequence, this is not possible to construct deterministic timed automata which accept the same languages as the two timed automata of Figures 6 and 7. And it is even possible to prove that this is not possible to decide whether a timed automaton is determinizable or not [Tri03,Fin06].

Finally it is interesting to mention that the reduction to prove the undecidability of the universality problem (proof of Theorem 3) builds a non deterministic timed automaton. And indeed the universality problem (and the inclusion problem) is decidable for the class of deterministic timed automata: to check for the universality of a given deterministic timed automaton $\mathcal{A}$, first build a (deterministic) timed automaton which accepts the complement of $L(\mathcal{A})$, and then check for emptiness of this automaton.

Determinizable classes of timed automata. As mentioned in the previous paragraph, not all timed automata can be determinized (*i.e.* there exist timed automata that accept timed languages which cannot be recognized by any deterministic timed automaton). However, deterministic (and hence effectively determinizable) timed automata enjoy nice closure (complementation) and decidability (universality, inclusion) properties. Verification can thus benefit of such properties.

One of the first determinizable classes of timed automata which have been investigated is the class of *event-clock timed automata* [AFH94]. In such an automaton every letter of the alphabet is associated two clocks, one which measures delays since the last occurrence of this action (those are called event-recording clock), and one which measures delays to the next occurrence of this action (those are called event-predicting clock). In the syntax of event-clock timed automata. resets of clocks are ommitted as they are implicitely given by actions.

Example 10. In Figure 11 we give two event-clock timed automata (we take the convention that x_a is the event-recording clock associated with a whereas y_b is the event-predicting clock associated with b). In the first automaton, the time between the last b and the unique



x_a: event-recording clock for a y_b: event-predicting clock for b

Fig. 11: *Two event-clock timed automata*

initial a is precisely one time unit (specified with the constraint on the last transition $x_a = 1$): when we do the last b, we know that the last a was precisely one time unit earlier. In the second automaton, the time between the first a and the unique final b is precisely

one time unit as well (specified with the constraint on the first transition $y_b = 1$): when the first a is done, we know that the next b has to be one time unit later.

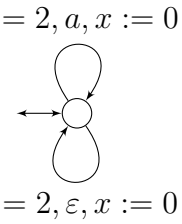
We give the intuition why an event-clock timed automaton with only event-recording clocks can be determinized (the case of event-predicting clocks is more involved and we refer the reader to [AFH94] for more details). The reason is that the timed behaviour of those automata is input-determined: given a timed word, the value of the clocks after each prefix of the timed word is determined by that prefix (and not by the run followed in the timed automaton). For that reason a subset construction can be done. This kind of arguments has later been used for more complex classes of timed systems [DT04].

Recently more determinizable classes of timed automata have been investigated [BBBB09], among which we can find the class of so-called strongly non-Zeno timed automata (we omit the definition of this class here, but basically it enforces time elapsing in a rather strong way) or more dedicated classes, e.g. corresponding to logical formalisms [NP10] or to simpler classes of timed systems [SP09].

6.4 What about ε-transitions?

Following classical automata theory, we assume some transitions are silent, and we denote them ε-transitions.

Exercise 18. Consider the following timed automaton.



Prove that the timed language recognized by the above timed automaton cannot be recognized by any classical timed automaton with no ε-transitions. ⌟

7 Implementation and data structures

In the original work by Alur and Dill, the so-called region automaton construction is an abstraction which proves the decidability of the model. However, whereas well-suited for establishing decidability of problems related to timed automata, the region automaton is highly impractical from a tool implementation point-of-view. Instead, most real-time verification tools (like CMC[10] [LL98], Kronos[11] [DOTY96], and Uppaal[12] [BDL+06]) apply abstractions based on so-called zones, which in practice provide much coarser abstractions.

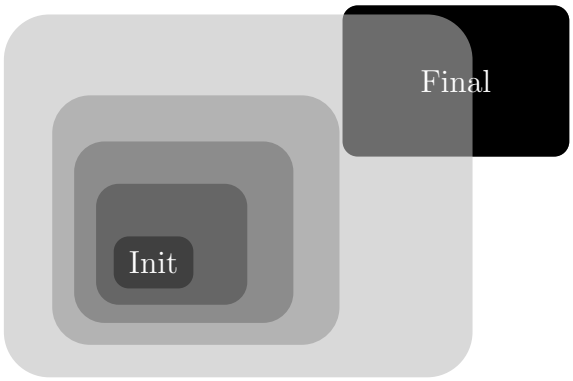
In this section, we describe methods that can be used for analysing reachability (or equivalently simple safety) properties in timed automata.

[10] http://www.lsv.ens-cachan.fr/ fl/cmcweb.html/
[11] http://www-verimag.imag.fr/TEMPORISE/kronos/
[12] http://www.uppaal.com/

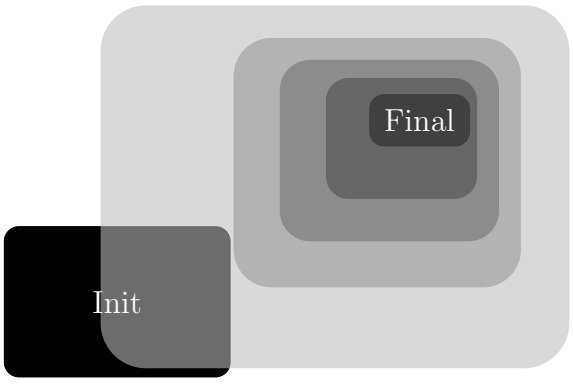
7.1 Checking reachability properties: two general methods

There are two main approaches for checking reachability (or safety) properties in systems (not only timed systems, but all kinds of systems). We describe these two approaches shortly and apply them to timed automata.

– *Forward analysis.* The general idea is to compute configurations which are reachable from the initial configuration within 1 step, 2 steps, *etc.* until final (or goal) configurations are computed, or until the computation terminates. The forward analysis computation can be schematized as below.



– *Backward analysis.* The general idea is to compute configurations from which we can reach final configurations within 1 step, 2 steps, *etc.* until the initial configuration is computed, or until the computation terminates. The backward analysis computation can be represented as below.



These two generic approaches are used in many contexts, including the analysis of models like counter machines, hybrid systems, *etc.* Of course, given a class of systems, specific techniques (*eg.* abstractions, widening operations, *etc.*) can be used for improving the computations. We will now focus on timed automata and explain how these two approaches can be implemented in that framework.

7.2 Reachability analysis in timed automata: the zone symbolic representation

Timed automata have infinitely (and even uncountably) many configurations, it is thus necessary to use symbolic representations for doing the computations. For the discussion which follows we fix a timed automaton $\mathcal{A} = (L, L_0, L_F, X, \Sigma, T)$. Given an edge $e = (\ell, g, a, Y, \ell') \in T$ of $\mathcal{A}$, we need to be able to compute its effect on a set of valuations. More precisely, if W is a set of valuations, we define the two following sets of valuations:

$$\begin{cases} \mathsf{Post}_e(W) = \{v' \in \mathbb{T}^X \mid \exists v \in W\ \exists t \in \mathbb{T} \text{ such that } v + t \models g \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{and } v' = [Y \leftarrow 0](v + t)\} \\ \mathsf{Pre}_e(W) \ = \{v \in \mathbb{T}^X \mid \exists v' \in W\ \exists t \in \mathbb{T} \text{ such that } v + t \models g \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{and } [Y \leftarrow 0](v + t) = v'\} \end{cases}$$

A valuation v' is in $\mathsf{Post}_e(W)$ whenever there exists some valuation $v \in W$ and some $t \in \mathbb{R}_{\geq 0}$ such that $(\ell, v) \xrightarrow{t,e} (\ell', v')$ is a mixed move in $\mathcal{T}_\mathcal{A}$. Similarly a valuation v is in $\mathsf{Pre}_e(W)$ whenever there exists some valuation $v' \in W$ and some $t \in \mathbb{R}_{\geq 0}$ such that $(\ell, v) \xrightarrow{t,e} (\ell', v')$ is a mixed move in $\mathcal{T}_\mathcal{A}$.

It is worth noticing that if W is a *zone*, *i.e.*, a set of valuations defined by a general clock constraint, then for every transition e of $\mathcal{A}$, $\mathsf{Post}_e(W)$ and $\mathsf{Pre}_e(W)$ are both zones. For analysing timed automata, zones are the most basic and commonly used *symbolic representation*.

In the following, we will need to decompose the computation of Post_e and Pre_e in several simpler steps, hence we define the following operations on zones (or more generally on sets of valuations):

- Future of W: $\overrightarrow{W} = \{v + t \mid v \in W \text{ and } t \in \mathbb{R}_{\geq 0}\}$
- Past of W: $\overleftarrow{W} = \{v - t \mid v \in W \text{ and } t \in \mathbb{R}_{\geq 0}\}$
- Intersection of W and W': $W \cap W' = \{v \mid v \in W \text{ and } v \in W'\}$
- Reset to zero of W with respect to the set of clocks Y:
$$[Y \leftarrow 0]W = \{[Y \leftarrow 0]v \mid v \in W\}$$
- Inverse reset to zero of W with respect to the set of clocks Y:
$$[Y \leftarrow 0]^{-1}W = \{v \mid [Y \leftarrow 0]v \in W\}$$

Exercise 19. Prove that those elementary operations preserve zones. ⌟

These operations allow to express the Post_e and Pre_e operators:

$$\begin{cases} \mathsf{Post}_e(W) = [Y_e \leftarrow 0](\overrightarrow{W} \cap [\![g_e]\!]) \\ \mathsf{Pre}_e(W) \ = \overleftarrow{[Y_e \leftarrow 0]^{-1}(W \cap [\![Y_e = 0]\!]) \cap [\![g_e]\!]} \end{cases}$$

7.3 The DBM data structure

The most common data structure for representing zones is the so-called DBM data structure. This data structure has been first introduced in [BM83] and then set in the framework of timed automata in [Dil90]. Several presentations of this data structure can be found in the literature, for example in [CGP99,Ben02,Bou04].

A *difference bound matrix*, we shall write DBM for short, for a set $X = \{x_1, \ldots, x_n\}$ of n clocks is an $(n+1)$-square matrix of pairs

$$(\prec, m) \in \mathbb{V} = (\{<, \leq\} \times \mathbb{Z}) \cup \{(<, \infty)\}.$$

A DBM $M = (\prec_{i,j}, m_{i,j})_{0 \leq i,j \leq n}$ defines the following subset of $\mathbb{R}^X_{\geq 0}$ (if $v \in \mathbb{R}^X_{\geq 0}$, $\overline{v}$ is the 'canonical' valuation over $X \cup \{x_0\}$ — where x_0 is a fresh clock — such that $\overline{v}(x) = v(x)$ for every $x \in X$, and $\overline{v}(x_0) = 0$; In the following we may write v instead of $\overline{v}$):

$$\{v : X \to \mathbb{R}_{\geq 0} \mid \forall\ 0 \leq i, j \leq n,\ \overline{v}(x_i) - \overline{v}(x_j) \prec_{i,j} m_{i,j}\}$$

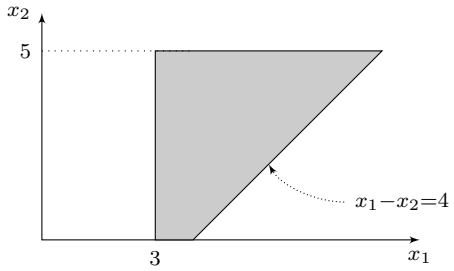
where $\gamma < \infty$ simply means that $\gamma \in \mathbb{R}_{\geq 0}$ (without any constraint on γ). This subset of $\mathbb{R}^X_{\geq 0}$ is a zone and will be denoted by $[\![M]\!]$. To simplify the notations, we now assume that all constraints are non-strict, so that coefficients of DBMs will simply be elements of $\mathbb{Z} \cup \{\infty\}$.

Example 11. We consider the zone over the set of clocks $X = \{x_1, x_2\}$ defined by the general clock constraint

$$(x_1 \geq 3)\ \wedge\ (x_2 \leq 5)\ \wedge\ (x_1 - x_2 \leq 4).$$

This zone, depicted on the next picture on the right, can be represented by the DBM on the left.

$$\begin{array}{c} \\ x_0 \\ x_1 \\ x_2 \end{array} \begin{array}{c} \begin{array}{ccc} x_0 & x_1 & x_2 \end{array} \\ \begin{pmatrix} \infty & -3 & \infty \\ \infty & \infty & 4 \\ 5 & \infty & \infty \end{pmatrix} \end{array}$$



⌟

A zone can have several representations using DBMs. For example, the zone of the previous example can equivalently be represented by the DBM

$$\begin{array}{c} \\ x_0 \\ x_1 \\ x_2 \end{array} \begin{array}{c} \begin{array}{ccc} x_0 & x_1 & x_2 \end{array} \\ \begin{pmatrix} 0 & -3 & 0 \\ 9 & 0 & 4 \\ 5 & 2 & 0 \end{pmatrix} \end{array}$$

Normal forms of DBMs. We define a total order on $\mathbb{V}$ in the following way: if $(\prec, m), (\prec', m') \in \mathbb{V}$, then

$$(\prec, m) \leq (\prec', m') \Leftrightarrow \begin{cases} m < m' \\ \text{or} \\ m = m' \text{ and either } \prec = \prec' \text{ or } \prec' = \leq . \end{cases}$$

Of course, for each $m \in \mathbb{Z}$, we have $m < \infty$. We define $>$, $\geq$ and $<$ in a natural way. These orders are extended to DBMs $M = (\prec_{i,j}, m_{i,j})_{i,j=0\ldots n}$ and $M' = (\prec'_{i,j}, m'_{i,j})_{i,j=0\ldots n}$ by

$$M \leq M' \Leftrightarrow \text{for every } i, j = 0 \ldots n,\ (\prec_{i,j}, m_{i,j}) \leq (\prec'_{i,j}, m'_{i,j}).$$

We also define an addition on the set $\mathbb{V}$ as follows:

$$(\prec, m) + (\prec', m') = (\prec'', m'')$$

where $m'' = m + m'$ and $\prec''$ is $\leq$ if both $\prec$ and $\prec'$ are $\leq$ and $\prec''$ is $<$ otherwise.

Let M be a DBM. We write G_M for the graph corresponding to that DBM. Then the following property follows:

Property 1 (Emptiness checking).

$$[\![M]\!] = \varnothing \text{ iff there is a cycle with weight strictly smaller than } (\leq, 0) \text{ in } G_M$$

Proof. The right-to-left implication is obvious.

We focus on the left-to-right implication. The proof relies on shortest paths properties [CLR90].

We first assume that all $\prec_{i,j}$ are $\leq$. In the graph G_M, there is a path from vertex x_0 to any other vertex. We can therefore define $\delta_(i)$ the shortest path in G_M from 0 to i (forgetting the comparison operator, since it is $\leq$). We have that for every j, $\delta(i) \leq \delta(j) + m_{j,i}$, that is $(-\delta(j)) - (-\delta(i)) \leq m_{j,i}$. Hence, $(-\delta(i))_i$ is a solution to the system of equations defined by M.

In the general case, we fix some $\epsilon_{i,j} \geq 0$ such that the following conditions hold:

- $\prec_{i,j} = <$ iff $\epsilon_{i,j} > 0$
- for every cycle $i_1 i_2 \ldots i_k$ of length at most $n + 1$,

$$\sum_{j=1}^{k-1} \left(m_{i_j, i_{j+1}} - \epsilon_{i_j, i_{j+1}} \right) \geq 0$$

This is possible to do so since there is no negative cycle in M. We define $M_\epsilon = ((m_{i,j} - \epsilon_{i,j}); \leq)_{i,j}$. Applying the previous result, we get that $[\![M_\epsilon]\!] \neq \varnothing$. Furthermore, $M_\epsilon \leq M$, which implies $[\![M]\!] \neq \varnothing$. ⌟

We apply the Floyd-Warshall algorithm to DBM M (seen as the adjacency matrix of G_M), yielding the DBM $\phi(M)$. The DBM $\phi(M)$ is the *normal form* of M, which will be justified later.

We can now state some (very useful) properties of normal forms of DBMs.

Property 2. If M and M' are DBMs, then:

(i) $[\![M]\!] = [\![\phi(M)]\!]$ and $\phi(M) \leq M$,
(ii) $[\![M]\!] \subseteq [\![M']\!] \Leftrightarrow \phi(M) \leq M' \Leftrightarrow \phi(M) \leq \phi(M')$.

The last point expresses the fact that the test for inclusion of zones can be checked syntactically on the normal forms of the DBMs (representing the zones).

Normal forms of DBMs can be characterized in a natural way.

Property 3. If $M = (\prec_{i,j}, m_{i,j})_{i,j=0\dots n}$ is a DBM such that $[\![M]\!] \neq \varnothing$, then the two following properties are equivalent:

(i) M is in normal form,
(ii) for every $i, j = 0 \dots n$, for every real $-m_{j,i} \prec_{j,i} r \prec_{i,j} m_{i,j}$, there exists a valuation $v \in [\![M]\!]$ such that $v(x_j) - v(x_i) = r$ (still assuming that $v(x_0) = 0$).

This property expresses the fact that if a DBM is in normal form, then no constraint of this DBM can be tightened using Floyd algorithm.

Computation of Some Operations on DBMs. As we argued at the beginning of the section, the data structure used to represent zones must also be appropriate to compute several oeprations.

Intersection. Let $M = (\prec_{i,j}, m_{i,j};)_{i,j=1\dots n}$ and $M' = (\prec'_{i,j}, m'_{i,j})_{i,j=1\dots n}$ be two DBMs and define $M'' = (\prec''_{i,j}, m''_{i,j})_{i,j=1\dots n}$ by

$$(\prec''_{i,j}, m''_{i,j}) = \min((\prec_{i,j}, m_{i,j}), (\prec'_{i,j}, m'_{i,j})) \text{ for all indices } i, j = 1 \dots n.$$

Then $[\![M'']\!] = [\![M]\!] \cap [\![M']\!]$. Note that it can be the case that M'' is not in normal form, even if M and M' are in normal form.

Future. Assume that $M = (\prec_{i,j}, m_{i,j})_{i,j=1\dots n}$ is a DBM in normal form. Define the DBM $\overrightarrow{M} = (\prec'_{i,j}, m'_{i,j})_{i,j=1\dots n}$ by:

$$\begin{cases} (\prec'_{i,j}, m'_{i,j}) = (\prec_{i,j}, m_{i,j}) & \text{if } j \neq 0 \\ (\prec'_{i,0}, m'_{i,0}) = (<, \infty) \end{cases}.$$

Then $[\![\overrightarrow{M}]\!] = \overrightarrow{[\![M]\!]}$ and the DBM $\overrightarrow{M}$ is in normal form.

Past. Assume that $M = (\prec_{i,j}, m_{i,j})_{i,j=1\dots n}$ is a DBM in normal form. Define the DBM $\overleftarrow{M} = (\prec'_{i,j}, m'_{i,j})_{i,j=1\dots n}$ by:

$$\begin{cases} (\prec'_{i,j}, m'_{i,j}) = (\prec_{i,j}, m_{i,j}) & \text{if } i \neq 0 \\ (\prec'_{0,j}, m'_{0,j}) = (\leq, 0) \end{cases}.$$

Then $[\![\overleftarrow{M}]\!] = \overleftarrow{[\![M]\!]}$ and the DBM $\overleftarrow{M}$ is in normal form.

Image by resets. Assume that $M = (\prec_{i,j}, m_{i,j})_{i,j=1\ldots n}$ is a DBM in normal form and define the DBM $M_{x_k:=0} = (\prec'_{i,j}, m'_{i,j})_{i,j=1\ldots n}$ by:

$$\begin{cases} (\prec'_{i,j}, m'_{i,j}) = (\prec_{i,j}, m_{i,j}) & \text{if } i,j \neq k \\ (\prec'_{k,k}, m'_{k,k}) = (\prec'_{k,0}, m'_{k,0}) = (\prec'_{0,k}, m'_{0,k}) = (\leq, 0) & \\ (\prec'_{i,k}, m'_{i,k}) = (\prec_{i,0}, m_{i,0}) & \text{if } i \neq k \\ (\prec'_{k,i}, m'_{k,i}) = (\prec_{0,i}, m_{0,i}) & \text{if } i \neq k \end{cases}$$

Then $[\![M_{x_k:=0}]\!] = [x_k \leftarrow 0][\![M]\!]$ and the DBM $M_{x_k:=0}$ is in normal form.

Let us just mention that the DBM data structure is the most basic data structure which is used for analysing timed systems, some more involved BDD-like data structures can also be used, for example CDDs (which stands for 'Clock Difference Diagrams') [LPWY99], or more recently federations [DHGP04,Dav05].

7.4 Backward analysis

We first focus on the backward analysis computation, which will surprisingly turn out to be the simplest to analyse. We fix a timed automaton $\mathcal{A} = (L, L_0, L_F, X, \Sigma, T)$.

Backward symbolic transition system. The *backward symbolic transition system* associated with $\mathcal{A}$ is denoted by '$\Leftarrow$' and is defined inductively as follows:

$$\frac{e = \left(\ell_1 \xrightarrow{g,a,Y} \ell_2\right) \in E \qquad W_1 = \mathsf{Pre}_e(W_2)}{(\ell_2, W_2) \Leftarrow (\ell_1, W_1)}$$

Obviously, if we write $\Leftarrow^*$ for the reflexive and transitive closure of $\Leftarrow$, we have that $(\ell', W') \Leftarrow^* (\ell, W)$ if and only if for every $v \in W$, there exists $v' \in W'$ and a run in $\mathcal{A}$ from (ℓ, v) to (ℓ', v').

The backward analysis algorithm then consists in computing iteratively the following sets of symbolic configurations:

$$\begin{aligned} \mathcal{S}^{\mathsf{b}}_0 &= \{(\ell, \mathbb{R}^X_{\geq 0}) \mid \ell \in L_F\} \\ \mathcal{S}^{\mathsf{b}}_1 &= \mathcal{S}^{\mathsf{b}}_0 \cup \{(\ell, W) \mid \exists (\ell', W') \in \mathcal{S}^{\mathsf{b}}_0 \text{ such that } (\ell', W') \Leftarrow (\ell, W)\} \\ &\vdots \\ \mathcal{S}^{\mathsf{b}}_{p+1} &= \mathcal{S}^{\mathsf{b}}_p \cup \{(\ell, W) \mid \exists (\ell', W') \in \mathcal{S}^{\mathsf{b}}_p \text{ such that } (\ell', W') \Leftarrow (\ell', W')\} \\ &\vdots \end{aligned}$$

until either (i) the computation stabilizes, or (ii) a symbolic state is computed, which contains the initial configuration of $\mathcal{A}$. To help event (i) happen, it is possible to add the following inclusion check: if $(\ell, W) \in \mathcal{S}^{\mathsf{b}}_{p+1}$ and if there exists $(\ell, W') \in \mathcal{S}^{\mathsf{b}}_p$ such that $W \subseteq W'$ (or even if there exist $(\ell, W_i) \in \mathcal{S}^{\mathsf{b}}_p$ for finitely many i's such that $W \subseteq \bigcup_i W_i$), then do not include (ℓ, W) in $\mathcal{S}^{\mathsf{b}}_{p+1}$. The procedure answers 'Yes' in case (ii) and 'No' in case $(i) \wedge \neg(ii)$.

Termination and correctness. The backward analysis computation enjoys the following nice property, which can be seen as a consequence of the correctness of the backward analysis algorithm for TCTL [HNSY94]. However, we will give below a simple and direct proof of that result.

Theorem 4. *The backward computation terminates and is correct with respect to reachability properties.*[13]

Proof. Correctness is immediate as the computation is *exact* (as opposed to over-(or under-)approximate). The inclusion check does not cause any trouble as any state reachable from a configuration belonging to a symbolic state that is not added due to that test is actually already reachable from the previous symbolic state that was computed.

Termination needs some additional argument, that we sketch here. Assume that R_i's (for $1 \leq i \leq p$) are regions of $\mathcal{A}$, then:

- $\overleftarrow{\bigcup_{i=1}^{p}} R_i$ is a finite union of regions;
- $[Y \leftarrow 0]^{-1}\left(\bigcup_{i=1}^{p} R_i\right)$ is a finite union of regions (for any set of clocks Y);
- $g \cap \left(\bigcup_{i=1}^{p} R_i\right)$ is a finite union of regions if g is a clock constraint of $\mathcal{A}$.

These properties altogether imply that each of the symbolic configurations (ℓ, W) that are added to $\mathcal{S}_i^{\mathsf{b}}$ is such that W is a finite union of regions. As there are finitely many regions, the sequence $(\mathcal{S}_i^{\mathsf{b}})_{i\geq 0}$ stabilizes, hence the termination of the backward computation. ⌟

Backward analysis may appear as an accurate method for analysing timed automata, but in practice, some tools (like Uppaal) prefer using a forward analysis computation. One of the reasons comes from the use of (bounded) integer variables that are really helpful for modelling real systems. Backward analysis is then not suitable for dealing with arithmetical operations: for example if we know in which interval lies the variable i and if we know that i is assigned the value $j.k + \ell.m$, it is not easy to compute the possible values of variables j, k, ℓ, m (apart from listing all possible tuples of values). For this kind of operations, forward analysis is much more suitable.

7.5 Forward analysis

In this section we focus on the forward analysis computation, which will require the development of abstractions, and more effort for proving its correctness. We fix a timed automaton $\mathcal{A} = (L, L_0, L_F, X, \Sigma, T)$.

[13] *I.e.* L_F is reachable if and only if it is declared as reachable by the backward computation.

Forward symbolic transition system. The *forward symbolic transition system* associated with $\mathcal{A}$ is denoted by '$\Rightarrow$' and is defined inductively as follows:

$$\frac{e = \left(\ell_1 \xrightarrow{g,a,Y:=0} \ell_2\right) \in E \qquad W_2 = \mathsf{Post}_e(W_1)}{(\ell_1, W_1) \Rightarrow (\ell_2, W_2)}$$

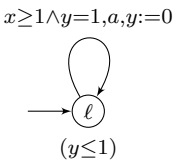
Obviously, if we write $\Rightarrow^*$ for the reflexive and transitive closure of $\Rightarrow$, we have that $(\ell, W) \Rightarrow^* (\ell', W')$ if and only if for every $v' \in W'$, there exists $v \in W$ and a run in $\mathcal{A}$ from (ℓ, v) to (ℓ', v').

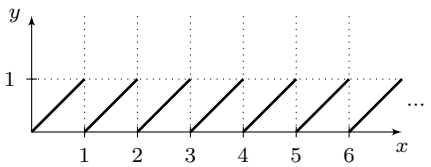
The forward analysis computation then consists in computing iteratively the following sets of symbolic configurations:

$$\begin{aligned}
\mathcal{S}^{\mathsf{f}}_0 &= \{(\ell_0, \mathbf{0}_X)\} \\
\mathcal{S}^{\mathsf{f}}_1 &= \mathcal{S}^{\mathsf{f}}_0 \cup \{(\ell', W') \mid \exists (\ell, W) \in \mathcal{S}^{\mathsf{f}}_0 \text{ such that } (\ell, W) \Rightarrow (\ell', W')\} \\
&\vdots \\
\mathcal{S}^{\mathsf{f}}_{p+1} &= \mathcal{S}^{\mathsf{f}}_p \cup \{(\ell', W') \mid \exists (\ell, W) \in \mathcal{S}^{\mathsf{f}}_p \text{ such that } (\ell, W) \Rightarrow (\ell', W')\} \\
&\vdots
\end{aligned}$$

until either (i) the computation stabilizes, or (ii) a symbolic state is computed, which contains a final configuration of the timed automaton (*i.e.*, a configuration of the form (f, v) with $f \in F$). To help event (i) happen, it is possible to add an inclusion check, as in the backward analysis computation. The procedure answers 'Yes' in case (ii) and 'No' in case $(i) \wedge \neg(ii)$.

Discussion on the termination and correctness. The forward analysis gives a correct answer, but it may not terminate. An example of automaton in which the forward computation does not terminate is given below. The zones that are computed by the above procedure are represented on the right part of the figure, and it is easy to check that the computation will never terminate.





To overcome this problem, it is necessary to use some abstraction operators. Several have been proposed in [DT98]: for instance, if Z and Z' are computed for the location ℓ, they are replaced by the smallest zone containing both Z and Z'; this approximation is called the *convex-hull* abstraction[14], it does not ensure termination and is only semi-correct, in

[14] It is a language abuse, because it is not really the convex hull of the two zones, but it is the smallest zone containing the convex-hull of the two zones.

the sense that a location announced as reachable might not be reachable (the convex-hull abstraction is an over-approximation). The most interesting abstraction studied in this paper is the *extrapolation* operator. We will present it now, but we first need to formalize a little more the forward analysis procedure. We follow the lines of [BBFL03,BBLP04] and define (abstract) symbolic transition systems.

Abstract forward symbolic transition systems. Let $\mathfrak{a}$ be an abstraction operator (possibly partially) defined on the sets of valuations ($\mathfrak{a}$ associates to sets of valuations sets of valuations). We define the *abstract forward symbolic transition system* '$\Rightarrow_{\mathfrak{a}}$' in the following way:

$$\frac{(\ell, W) \Rightarrow (\ell', W') \qquad W = \mathfrak{a}(W)}{(\ell, W) \Rightarrow_{\mathfrak{a}} (\ell', \mathfrak{a}(W'))}$$

This transition system gives naturally rise to the following forward computation in $\mathcal{A}$.

$$\begin{aligned}
\mathcal{S}_0^{\mathsf{f},\mathfrak{a}} &= \{(\ell_0, \mathfrak{a}(\{\mathbf{0}_X\}))\} \\
\mathcal{S}_1^{\mathsf{f},\mathfrak{a}} &= \mathcal{S}_0^{\mathsf{f},\mathfrak{a}} \cup \{(\ell', W') \mid \exists (\ell, W) \in \mathcal{S}_0^{\mathsf{f},\mathfrak{a}} \text{ such that } (\ell, W) \Rightarrow_{\mathfrak{a}} (\ell', W')\} \\
&\vdots \\
\mathcal{S}_{p+1}^{\mathsf{f},\mathfrak{a}} &= \mathcal{S}_p^{\mathsf{f},\mathfrak{a}} \cup \{(\ell', W') \mid \exists (\ell, W) \in \mathcal{S}_p^{\mathsf{f},\mathfrak{a}} \text{ such that } (\ell, W) \Rightarrow_{\mathfrak{a}} (\ell', W')\} \\
&\vdots
\end{aligned}$$

with the same halting conditions (and inclusion checks) as previously.

Soundness criteria. The abstraction operator $\mathfrak{a}$ is said *correct* with respect to reachability properties in $\mathcal{A}$ whenever the following holds:

$$\begin{gathered}
\text{if } (\ell_0, \mathfrak{a}(\{\mathbf{0}_X\})) \Rightarrow_{\mathfrak{a}}^* (\ell, W) \text{ then there exists a run} \\
(\ell_0, \mathbf{0}_X) \rightarrow^* (\ell, v) \text{ with } v \in W \text{ in } \mathcal{A}
\end{gathered}$$

The abstraction operator $\mathfrak{a}$ is said *complete* with respect to reachability properties whenever the following holds in $\mathcal{A}$:

$$\begin{gathered}
\text{if } (\ell_0, \mathbf{0}_X) \rightarrow^* (\ell, v) \text{ is a run in } \mathcal{A} \text{ then} \\
(\ell_0, \mathfrak{a}(\{\mathbf{0}_X\})) \Rightarrow_{\mathfrak{a}}^* (\ell, W) \text{ for some } W \text{ with } v \in W
\end{gathered}$$

Remark 9. Note that these two notions could be generalized to more general properties than reachability properties, but we follow our lines and concentrate on reachability properties. ⌟

Our aim is to define abstraction operators $\mathfrak{a}$ such that the four following properties hold:

(Finiteness) $\{\mathfrak{a}(W) \mid \mathfrak{a} \text{ defined on } W\}$ is finite
(this ensures termination of the "abstract" forward computation)
(Correctness) $\mathfrak{a}$ is correct with respect to reachability

(Completeness) $\mathfrak{a}$ is complete with respect to reachability
(Effectiveness) $\mathfrak{a}$ is "effective"

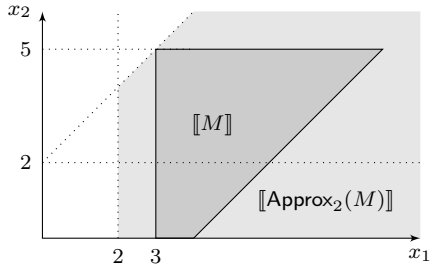
The three first properties are properly defined, the last one is more informal. The effectiveness criterion expresses that the abstraction has to be easily computable. In timed automata literature this is most of the time interpreted as "$\mathfrak{a}$ has to be defined for all zones and $\mathfrak{a}(Z)$ has to be a zone when Z is a zone". Note that other effectiveness criteria could be proposed, but that is the one we choose here.

The extrapolation operator. The abstraction operator which is commonly used is called *extrapolation*, and sometimes *normalization* [Ben02] or *approximation* [Bou04]. We will note it here Approx_K, it is defined up to a constant K as follows: if Z is a zone, $\mathsf{Approx}_K(Z)$ is the smallest K-bounded zone[15] which contains Z. This operation is well-defined on DBMs: if M is a DBM in normal form representing Z, a DBM representing $\mathsf{Approx}_K(Z)$ is obtained from M where each coefficient $(\prec; m)$ with $m < -K$ is replaced by $(<; -K)$ and all coefficients $(\prec; m)$ with $m > K$ is replaced by $(<; \infty)$, all other coefficients are unchanged. We write $\mathsf{Approx}_K(M)$ for this transformed DBM: it holds that $[\![\mathsf{Approx}_K(M)]\!] = \mathsf{Approx}_K([\![M]\!])$.

Example 12. Consider again the zone introduced in example 11. As we have already mentioned, it can be represented by the DBM in normal form on the left and its 2-extrapolation is the DBM on the right (where we again do not mention the comparison operators):

$$M = \begin{pmatrix} 0 & -3 & 0 \\ 9 & 0 & 4 \\ 5 & 2 & 0 \end{pmatrix} \quad \text{and} \quad \mathsf{Approx}_2(M) = \begin{pmatrix} 0 & -2 & 0 \\ +\infty & 0 & +\infty \\ +\infty & 2 & 0 \end{pmatrix}$$

They are both represented on the picture below.



Obviously, ⌟

- Approx_K is a finite abstraction operator because there are finitely many DBMs whose coefficients are either $(<; \infty)$ or some $(\prec; m)$ with $\prec \in \{<; \leq\}$ and $-K \leq m \leq K$;
- the computation of Approx_K is effective and can easily be done using DBMs;
- Approx_K is a complete abstraction with respect to reachability because for every zone Z, $Z \subseteq \mathsf{Approx}_K(Z)$.

[15] A *K-bounded zone* is a zone defined by a K-bounded clock constraint.

The only point that needs to be carefully studied is the correctness of Approx_K: of course, not all constants K yield correctness, but we have to choose such a constant K carefully, so that the abstraction operator be correct with respect to reachability properties. We now discuss in details this important aspect.

Correctness of the extrapolation operator. It is far from being obvious! We can state it as follows.

Theorem 5 ([Bou04]). *Let $\mathcal{A}$ be a **diagonal-free** timed automaton. Take K the maximal constant appearing in the constraints of $\mathcal{A}$. Then Approx_K is a correct abstraction with respect to reachability properties in $\mathcal{A}$.*

Proof. We prove the following property: for every region R (with maximal constant K), for every zone Z, $R \cap Z = \varnothing$ implies $R \cap \mathsf{Approx}_K(Z) = \varnothing$.

Let $R = ((I_{x_i})_{x_i \in X}; \prec)$ be a region and let $M_R = ((r_{i,j}; \prec_{i,j})_{i,j=0\ldots n})$ be a DBM representing R such that:

$$
\begin{cases}
(r_{i,0}; \prec_{i,0}) = \begin{cases} (c+1; <) & \text{if } I_{x_i} =]c; c+1[\\ (c; \leq) & \text{if } I_{x_i} = \{c\} \\ (+\infty; <) & \text{if } I_{x_i} =]K; +\infty[\end{cases} \\
(r_{0,i}; \prec_{0,i}) = \begin{cases} (-c; <) & \text{if } I_{x_i} =]c; c+1[\\ (-c; \leq) & \text{if } I_{x_i} = \{c\} \\ (-K; <) & \text{if } I_{x_i} =]K; +\infty[\end{cases} \\
(r_{i,j}; \prec_{i,j}) = \begin{cases} (c-d+1; <) & \text{if } I_{x_i} =]c; c+1[,\ I_{x_j} =]d; d+1[\text{ and } x_j \prec x_i,\ x_i \not\prec x_j \\ (c-d; <) & \text{if } I_{x_i} =]c; c+1[,\ I_{x_j} =]d; d+1[\text{ and } x_i \prec x_j,\ x_j \not\prec x_i \\ (c-d; \leq) & \text{if } I_{x_i} =]c; c+1[,\ I_{x_j} =]d; d+1[\text{ and } x_i \prec x_j,\ x_j \prec x_i \\ (c-d; \leq) & \text{if } I_{x_i} = \{c\} \text{ and } I_{x_j} = \{d\} \\ (c+1-d; <) & \text{if } I_{x_i} =]c; c+1[,\ I_{x_j} = \{d\} \\ (c-d-1; <) & \text{if } I_{x_i} = \{c\} \text{ and } I_{x_j} =]d; d+1[\\ (+\infty; <) & \text{in all other cases} \end{cases}
\end{cases}
$$

Note that the diagonal-free hypothesis has the following important consequence on the above representation of R: if $(r_{i,0}; \prec_{i,0}) = (+\infty; <)$, then for every $j \neq 0$, $(r_{i,j}; \prec_{i,j}) = (r_{j,i}; \prec_{j,i}) = (+\infty; <)$. This property will be the core of the proof.

Let Z be a zone and $M = ((m_{i,j}; <_{i,j})_{i,j})$ a DBM in normal form representing zone Z. Assume that $Z \cap R = \varnothing$. It means that there exists a sequence of distinct indices $(i_1, i_2, \ldots, i_l = i_1)$ such that

$$\alpha_{i_1,i_2} + \alpha_{i_2,i_3} + \cdots + \alpha_{i_{l-1},i_l} < (0; \leq) \tag{1}$$

where $\alpha_{i_j,i_k} = \min((m_{i_j,i_k}; <_{i_j,i_k}); (r_{i_j,i_k}; \prec_{i_j,i_k}))$. Since M is in normal form, we can assume that two successive $\alpha_{h,\ell}$ do not come from M, otherwise we could simplify the sum (1).

Assume that $(m_{h,\ell}, <_{h,\ell}) < (r_{h,\ell}, \prec_{h,\ell})$ for some h and some ℓ such that I_{x_h} and I_{x_ℓ} are bounded. We distinguish between two cases:

– $\prec_{h,\ell}{=}\leq$, which implies $r_{h,\ell} = -r_{\ell,h}$ and $\prec_{\ell,h}{=}\leq$ (by construction of M_R). In that case, $(m_{h,\ell}, <_{h,\ell}) \leq (r_{h,\ell}, <)$, and therefore $Z \subseteq (x_{i_h} - x_{i_\ell} < r_{h,\ell})$. This is K-bounded, hence $\mathsf{Approx}_K(Z) \subseteq (x_{i_h} - x_{i_\ell} < r_{h,\ell})$. In particular, as $R \subseteq (x_{i_h} - x_{i_\ell} = r_{h,\ell})$, $\mathsf{Approx}_K(Z) \cap R = \varnothing$.
– $\prec_{h,\ell}{=}<$, which implies that $-r_{h,\ell} = r_{\ell,h} - 1$ and $\prec_{\ell,h}{=}<$. Also, $m_{h,\ell} < r_{h,\ell}$, hence $m_{h,\ell} \leq r_{h,\ell} - 1 = -r_{\ell,h}$. This implies $(m_{h,\ell}, <_{h,\ell}) + (r_{\ell,h}, <) < (0, \leq)$: $Z \cap (x_{i_\ell} - x_{i_h} < r_{\ell,h}) = \varnothing$, that is $Z \subseteq (x_{i_\ell} - x_{i_h} \geq r_{\ell,h})$. This is K-bounded, hence $\mathsf{Approx}_K(Z) \subseteq (x_{i_\ell} - x_{i_h} \geq r_{\ell,h})$. In particular, as $R \subseteq (x_{i_\ell} - x_{i_h} < r_{\ell,h})$, $\mathsf{Approx}_K(Z) \cap R = \varnothing$.

Let us now assume that the above case does not happen: in the above sum, if $(m_{i_j,i_{j+1}}, <_{i_j,i_{j+1}}) < (r_{i_j,i_{j+1}}, \prec_{i_j,i_{j+1}})$, then either $I_{x_{i_j}} =]K, +\infty[$ or $I_{x_{i_{j+1}}} =]K, +\infty[$.

– Assume that $I_{x_{i_j}} =]K; +\infty[$. If $i_{j-1} \neq 0$, then by hypothesis, $(r_{i_{j-1},i_j}; \prec_{i_{j-1},i_j}) = (+\infty; <)$, which is not possible (it cannot appear in the sum). Therefore, $i_{j-1} = 0$.
– Assume that $I_{x_{i_{j+1}}} =]K; +\infty[$. By definition of M_R we get that every coefficient $r_{i_{j+1},*}$ is infinite, which is not possible in the sum. This case cannot happen.

After this analysis, we know that in the sum (1), we cannot have two coefficients coming from M (otherwise index 0 would appear several times, which is not allowed). Thus (since R is not empty) we have exactly one coefficient coming from M in the sum (1). Collecting all these informations, we obtain a sequence of indices $(k_1, \ldots, k_p)$ such that

$$(r_{0,k_1}; \prec_{0,k_1}) + (m_{k_1,k_2}; <_{k_1,k_2}) + (r_{k_2,k_3}; \prec_{k_2,k_3}) + \cdots + (r_{k_p,0}; \prec_{k_p,0}) < (0; \leq)$$

(and $I_{x_{k_1}} =]K; +\infty[$).

Now, we can notice that all coefficients $(r_{k_i,k_{i+1}}, \prec_{k_i,k_{i+1}})$ (with $2 \leq i < p$) of M_R are normalized (they cannot be tightened). Therefore, $(r_{k_2,k_3}; \prec_{k_2,k_3}) + \cdots + (r_{k_{p-1},k_p}; \prec_{k_{p-1},k_p}) \geq (r_{k_2,k_p}; \prec_{k_2,k_p})$. The sum can therefore be simplified into:

$$(r_{0,k_1}; \prec_{0,k_1}) + (m_{k_1,k_2}; <_{k_1,k_2}) + (r_{k_2,0}; \prec_{k_2,0}) < (0; \leq)$$

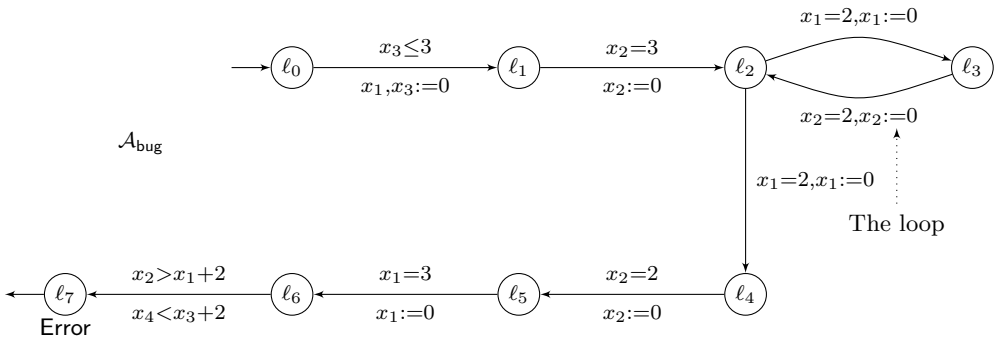
We can also notice that $(r_{0,k_1}; \prec_{0,k_1}) = (-K; <)$ otherwise this sum could be simplified once more (both $I_{x_{k_1}}$ and $I_{x_{k_2}}$ would then be bounded).

We set $(c; \prec) = (r_{k_2,0}; \prec_{k_2,0}) + (r_{0,k_1}; \prec_{0,k_1})$, and we get

$$\begin{cases} Z \subseteq (x_{k_1} - x_{k_2} \prec -c) \\ R \cap (x_{k_1} - x_{k_2} \prec -c) = \varnothing \end{cases}$$

Then $-K \leq c \leq K$ implies that the zone $(x_{k_1} - x_{k_2} \prec -c)$ is K-bounded. As previously, we conclude that $\mathsf{Approx}_K(Z) \subseteq (x_{k_1} - x_{k_2} \prec -c)$ and thus $R \cap \mathsf{Approx}_K(Z) = \varnothing$. ⌟

Surprisingly this theorem does not extend to timed automata with general clock constraints. Indeed, consider the timed automaton $\mathcal{A}_{\mathsf{bug}}$ depicted below. For every integer k, the extrapolation operator Approx_k is not correct with respect to reachability properties for $\mathcal{A}_{\mathsf{bug}}$. One can even also prove that, for automaton $\mathcal{A}_{\mathsf{bug}}$, there is no abstraction operator Abs satisfying the four above-mentioned criteria (finiteness, correctness, completeness and effectiveness).



Proposition 8. *Consider the timed automaton $\mathcal{A}_{bug}$ defined before. For every integer k, Approx_k is not a correct abstraction with respect to reachability properties in $\mathcal{A}_{bug}$. Furthermore, there is no abstraction operator $\mathfrak{a}$ that over-approximates zones, and that can be finite, effective and correct with respect to reachability properties in $\mathcal{A}_{bug}$.*

Proof. We explain the problem with automaton $\mathcal{A}_{\mathsf{bug}}$. The zone Z_α which is computed by a forward analysis when reaching the location 'Error' after having taken α times the loop is defined by the constraints below (on the left). Fixing an integer k, taking α large enough the extrapolated zone is also described below (on the right).

$$Z_\alpha : \begin{cases} 1 \leq x_2 - x_1 \leq 3 \\ 1 \leq x_4 - x_3 \leq 3 \\ x_3 - x_1 = 2\alpha + 5 \\ x_4 - x_2 = 2\alpha + 5 \end{cases} \qquad \mathsf{Approx}_k(Z_\alpha) : \begin{cases} 1 \leq x_2 - x_1 \leq 3 \\ 1 \leq x_4 - x_3 \leq 3 \\ x_3 - x_2 > k \end{cases}$$

Note that if $v \in Z_\alpha$, then in particular $v(x_2) - v(x_1) = v(x_4) - v(x_3)$. On the other hand, there are valuations $v \in \mathsf{Approx}_k(Z_\alpha)$ such that $v(x_2) - v(x_1) \neq v(x_4) - v(x_3)$. Obviously, the zone Z_α does not intersect the constraint $x_2 - x_1 > 2 \wedge x_4 - x_3 < 2$, which implies that the location 'Error' is not reachable. However, $\mathsf{Approx}_k(Z_\alpha)$ intersects the constraint $x_2 - x_1 > 2 \wedge x_4 - x_3 < 2$ (for α large enough), which implies that the location 'Error' is computed as reachable by the abstract forward analysis that uses the abstraction operator Approx_k (for any integer k). The problem with automaton $\mathcal{A}_{\mathsf{bug}}$ comes from the use of diagonal constraints on the transition leading to location 'Error'. ⌟

Note however that for timed automata with three clocks (and possibly diagonal constraints), it is possible to find a constant K such that Approx_K is correct with respect to reachability properties (the constant K may however be larger than the maximal constant appearing in a constraint of the automaton) [Bou04]. In the general case (more than three clocks), a way of handling diagonal constraints is to remove first (or on-the-fly) diagonal constraints as we know they can be removed (see [BDGP98]). However this leads to an unavoidable exponential blowup in the size of the model [BC05].

Remark 10. Several improvements on that extrapolation operator have been made, like distinguish locations, distinguish lower- and upper-bounds.

8 The task-graph example

8.1 Description of the problem

Compute $D\times(C\times(A+B))+(A+B)+(C\times D)$ using two processors:

P_1 (fast):

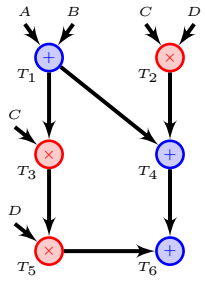
time	
+	2 picoseconds
×	3 picoseconds

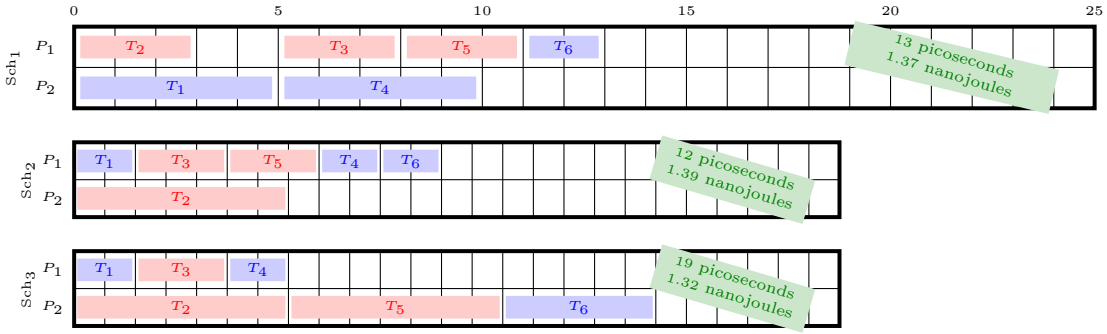
energy	
idle	10 Watt
in use	90 Watts

P_2 (slow):

time	
+	5 picoseconds
×	7 picoseconds

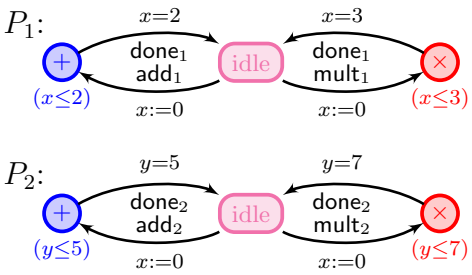
energy	
idle	20 Watts
in use	30 Watts



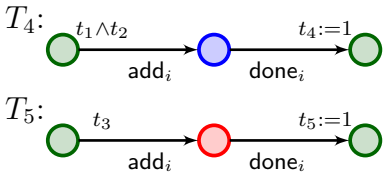


8.2 Modelization of the problem

– Processors



– Tasks



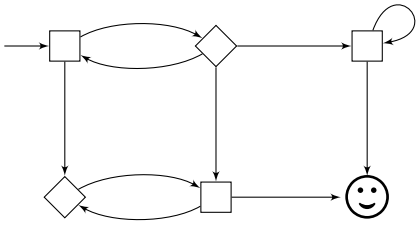
8.3 How to model more?

How should we take into account:

– uncertainties on delays?
– power consumption?
– job preemption?

9 Timed games: controller synthesis

Example 13 (Finite untimed games). Recall the bases.



Motivation for timed games. Uncertainty over delays, controllers. We choose here the model of control games, which is not used in a game-theoretic perspective.

A timed game $\mathcal{G}$ is a timed automaton $(L, L_0, L_F, X, \Sigma, T)$ in which we assume that edges are partitioned into controllable and uncontrollable edges, $T = T_c \cup T_u$. We assume that L_F are sink locations, with an unconstrained controllable self-loop, and we will consider *reachability games*. Its dual, *safety games*, are useful to model *eg* control problems.

9.1 Reachability games

A *strategy* for the controller from state $s = (\ell, v)$ is a partially defined function $f : \bigcup_s \mathsf{Runs}(\mathcal{G}, s) \to (\mathbb{R}_+ \times T_c)$ (where $\mathsf{Runs}(\mathcal{G}, s)$ is the set of finite runs from configuration s) such that for every run ϱ on which f is defined, if $f(\varrho) = (t, e)$, then (t, e) is a valid mixed move after ϱ. The set of possible *outcomes* (or *plays*) of f from s in $\mathcal{G}$ is denoted $\mathsf{Out}_f(\mathcal{G}, s)$ and is defined inductively as follows:

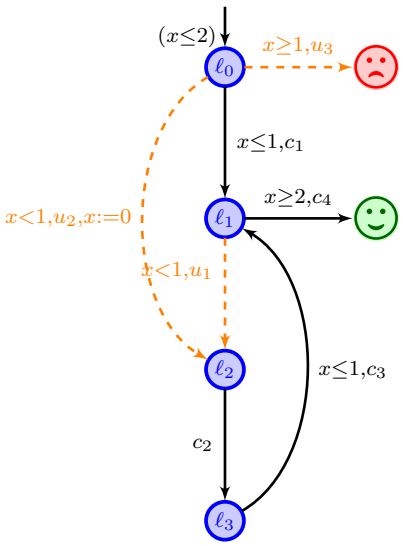
- the run s is in $\mathsf{Out}_f(\mathcal{G}, s)$;
- if ϱ is in $\mathsf{Out}_f(\mathcal{G}, s)$ and if $(t, e) = f(\varrho)$, then $\varrho \xrightarrow{t,c} s'$ is in $\mathsf{Out}_f(\mathcal{G}, s)$, and for every $0 \leq t' \leq t$ and every $e' \in T_u$, if $\varrho \xrightarrow{t',e'} s'$ is a possible extension of ϱ, then $\varrho \xrightarrow{t',e'} s'$ is in $\mathsf{Out}_f(\mathcal{G}, s)$.

An outcome $\varrho \in \mathsf{Out}_f(\mathcal{G}, s)$ is *f-maximal* if it cannot be extended by the strategy f (its image by f is not defined). A run is *winning* if it ends up in $L_F \times \mathbb{R}_+^X$. A strategy f is *winning* if all f-maximal outcomes of f are winning.

Remark 11. Any reasonable change in this definition (only infinite runs, uncontrollable edges should be taken, *etc*) can be taken into account. *Explain turn-based games, and changes that should be made.*

Given a timed game, we want to synthesize winning strategies, if possible, or assert it is not possible to do so.

Example 14. We consider the following timed game:



Is the configuration $(\ell_0, 0)$ winning? [objective: reach the happy location] ⌟

Theorem 6. *The controller synthesis problem is computable/decidable.*

We define the following operators, defined on sets of configurations:

$$\mathsf{Pred_t}(W, W') = \{(\ell, v) \in L \times \mathbb{R}_+^X \mid \exists t \in \mathbb{R}_+ \text{ s.t. } (\ell, v+t) \in W \text{ and } \forall 0 \leq t' \leq t,\ (\ell, v+t') \notin W'\}$$

We also define

$$\mathsf{cPred}(W) = \{(\ell, v) \in L \times \mathbb{R}_+^X \mid \exists e \in T_c \text{ s.t. } (\ell, v) \xrightarrow{0,e} (\ell', v') \text{ and } (\ell', v') \in W\}$$

$$\mathsf{uPred}(W) = \{(\ell, v) \in L \times \mathbb{R}_+^X \mid \exists e' \in T_u \text{ s.t. } (\ell, v) \xrightarrow{0,e'} (\ell', v') \text{ and } (\ell', v') \in W\}$$

Finally, we define

$$\pi(W) = W \cup \mathsf{Pred_t}(\mathsf{cPred}(W), \mathsf{uPred}(W^c))$$

This is the set of controllable predecessor of W: from $\pi(W)$, one can ensure a move to W, without being kicked out by the environment.

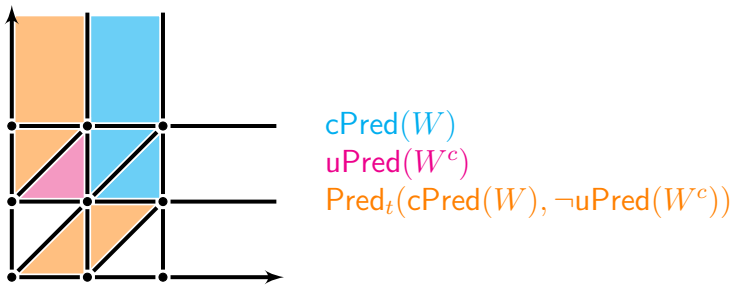
Lemma 7. *A configuration (ℓ, v) is in $\pi(W) \setminus W$ iff there exists $t \in \mathbb{R}_+$ and $e \in T_c$ such that $(\ell, v) \xrightarrow{t,e} W$, and for every $0 \leq t' \leq t$, for every $e' \in T_u$, $(\ell, v) \xrightarrow{t',e'} W$.*

Proof. This is a direct application of the definitions. ⌟

Lemma 8. *Let $\mathsf{Goal} = \{(\ell, v) \in L \times \mathbb{R}_+^X \mid \ell \in L_F\}$. The non-decreasing sequence $(\pi^i(\mathsf{Goal}))_{i \in \mathbb{N}}$ stabilizes in a finite number of steps.*

Proof. Operators cPred, uPred and Pred_t preserve finite unions of extended regions (an extended region is a pair (ℓ, R) where ℓ is a location of the automaton, and R is a region). This can be done as an exercise. ⌟

Remark 12. This fixpoint operator π does not preserve zones in general:



The limit of the sequence $(\pi^i(\mathsf{Goal}))_{i\in\mathbb{N}}$, denoted $\pi^*(\mathsf{Goal})$, is the *attractor* of Goal. Note that, due to the convergence of the iterative sequence, $\pi(\pi^*(\mathsf{Goal})) = \pi^*(\mathsf{Goal})$.

Proposition 9. *A configuration (ℓ, v) is in $\pi^*(\mathsf{Goal})$ iff the controller has a winning strategy for the corresponding reachability game.*

Proof. We will define a memoryless strategy for the controller, which will be winning for all configurations in $\pi^*(\mathsf{Goal})$. For every configuration $s \in \pi^*(\mathsf{Goal})$, there is a unique index $i \in \mathbb{N}$ such that $s \in \pi^i(\mathsf{Goal}) \setminus \pi^{i-1}(\mathsf{Goal})$. We will call this index the *rank* of s in the game. We construct the strategy f, inductively on the index of the configurations, as follows.

For every $i \in \mathbb{N}$ we construct a memoryless strategy f_i defined over all configurations in $\pi^i(\mathsf{Goal}) \setminus \mathsf{Goal}$, and such that:

- f_i restricted to $\pi^{i-1}(\mathsf{Goal})$ coincides with f_{i-1}
- for every configuration $s_0 \in \pi^i(\mathsf{Goal})$, if $\varrho \in \mathsf{Out}_{f_i}(\mathcal{G}, s_0)$, $\varrho = s_0 \xrightarrow{t_1,a_1} s_1 \xrightarrow{t_2,a_2} s_2 \cdots \xrightarrow{t_p,a_p} s_p$ implies $\mathsf{rank}(s_{j+1}) < \mathsf{rank}(s_j)$ for every $0 \leq j < p$, and furthermore, if ϱ is f_i-maximal, then $\mathsf{rank}(s_p) = 0$ (the run is winning).

Only configurations in Goal have rank 0, hence we define f_0 as being undefined everywhere. It satisfies the above conditions over $\pi^0(\mathsf{Goal}) \setminus \mathsf{Goal} = \varnothing$.

Assume now the induction hypothesis for all configurations of rank at most $i-1$. Pick $(\ell, v) \in \pi^i(\mathsf{Goal}) \setminus \pi^{i-1}(\mathsf{Goal})$. There exists $t \in \mathbb{R}_+$, $e \in T_c$ such that $(\ell, v) \xrightarrow{t,e} (\ell', v')$ implies (ℓ', v') has at most rank $i-1$, and for all $0 \leq t' \leq t$ and $e' \in T_u$, $(\ell, v) \xrightarrow{t',e'} (\ell'', v'')$ implies that (ℓ'', v'') has rank at most $i-1$. So if we assume that we have already defined the strategy f_i from configurations of rank no more than $i-1$, we define the strategy in (ℓ, v) as the move (t, e). Now if we take a (maximal) run ϱ generated by the strategy f_i, $\varrho = (\ell, v) \xrightarrow{t_1,a_1} (\ell_1, v_1) \xrightarrow{t_2,a_2} (\ell_2, v_2) \cdots$, we have that the sub-run $(\ell_1, v_1) \xrightarrow{t_2,a_2} (\ell_2, v_2) \cdots$ is generated by the strategy f_{i-1} and is thus finite and winning.

Conversely, assume that $(\ell, v) \notin \pi^*(\mathsf{Goal}) = \pi(\pi^*(\mathsf{Goal}))$, but that the controller has a strategy f (possibly with memory) to win the game from (ℓ, v). We will build a run starting in (ℓ, v), generated by the strategy f, and which only visits configurations outside $\pi^*(\mathsf{Goal})$ (hence never winning). Assume we have constructed such a run, say ϱ, of length i. We will extend this run into a run of length $i+1$. Assume that ϱ ends in configuration

$(\ell', v') \notin \pi^*(\mathsf{Goal})$ and that $f(\varrho) = (t, e)$. There are two cases: either $(\ell', v') \xrightarrow{t,e} (\ell'', v'')$ is such that $(\ell'', v'') \notin \pi^*(\mathsf{Goal})$, in which case we are done, or there exists $0 \leq t' \leq t$ and $e' \in T_u$ such that $(\ell', v') \xrightarrow{t,e'} (\ell''', v''')$ implies $(\ell''', v''') \notin \pi^*(\mathsf{Goal})$, in which case the extended run is $\varrho \xrightarrow{t',e'} (\ell''', v''')$. That way, we build an infinite run which is generated by f but is not winning... contradiction. ⌟

Remark 13. These games are *memoryless determined*: from any state, one of the players wins (from $\pi^*(\mathsf{Goal})$, the controller wins, from outside $\pi^*(\mathsf{Goal})$, the environment wins), and memoryless strategies are sufficient to win (for both players).

Exercise 20. Apply the algorithm to the game of Example 14.

Exercise 21. Compute the complexity of the above algorithm. Think of (and prove) a matching lower bound. Conclude.

Proof. This problem is EXPTIME-complete, and EXPTIME-hardness follows from a reduction from the halting problem of alternating linearly-bounded Turing machines. ⌟

9.2 Safety games

In that case, a run is winning if it avoids L_F. A strategy f is winning if all f-maximal outcomes of f are winning.

Those safety games can be solved using a fixpoint operator as well:

$$\pi'(W) = W \cap \mathsf{Pred}_t(\mathsf{cPred}(W), \mathsf{uPred}(W^c))$$

starting from $\mathsf{Safe} = \{(\ell, v) \in L \times \mathbb{R}_+^X \mid \ell \notin L_F\}$.

Those games are also EXPTIME-complete.

9.3 Imperfect information in reachability games

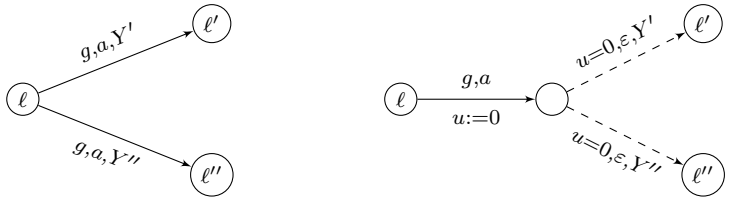
Instead of observing the runs, we observe timed words, that is the strategies only depend on the underlying timed word, not on the real run. The set of actions is $\Sigma = \Sigma_c \cup \Sigma_u$. This allows to have a partial observation of the system.

Theorem 7. *Reachability timed games under the imperfect information hypothesis are undecidable.*

Proof (Sketch). By reduction from the universality problem of timed automata. Let $\mathcal{A} = (L, L_0, L_F, X, \Sigma, T)$ be a timed automaton. W.l.o.g. we assume that if $\ell \xrightarrow{g',a,Y'} \ell'$ and $\ell \xrightarrow{g'',a,Y''} \ell''$ then either $[\![g']\!] = [\![g'']\!]$ or $[\![g']\!] \cap [\![g'']\!] = \varnothing$. We also assume the automaton is complete. We construct the game $\mathcal{G} = (L', L'_0, L'_F, X \cup \{u\}, \Sigma', T')$ where:

- $L' = L \cup T \cup \{\mathsf{win}, \mathsf{lose}\}$;
- $L'_0 = L_0$;

– $L'_F = \{\mathsf{win}\}$;
– $\Sigma' = \Sigma \cup \{\varepsilon, \#\}$;
– T' is defined as follows:



We also add transitions $\ell \xrightarrow{u=0,\#} \mathsf{win}$ if $\ell \in L \setminus L_F$, and $\ell \xrightarrow{u=0,\#} \mathsf{lose}$ if $\ell \in L_F$.

It is not difficult to prove that the controller has a strategy to ensure $\mathsf{Goal} = L'_F \times \mathbb{R}_+^X$ iff the timed automaton $\mathcal{A}$ is not universal. The idea is to play a timed word w which is not accepted by $\mathcal{A}$: in $\mathcal{A}$, all runs that read w leads to a location in $L \setminus L_F$, from which we can play the action $\#$ and reach location win. ⌟

9.4 Application of games to checking strong timed bisimulation

Theorem 8. *Strong timed bisimulation between timed automata can be decided in EXPTIME.*

Proof. The proof of this theorem can be done using timed games!

Assume $\mathcal{A}_1 = (L_1, L_{1,0}, L_{1,F}, X_1, \Sigma, T_1)$ and $\mathcal{A}_2 = (L_2, L_{2,0}, L_{2,F}, X_2, \Sigma, T_2)$ are two timed automata over the same alphabet.

We construct a two-player turn-based[16] timed game $\mathcal{G} = (Q, Q_0, Q_{\text{tester}}, Q_{\text{prover}}, Q_F, X'', \Sigma, T'')$ as follows:

– $Q_{\text{tester}} = L_1 \times L_2$
– $Q_{\text{prover}} = \{(\ell_1, e_2) \mid \ell_1 \in L_1,\ e_2 \in T_2\} \cup \{(\ell_2, e_1) \mid \ell_2 \in L_2,\ e_1 \in T_1\}$
– $Q = Q_{\text{tester}} \cup Q_{\text{prover}} \cup \{\mathsf{Bad}\}$
– $Q_0 = L_{1,0} \times L_{2,0}$
– $Q_F = \{\mathsf{Bad}\}$
– $X'' = X \cup X' \cup \{z\}$
– T'' is composed of the following transitions:
  - $(\ell_1, \ell_2) \xrightarrow{g_1,a,Y_1 \cup \{z\}} (\ell_2, e_1)$ where $(\ell_1, \ell_2) \in L_1 \times L_2$, $e_1 = (\ell_1, g_1, a, Y_1, \ell'_1) \in T_1(\ell_1)$
  - $(\ell_1, \ell_2) \xrightarrow{g_2,a,Y_2 \cup \{z\}} (\ell_1, e_2)$ where $(\ell_1, \ell_2) \in L_1 \times L_2$, $e_2 = (\ell_2, g_2, a, Y_2, \ell'_2) \in T_2(\ell_2)$
  - $(\ell_2, e_1) \xrightarrow{g_2 \wedge (z=0),a,Y_2} (\ell'_1, \ell'_2)$ where $(\ell'_1, \ell_2) \in L_1 \times L_2$, $e_1 = (\ell_1, g_1, a, Y_1, \ell'_1) \in T_1(\ell_1)$, and $e_2 = (\ell_2, g_2, a, Y_2, \ell'_2) \in T_2(\ell_2)$
  - $(\ell_1, e_2) \xrightarrow{g_1 \wedge (z=0),a,Y_1} (\ell'_1, \ell'_2)$ where $(\ell_1, \ell'_2) \in L_1 \times L_2$, $e_2 = (\ell_2, g_2, a, Y_2, \ell'_2) \in T_2(\ell_2)$, and $e_1 = (\ell_1, g_1, a, Y_1, \ell'_1) \in T_1(\ell_1)$
  - $(\ell_2, e_1) \xrightarrow{(z=0) \wedge \neg g, a} \mathsf{Bad}$ where $g = \bigvee_{(\ell_2, g_2, a, Y_2, \ell'_2) \in T_2(\ell_2)} g_2$
  - $(\ell_1, \ell'_2, e_2) \xrightarrow{(z=0) \wedge \neg g, a} \mathsf{Bad}$ where $g = \bigvee_{(\ell_1, g_1, a, Y_1, \ell'_1) \in T_1(\ell_1)} g_1$

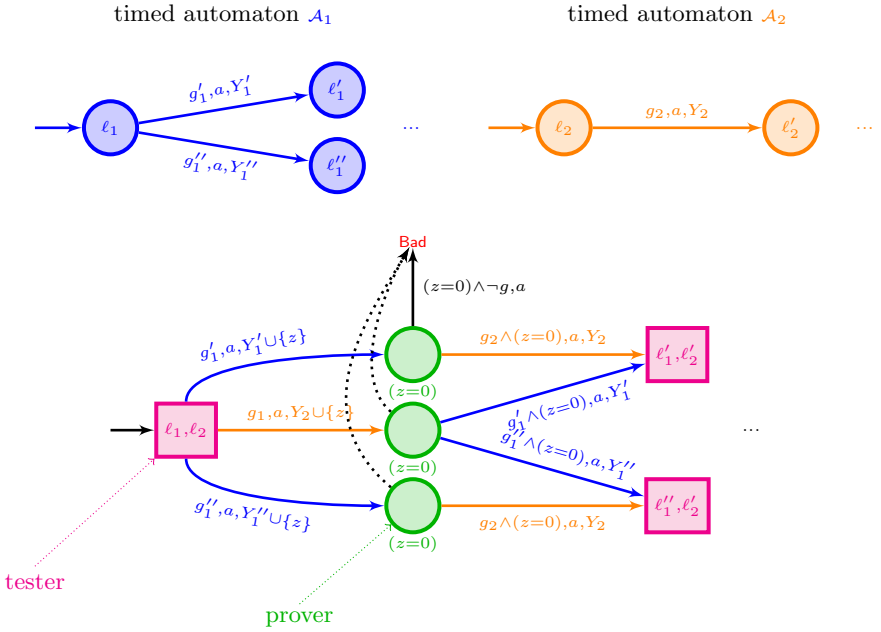


Fig. 12: *Bisimulation game*

This is depicted on Figure 12. Then, (ℓ_1, v_1) and (ℓ_2, v_2) are timed bisimilar iff the prover has a winning strategy to avoid the Bad state from $((\ell_1, \ell_2), v_1v_2)$.

Define $\mathcal{R}$ the relation defined by $(\ell_1, v_1)\mathcal{R}(\ell_2, v_2)$ iff the prover has a winning strategy to avoid the Bad state from $((\ell_1, \ell_2), v_1v_2)$. It is easy to prove that $\mathcal{R}$ is a bisimulation relation by playing the winning strategy.

Assume now that there is a bisimulation relation $\mathcal{R}$ such that $(\ell_1, v_1)\mathcal{R}(\ell_2, v_2)$. We define a winning memoryless strategy for the player as follows: for all states $(\ell_1, v_1)\mathcal{R}(\ell_2, v_2)$, for all t, from states $((\ell_2, e_1), (v_1v_2) + t)$, play the corresponding matching move witnessing the fact that there is an edge e_2 labelled with a such that $(\ell_2, v_2) \xrightarrow{t,a} (\ell'_2, v'_2)$. This yields a winning strategy... ⌟

[16] To be explained.

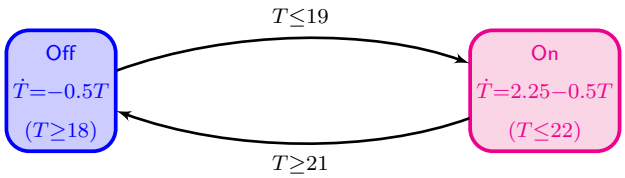
10 Hybrid systems

Hybrid systems are convenient models for representing general continuous behaviours. They have application in control theory, biology, etc.

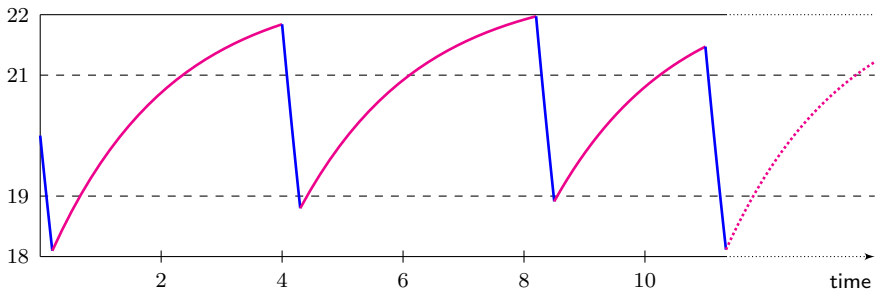
Example 15. We want to model a thermostat. If T is the temperatur, it can written as a list of rules:

- when the heater is Off, the room cools down: $\dot{T} = -0.5T$;
- when the heater is On, the room heats: $\dot{T} = 2.25 - 0.5T$;
- when $21 \leq T \leq 22$, it switches Off;
- when $18 \leq T \leq 19$, it switches On.

This thermostat can be modelled by the following automaton:



A possible behaviour for that system is as follows:



A behaviour can also be seen as a sequence of delays and jumps.

Theorem 9. *The reachability problem is undecidable in hybrid automata.*

Therefore, to be able to analyze such systems, one needs to consider subclasses:

- dimension (number of variables), continuous- or discrete time
- what kind of dynamics
- what kind of guards, invariants, jumps

A zoology of subclasses:

- Linear hybrid automata: dynamics described by $\dot{x} = c$, polyhedral guards and invariants, linear resets
  - memory cell: $\dot{x} = 0$ (but x can be reset)
  - clock: $\dot{x} = 1$

  - stopwatch: in some locations, $\dot{x} = 1$, in others $\dot{x} = 0$
  - skewed clock: $\dot{x} = c$ in all locations
- Rectangular hybrid automata: dynamics described by $\dot{x} \in [c, d]$, guards and invariants described by $x \in [c, d]$, resets described by $x \in [c, d]$
  - Initialized rectangular hybrid automata: each time a variable changes its slope, it should be reset
  - Linear hybrid automata are rectangular hybrid automata
- O-minimal hybrid automata: strong resets between locations, everything described in an o-minimal theory
- Piecewise affine maps
- Piecewise constant derivatives (PCD):
- Polygonal differential inclusion systems (PDIS):
- *etc...*

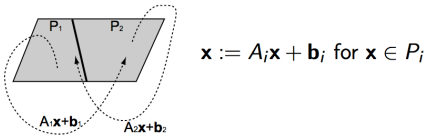
Exercise 22. Back to the task-graph scheduling problem, how would you model the power consumption? And how would you model job preemption?

In this course we will study:

① Piecewise affine maps: rich jumps, poor dynamics
② (Initialized) Rectangular hybrid automata: decorrelated variables (rather poor jumps)
③ Piecewise constant derivative systems: poor jumps, richer dynamics
④ A generic semi-algorithm

10.1 Piecewise affine maps (PAMs)

(Pictures borrowed from Eugene Asarin)
A d-dimensional *piecewise affine map* (PAM) is a discrete-time dynamical system defined as follows: we are given a bounded polyhedron $\mathcal{P}$ of $\mathbb{R}^d$ (e.g. $\mathcal{P} = [0, 1]^d$); we are given a polyhedral partition of $\mathcal{P}$, and for each piece P of this partition, we are given a function $f : \mathcal{P} \to \mathcal{P}$ such that $f_{|P}$ is affine. The reachability problem asks whether, given two points $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{P}$, there exists an index n such that $f^n(\mathbf{x}_1) = \mathbf{x}_2$.



Theorem 10 ([KCG94]). *Two-dimensional PAMs are undecidable.*

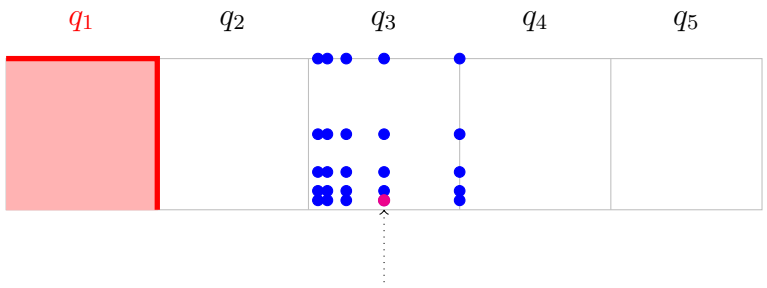
Note that we cannot directly express a counter machine using a PAM since in a PAM, the domain is bounded.

However we can simulate a deterministic two-counter machine as follows: A state (q_i, c, d) will be represented by state $(i-1+2^{-c}, 2^{-d})$. The domain is therefore $(0, N] \times [0, 1)$

where N is the number of states of the two-counter machine. State q_i is encoded by the polyhedron $P_i = (i-1, i] \times [0, 1)$ (we may have to split this polyhedron for defining the affine functions). We now define the affine functions, depending on the instruction starting at q_i:

- if q_i : $c++$; `goto` q_j, then we should go from $(i - 1 + 2^{-c}, 2^{-d})$ to $(j - 1 + 2^{-c-1}, 2^{-d})$, that is $y := y$ and $x := j - 1 + \frac{x-(i-1)}{2}$
- if q_i : $c--$; `goto` q_j, then we should go from $(i - 1 + 2^{-c}, 2^{-d})$ to $(j - 1 + 2^{-c+1}, 2^{-d})$ in case $c > 0$, that is from $P_i \cap (x < i)$. The affine function is then defined as: $x := j - 1 + 2(x - (i-1))$ and $y := y$
- if q_i : `if` $c > 0$ `then goto` q_j `else goto` q_k, then we should go from $P_i \cap (x < i)$ to P_j using $x := j - i + x$ and $y := y$ and from $P_i \cap (x = i)$ to P_k using $x := k - i + x$ and $y := y$
- if q_i : $d++$; `goto` q_j, then we should go from $(i - 1 + 2^{-c}, 2^{-d})$ to $(j - 1 + 2^{-c}, 2^{-d-1})$, that is $x := x + j - i$ and $y := \frac{y}{2}$
- if q_i : $d--$; `goto` q_j, then we should go from $(i - 1 + 2^{-c}, 2^{-d})$ to $(j - 1 + 2^{-c}, 2^{-d+1})$ on $P_i \cap (y < 1)$, that is $x := x + j - i$ and $y := 2y$
- if q_i : `if` $d > 0$ `then goto` q_j `else goto` q_k, then we should go from $P_i \cap y < 1$ to P_j using $x := j - i + x$ and $y := y$, and from $P_i \cap y = 1$ to P_k using $x := k - i + x$ and $y := y$.

The encoding is illustrated on the following picture. Blue dots represent counter values.



Corollary 2. *Reachability is undecidable for linear hybrid automata.*

Remark 14 (An example open problem). It is unknown whether we can decide the reachability problem in one-dimensional PAMs.

10.2 Rectangular hybrid automata

We write $\mathcal{I}$ for the set of intervals of $\mathbb{R}$ with rational bounds.

A *rectangular (hybrid) automaton* is a tuple $\mathcal{H} = (L, \ell_0, X, E, \mathsf{Inv}, \mathsf{Act}, \mathsf{Pre}, \mathsf{Post}, \mathsf{Upd})$ where:

- L is a finite set of locations;
- $\ell_0 \in L$ is the initial location;

- $X = \{x_1, \dots, x_n\}$ is a finite set of variables;
- E is a finite set of edges, each one is characterized by $\mathsf{src}(t) \in L$ and $\mathsf{target}(t) \in L$;
- $\mathsf{Inv} : L \to \mathcal{I}^X$ assigns an invariant to every location;
- $\mathsf{Act} : L \to \mathcal{I}^X$ indicates the evolution (aka activity) of the variables in each location;
- $\mathsf{Pre} : E \to \mathcal{I}^X$ and $\mathsf{Post} : E \to \mathcal{I}^X$ are the pre- and post-conditions to be fulfilled when firing transitions;
- $\mathsf{Upd} : E \to 2^X$ indicates which variables should be updated.

It is said *initialized* whenever for every $e \in E$ for every $x \in X$, either $x \in \mathsf{Upd}(e)$, or $\mathsf{Act}(\mathsf{src}(e)) = \mathsf{Act}(\mathsf{target}(e))$.

We assume $\mathcal{H}$ is equipped with an extra variable t which is a clock.

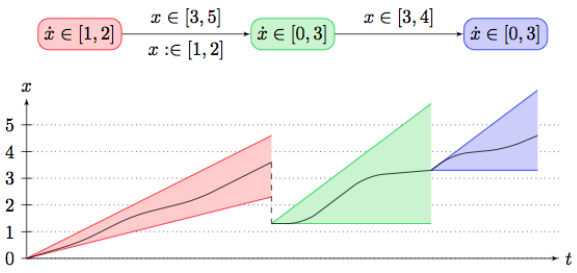
The semantics of $\mathcal{H}$ is given as a timed transition system $(S, s_0, \to)$ defined by $S = L \times \mathbb{R}^X$, $s_0 = (\ell_0, \mathbf{0})$, and $\to$ is defined as follows: $(\ell, \nu) \to (\ell', \nu')$ iff one of the two following conditions are met:

- [**continuous step**] $\ell = \ell'$, and either $\nu = \nu'$, or $\nu'(t) > \nu(t)$, and for every $x \in X$,

$$\frac{\nu'(x) - \nu(x)}{\nu'(t) - \nu(t)} \in \mathsf{Act}(\ell)(x)$$

- [**discrete step (or jump)**] there is a transition $e \in E$ such that $\mathsf{src}(e) = \ell$, $\mathsf{target}(e) = \ell'$, $\nu \models \mathsf{Pre}(e)$, $\nu' \models \mathsf{Post}(e)$, $\nu'(t) = \nu(t)$, for all $x \in X$, $\nu'(x) = \nu(x)$ unless $x \in \mathsf{Upd}(e)$.

Example 16. The following is an example of an initialized rectangular automaton:
(Picture taken from Nicolas Markey)



Remark 15. Obviously any timed automaton is an initialized rectangular automaton.

Undecidability of rectangular automata. A rectangular automaton is said *simple* whenever:

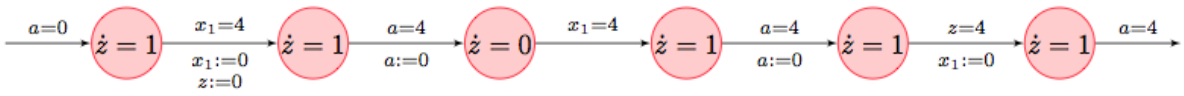
- only one variable, say z, is not a clock;
- the activity of z is singular in each location;
- Inv, Pre, Post have compact values;
- $\mathsf{Post}(e)(x) = [0,0]$ if $x \in \mathsf{Upd}(e)$, and $\mathsf{Post}(e)(x) = \mathsf{Pre}(e)(x)$ otherwise.

Such a simple automaton is said 2-slope whenever the only non-clock variable can have take two values.

Theorem 11. *The reachability problem is undecidable for simple 2-slope rectangular automata.*

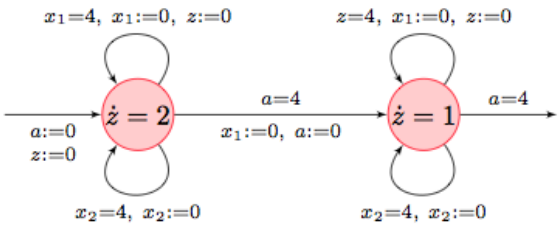
Proof. The result holds for any two different rationals serving as the slopes of the 2-slope variable, but we only prove it for the special case where the slopes are 0 and 1 on one hand, and 1 and 2 on the other hand.

We begin with the case of stopwatches. We encode the halting problem for two-counter machines. Counters c_1 and c_2 are encoded by clocks x_1 and x_2, with $x_i = 2^{1-c_i}$. We now have to be able to double and halve the value of a clock, preserving the value of the other clock at the same time. Doubling the value of a clock is achieved by the automaton depicted below:



Notice that, in this figure, we omit self-loops on each location resetting clock x_2 when it reaches 4, as well as invariants in each state, imposing each variable to be bounded by 4. Then it can be checked that the value of c_1 is doubled between the entry and exit of this automaton, while the value of c_2 is preserved. Halving the clock is achieved in a similar way, which can be done as an exercise.

The encoding in the case of slopes 1 and 2 follows similar ideas. The module for doubling the value of c_1 is depicted below:



Exercise 23. Show that we can decide the reachability problem for stopwatch automata with only two variables (we assume only resets to zero for the variables and rectangular guards). What happens if we allow diagonal constraints? Show the undecidability for three stopwatches and diagonal constraints.

Decidability of initialized rectangular automata.

Theorem 12. *The reachability problem is decidable in initialized rectangular automata. It is even PSPACE-complete.*

Proof. We assume $\mathcal{H}$ is a rectangular automataton. W.l.o.g. we assume that:

- $\mathsf{Pre}(e) \subseteq \mathsf{Inv}(\mathsf{src}(e))$;
- $\mathsf{Post}(e) \subseteq \mathsf{Inv}(\mathsf{target}(e))$;

– $\forall x \notin \mathsf{Upd}(e)$, $\mathsf{Pre}(e)(x) = \mathsf{Post}(e)(x)$.

W.l.o.g. we assume that Inv is always true (this does not affect reachability properties, because of the above assumptions). We furthermore assume that all intervals are compact. It will be clear later how we relax this restriction.

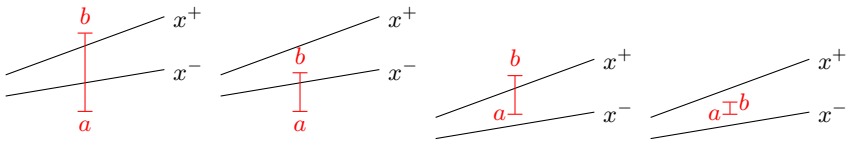
We first construct a multi-rate automaton $\mathcal{M}$ (activities are singular). Locations are the same as in $\mathcal{H}$. We have that:

– its set of variables is $X' = \{x^-, x^+ \mid x \in X\}$;
– activites are defined by $\mathsf{Act}'(\ell)(x^+) = b$ and $\mathsf{Act}'(\ell)(x^-) = a$ whenever $\mathsf{Act}(\ell)(x) = [a, b]$;
– We will have several copies of every edge in $\mathcal{H}$: $E' = E \times 4^X$. They are defined as follows.
  • Assume $x \in \mathsf{Upd}(e)$. Then any transition $e' = (e, \sigma) \in E'$ will update x^- and x^+ ($\{x^-, x^+\} \subseteq \mathsf{Upd}(e)$). Furthermore:

$$\mathsf{Pre}(e)(x) = [a, b] \implies \begin{cases} \mathsf{Pre}'(e')(x^-) = (-\infty, b] \\ \mathsf{Pre}'(e')(x^+) = [a, +\infty) \end{cases}$$

$$\mathsf{Post}(e)(x) = [m, M] \implies \begin{cases} \mathsf{Post}'(e')(x^-) = [m, m] \\ \mathsf{Post}'(e')(x^+) = [M, M] \end{cases}$$

  • Assume $x \notin \mathsf{Upd}(e)$. We write $\mathsf{Pre}(e)(x) = [a, b]$ and $\mathsf{Post}(e)(x) = [m, M]$. Note that by hypothesis, $[a, b] = [m, M]$. These constraints on x must be transferred to x^- and x^+. The way it will be handled depends on the relative positions of x^- and x^+ with regards to constants a and b.



(a) Case 1 (b) Case 2 (c) Case 3 (d) Case 4

  ∗ Case 1: We fix σ such that $\sigma(x) = 1$, and define $e' = (e, \sigma) \in E'$.

$$\begin{cases} \mathsf{Pre}'(e')(x^-) = [a, b] \\ \mathsf{Pre}'(e')(x^+) = [a, b] \end{cases} \qquad \begin{cases} \mathsf{Post}'(e')(x^-) = [a, b] \\ \mathsf{Post}'(e')(x^+) = [a, b] \end{cases}$$

    and variables x^- and x^+ are unchanged: $x^-, x^+ \notin \mathsf{Upd}(e')$.
  ∗ Case 2: We fix σ such that $\sigma(x) = 2$, and define $e' = (e, \sigma) \in E'$.

$$\begin{cases} \mathsf{Pre}'(e')(x^-) = [a, b] \\ \mathsf{Pre}'(e')(x^+) = (b, +\infty) \end{cases} \qquad \begin{cases} \mathsf{Post}'(e')(x^-) = [a, b] \\ \mathsf{Post}'(e')(x^+) = [b, b] \end{cases}$$

    variable x^- is unchanged whereas variable x^+ is changed: $x^- \notin \mathsf{Upd}(e')$ but $x^+ \in \mathsf{Upd}(e')$.

* Case 3: We fix σ such that $\sigma(x) = 3$, and define $e' = (e, \sigma) \in E'$.

$$\begin{cases} \mathsf{Pre}'(e')(x^-) = (-\infty, a) \\ \mathsf{Pre}'(e')(x^+) = [a, b] \end{cases} \qquad \begin{cases} \mathsf{Post}'(e')(x^-) = [a, a] \\ \mathsf{Post}'(e')(x^+) = [a, b] \end{cases}$$

variable x^+ is unchanged whereas variable x^- is changed: $x^+ \notin \mathsf{Upd}(e')$ but $x^- \in \mathsf{Upd}(e')$.

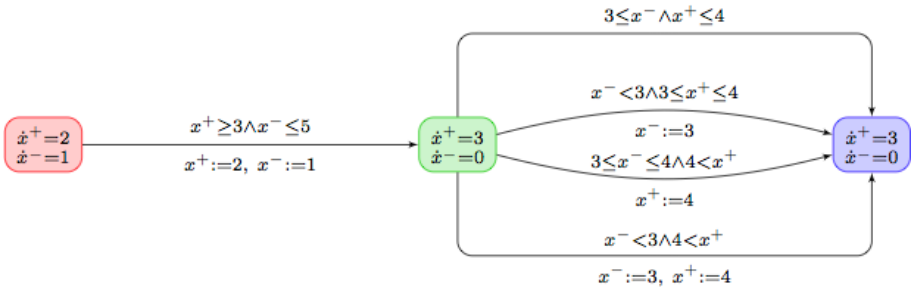
* Case 4: We fix σ such that $\sigma(x) = 4$, and define $e' = (e, \sigma) \in E'$.

$$\begin{cases} \mathsf{Pre}'(e')(x^-) = (-\infty, a) \\ \mathsf{Pre}'(e')(x^+) = (b, +\infty) \end{cases} \qquad \begin{cases} \mathsf{Post}'(e')(x^-) = [a, a] \\ \mathsf{Post}'(e')(x^+) = [b, b] \end{cases}$$

variables x^+ and x^- are changed: $\{x^-, x^+\} \subseteq \mathsf{Upd}(e')$.

Note that if $\mathcal{H}$ is initialized, then so is $\mathcal{M}$.

Example 17. The transformation applied to the previous example yields:



We now discuss the correctness of the construction of $\mathcal{M}$. For every configuration (ℓ, ν) of $\mathcal{M}$ we define the following set of configurations of $\mathcal{H}$:

$$\chi(\ell, \nu) = \{\ell\} \times \prod_{x \in X} [\nu(x^-), \nu(x^+)]$$

We define the following set of states of $\mathcal{M}$: $U = \{(\ell, \nu) \mid \forall x,\ \nu(x^-) \leq \nu(x^+)\}$. Obviously, $\chi(\ell, \nu) \neq \varnothing$ iff $(\ell, \nu) \in U$.

We can easily prove:

Lemma 9. *Let $(\ell, \nu) \in U$, $t \in \mathbb{R}_+$ and $e \in E$. Then:*

- $\mathsf{timePost}^t_{\mathcal{H}}\Big(\chi(\ell, \nu)\Big) = \chi\Big(\mathsf{timePost}^t_{\mathcal{M}}(\ell, \nu)\Big)$;
- $\mathsf{discPost}^e_{\mathcal{H}}\Big(\chi(\ell, \nu)\Big) = \chi\Big(\mathsf{discPost}^{\{(e,\sigma) \in E'\}}_{\mathcal{M}}(\ell, \nu)\Big)$.

As a consequence, the set of reachable states in $\mathcal{H}$ is the image by χ of the set of reachable states in $\mathcal{M}$.

We now transform the initialized multi-rate automaton $\mathcal{M}$ into a(n initialized) stopwatch automaton $\mathcal{S}$. The idea is to scale the variables by a correct factor. We assume the

set of variable of $\mathcal{M}$ is $X' = \{y_1, \ldots, y_p\}$. For every location ℓ of $\mathcal{M}$, we define m_i^ℓ by $\mathsf{Act}'(\ell)(y_j)$ if $\neq 0$, and 1 otherwise. We define $\alpha_\ell : \mathbb{R}^p \to \mathbb{R}^p$ such that $\alpha_\ell(y_1, \ldots, y_p) = (y_1/m_1^\ell, \ldots, y_p/m_p^\ell)$. The stopwatch automaton is obtained by applying α_ℓ to $\mathsf{Act}'(\ell)$ and $\mathsf{Pre}'(e')$ when $\ell = \mathsf{src}(e')$, and $\mathsf{Post}'(e')$ when $\ell = \mathsf{target}(e')$. We then define $\alpha(\ell, \nu) = (\ell, \alpha_\ell(\nu))$ when (ℓ, ν) is a state of $\mathcal{M}$.

Lemma 10. *For every set of states Z of $\mathcal{M}$,*

$$\alpha(\mathsf{Post}^*_{\mathcal{M}}(Z)) = \mathsf{Post}^*_{\mathcal{S}}(\alpha(Z))$$

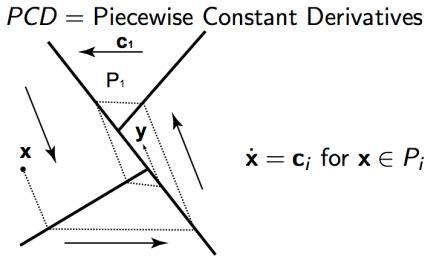
We finally transform $\mathcal{S}$ into a timed automaton $\mathcal{A}$, under the assumption that $\mathcal{S}$ is initialized. We let $K = \{\text{finite endspoints of intervals in } \mathcal{S}\}$. The set of locations of $\mathcal{A}$ is $L \times (K \cup \{\bot\})^{X'}$ ($\bot$ represents active clocks, whereas k represents stopped clocks at value k).If $e = \ell \xrightarrow{\mathsf{Pre},\mathsf{Post},\mathsf{Upd}} \ell'$ is an edge of $\mathcal{S}$, then we will have transitions $(\ell, \alpha) \xrightarrow{g,up} (\ell', \alpha')$ where g encodes Pre, if $x \in \mathsf{Upd}(e)$ then up is an update $x := cte$. Otherwise Post is redundant with Pre. If the activity of x in ℓ' is 0, then we set $\alpha'(x)$ to cte.

Reachability is decidable in PSPACE in timed automata and in space linear in the number of states of the automaton. We get a PSPACE algorithm. □

Exercise 24. Show that we can remove updates of the form $x := c$ in timed automata. However notice that this yields an exponential blowup in the size of the system. Therefore, show that standard region automata construction can be used to prove decidability of this class of systems (adapt the compatibility of the regions with resets).

10.3 Piecewise constant derivatives systems (PCDs)

(Pictures borrowed from Eugene Asarin)



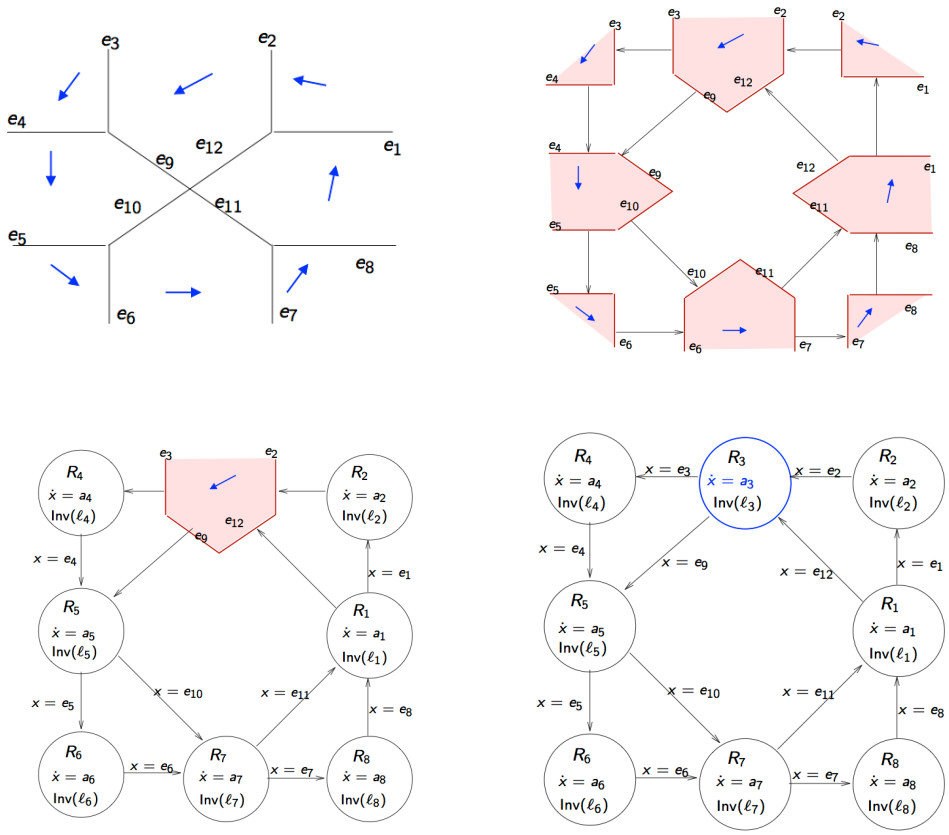
A *d-dimensional piecewise constant derivative system* is described by a pair $\mathcal{H} = (\mathcal{P}, \varphi)$ where:

- $P = (P_s)_{s \in S}$ is a partition of $\mathbb{R}^d$ such that each piece P_s is a polyhedron defined with rational constants, and
- $\varphi : S \mapsto \mathbb{Q}^d$ maps each piece of the partition to a vector in $\mathbb{Q}^d$.

By extension if $\mathbf{x} \in P_s$ we define $\varphi(\mathbf{x}) = \varphi(P_s)$. Under some well-formedness conditions for $\mathcal{H}$, if $\mathbf{x}_0 \in \mathbb{Q}^d$, we define the trajectory ϱ from $\mathbf{x}_0$ generated by $\mathcal{H}$ as the piecewise-derivable mapping from $\mathbb{R}_{\geq 0}$ to $\mathbb{R}^d$ such that:

– $\varrho(0) = \mathbf{x}_0$;
– the right-derivative of ϱ at t is given by $\varphi(\varrho(t))$.

Proposition 10. *PCDs are linear hybrid automata:*



Three-dimensional PCDs

Theorem 13 ([AMP95]). *The reachability problem in three-dimensional PCDs is undecidable.*

However we only show the result for four-dimensional PCDs.

Proof. We first show how to simulate a deterministic finite automaton.

Being at state q_i is encoded as being at position $(i, 0, 0)$. A transition from q_i to q_j is mimicked by first targetting state $(i, 1, 0)$ (using vector $(0, 1, 0)$) and then going up to $(i, 1, j)$ (using vector $(0, 0, 1)$). On the plane $z = j$, we target point $(j, 0, j)$ (using vector $(j - i, -1, 0)$), and then we target $(j, 0, 0)$ (using vector $(0, 0, -1)$). This is depicted on Figure 13.

We then encode a counter using an idea similar to PAMs. We first explain one counter. Being at state q_i with counter value c will correspond to being at state $(i - 1 + \frac{1}{2^c}, 0, 0)$. Mimicking a discrete change of the state can be done as before. Modules for incrementing

Table 4

Region	Defining conditions	$c = (\dot{x}, \dot{y}, \dot{z})$
F	$(z = 0) \wedge (y < 1)$	$(0, 1, 0)$
U_{ij}	$(x = i) \wedge (y = 1) \wedge (z < j)$	$(0, 0, 1)$
B_{ij}	$(z = j) \wedge (x + (j - i)y = j) \wedge (y > 0)$	$(j - i, -1, 0)$
D	$(z > 0) \wedge (y = 0)$	$(0, 0, -1)$

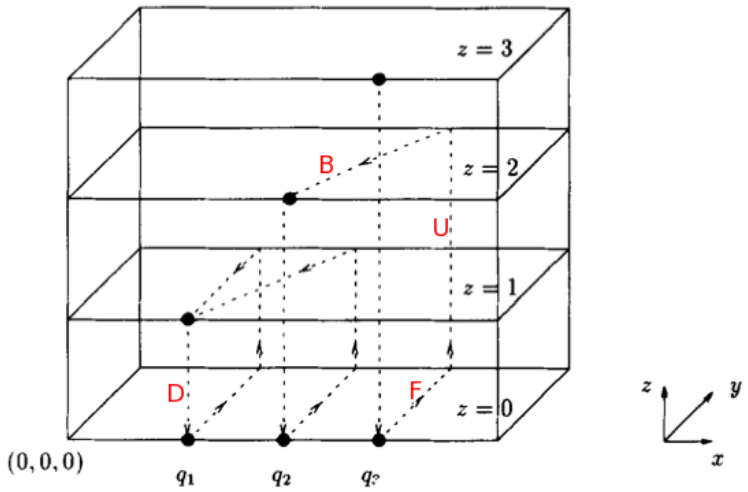


Fig. 14. Simulating a three-state automaton with $\delta(q_1) = \delta(q_2) = q_1$ and $\delta(q_3) = q_2$.

Fig. 13:

or decrementing the counter will consist in using the modules depicted on Figure 14. These modules are plugged on the ground level of Figure 13.

Globalement, this looks like on Figure 15.

To mimick a second counter we can use a fourth variable whose value will be $\frac{1}{2^d}$. A complicated reduction can be used to use only three variables... □

Two-dimensional PCDs

Theorem 14 ([AMP95]). *The reachability problem in two-dimensional PCDs is decidable.*

Proof. this is in my notes

Remark 16. Similar (but more involved) arguments can be used to prove the decidability of simple polygonal differential inclusions (that is, slopes lie between two vectors).

Fig. 14:

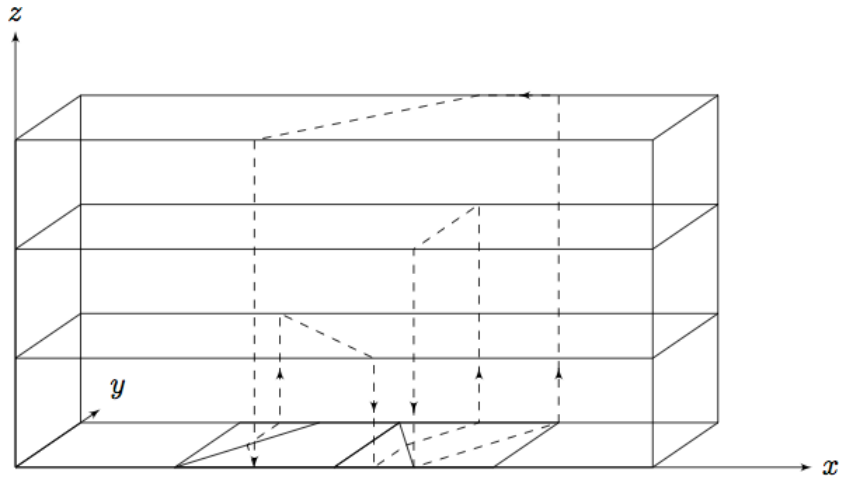


Fig. 15:

10.4 A generic semi-algorithm

The procedure. The following is a generic forward semi-algorithm for checking reachability properties in a hybrid automaton $\mathcal{H}$:

$F = \mathsf{Init}$;
repeat
$F = F \cup \mathsf{SuccFlow}(F) \cup \mathsf{SuccJump}(F)$
until $(F \cap \mathsf{Final} \neq \varnothing)$ or fixpoint reached or tired
in the first case, **say** reachable
in the second case, **say** unreachable
otherwise, **say** timeout

Remark 17. We can imagine any variant (like backward algorithm).

The above is a semi-algorithm as soon as we can compute the various operations in the above procedure, and test for inclusion (fixpoint). We therefore need a data structure for

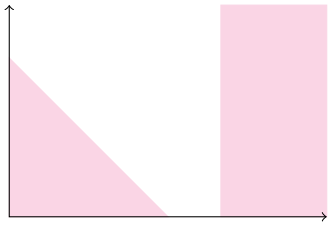
representing subsets of $\mathbb{R}^n$, and for efficiently computing unions, intersections, SuccFlow and SuccJump.

A possible data structure. We want to represent subsets of $L \times \mathbb{R}^n$. We will represent such subsets as polyhedra P, that is finite conjunctions of half-spaces defined by $Ax \leq b$ or $Ax < b$, where A is an n-square matrix and b is an n-vector with rational constants. A symbolic state is therefore a pair (ℓ, P) where $\ell \in L$ is a location of $\mathcal{H}$, and P is a polyhedron. If (ℓ, P) and (ℓ, P') have been computed, and P and P' are incomparable (for the inclusion), then this means that we take the union of both.

We will show that all operations can be performed using this data structure.

A linear constraint γ on $\mathbb{R}^n$ is a constraint of the form $\sum_{i=1}^n a_i x_i \leq b$ or $\sum_{i=1}^n a_i x_i < b$. A polyhedron is a finite conjunction of linear constraints: $\bigwedge_i \gamma_i$. A union of polyhedra is of the form $\bigvee_i \bigwedge_j \gamma_{i,j}$.

Example 18. For instance, $(x_1 > 0 \wedge x_2 > 0 \wedge x_1 + x_2 < 3) \vee x_1 > 4$ defines a union of polyhedra:



We define the logic FOLC (first-order logic of linear constraints) as:

$$\phi ::= \sum_{i=1}^{n} a_i x_i \leq c \mid \neq \phi \mid \phi \vee \phi \mid \exists x.\phi$$

We can obviously express $<$, $=$, $\wedge$ and $\forall$.

The semantics is obvious... and we can express union of polyhedra.

Example 19. $Q(u, v) = \exists x. \forall y. (x < 3 \vee y > 2 \vee 2u + x + v < 17)$

Theorem 15. *FOLC admits quantifier elimination: every formula is equivalent to a DNF of linear constraints, that is a union of polyhedra.*

Lemma 11 (Fourier-Motzkine elimination). *$\exists y. \bigwedge_j C_j$ can be written as $\bigwedge_j C'_j$. That is, the projection of a polyhedron in $\mathbb{R}^n$ onto $\mathbb{R}^n$ is a polyhedron.*

Example 20 (Fourier-Motzkine algorithm). We want to compute the closed form for the following polyhedron:

$$\phi = \exists z. \left\{ \begin{array}{r} 2x + y + z \leq 10 \\ 3x + 2y \leq 11 \\ -x - y + z \leq 4 \\ 6x - y - z \leq 9 \end{array} \right\}$$

We isolate variable z:

$$\begin{cases} z \leq 10 - 2x - 2y \\ 3x + 2y \leq 11 \\ z \leq 4 + x + y \\ 6x - y - 9 \leq z \end{cases}$$

There are three groups: no variable z, lower bounds on z, upper bounds on z.

$$\{ 3x + 2y \leq 11 \qquad \{ 6x - y - 9 \leq z \qquad \begin{cases} z \leq 10 - 2x - 2y \\ z \leq 4 + x + y \end{cases}$$

Then eliminate z! We get that $[\![\phi]\!]$ is equivalent to:

$$\begin{cases} 3x + 2y \leq 11 \\ 6x - y - 9 \leq 10 - 2x - 2y \\ 6x - y - 9 \leq 4 + x + y \end{cases}$$

Computing basic operations. We now explain why this data structure is adapted to the analysis of hybrid automata. Assume D_1 and D_2 are two DNFs of linear constraints, that is unions of polyhedra. Then we can compute:

- Union: $D_1 \vee D_2$ is already a finite union of polyhedra;
- Intersection: $D_1 \wedge D_2$ can be transformed to DNF using standard boolean logic;
- Emptiness check: to know if $[\![D_1]\!] = \varnothing$, eliminate quantifiers from $\exists \mathbf{x}.D_1(\mathbf{x})$;
- Inclusion test: to know if $[\![D_1]\!] \subseteq [\![D_2]\!]$, eliminate quantifiers from $\forall \mathbf{x}.(D_1(\mathbf{x}) \Rightarrow D_2(\mathbf{x}))$.

Let D be a DNF of linear constraints. Then:

- Flow successor: for a state with dynamics $\dot{\mathbf{x}} = \mathbf{c}$ and invariant I, the flow successor of D is given by the formula:

$$\exists \mathbf{x}.\exists t.(D(\mathbf{x}) \wedge t > 0 \wedge \mathbf{y} = \mathbf{x} + t\mathbf{c} \wedge I(\mathbf{x}) \wedge I(\mathbf{y}))$$

  Eliminate quantifiers and obtain the DNF for the flow successor.
- Jump successor: for a transition with guard G and reset R, the jump successor of D is given by the formula:

$$\exists \mathbf{x}.(D(\mathbf{x}) \wedge G(\mathbf{x}) \wedge \mathbf{y} = R(\mathbf{x}))$$

  Eliminate quantifiers and obtain the DNF for the successor.

We have shown how to implement all the operations for LHA, PCDs, PAMs, rectangular automata, *etc.* using finite unions of polyhedra (given as DNFs of linear constraints) as a data structure.

Remark 18. Fourier-Motzkine is very costly, hence this method is costly... Some other data structures are investigated...

11 Weighted timed automata and games

11.1 The models

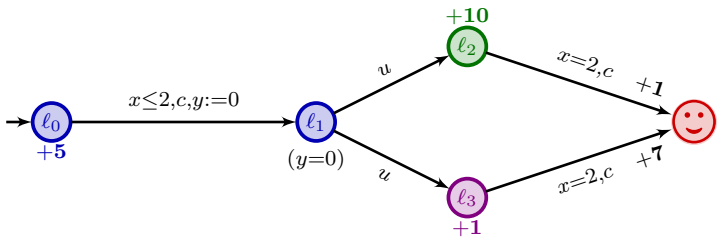
Weighted timed automaton. A *weighted timed automaton* is a timed automaton with an observer weight variable. It is defined as a tuple $\mathcal{A} = (L, \ell_0, L_F, X, \Sigma, T, \mathsf{weight})$, where:

- $(L, \ell_0, L_F, X, \Sigma, T)$ is a standard (diagonal-free) timed automaton, and
- $\mathsf{weight} : L \cup T \to \mathbb{Z}$ assigns a value to each location and to each transition.

We assume $\Sigma = T$, that is each transition can be identified with its label.

The weight of a delay move $(\ell, v) \xrightarrow{d} (\ell, v + d)$ is given by $d \cdot \mathsf{weight}(\ell)$: $\mathsf{weight}(\ell)$ is the rate in ℓ. The weight of a discrete move $(\ell, v) \xrightarrow{e} (\ell', v')$ is given by $\mathsf{weight}(e)$. The weight of a finite run ϱ is the accumulated weight of all moves that compose the run ϱ, we write $\mathsf{cost}(\varrho)$ for that value, and we call it the cost of ϱ.

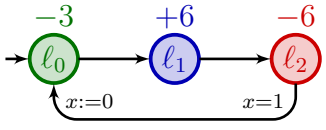
Example 21. We consider the following weighted timed automaton:

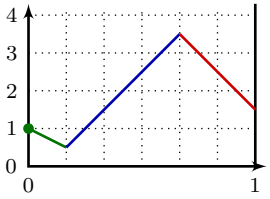


A possible execution for that system is:

	ℓ_0	$\xrightarrow{1.3}$	ℓ_0	$\xrightarrow{c}$	ℓ_1	$\xrightarrow{u}$	ℓ_3	$\xrightarrow{0.7}$	ℓ_3	$\xrightarrow{c}$	☺	
x	0		1.3		1.3		1.3		2			
y	0		1.3		0		0		0.7			
cost :		6.5	+	0	+	0	+	0.7	+	7		= 14.2

Example 22. We consider another weighted timed automaton:

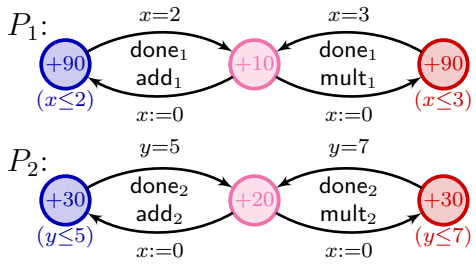




This corresponds to run $(\ell_0, 0) \xrightarrow{\text{delay}(\frac{1}{6})} (\ell_0, \frac{1}{6}) \to (\ell_1, \frac{1}{6}) \xrightarrow{\text{delay}(\frac{1}{2})} (\ell_1, \frac{2}{3}) \to (\ell_2, \frac{2}{3}) \dots$

Remark 19. Note that weighted timed automata can be seen as linear hybrid automata, or as rectangular automata. However the weight variable is never constrained, that is, the behaviour of the weighted timed automaton is that of the underlying timed automaton.

Example 23. Back to the taskgraph scheduling problem, we can refine the models for the processors and get:

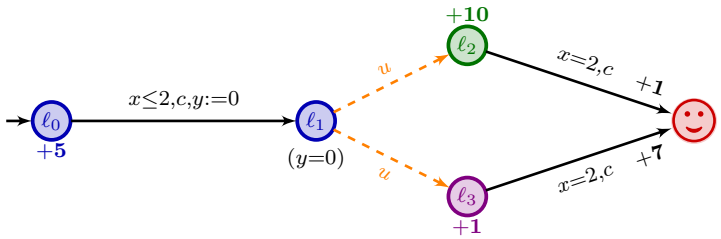


Weighted timed games. We will also be interested in reachability games. Weighted timed games are defined in a natural way. Given a winning strategy f for the controller (for the reachability objective), the cost of f is defined by:

$$\mathsf{cost}(f) = \sup\{\mathsf{cost}(\varrho) \mid \varrho \text{ ends in } L_F\}$$

If f is not a winning strategy, we assign it cost $+\infty$. Note that even if f is winning, it might be the case that $\mathsf{cost}(f) = +\infty$.

Example 24. The following is an example of weighted timed game:



Consider the strategy which consists in firing the first transition when $x = 0.7$. Its cost can be computed by:

$$5 \times (0.7) + \max(10 \times (2 - 0.7) + 1, 1 \times (2 - 0.7) + 7) = 3.5 + \max(14, 8.3) = 17.5$$

We will be interested in the following questions:

① Optimization questions
② Resource management questions

11.2 Optimal reachability in weighted timed automata

We first focus on the optimal reachability question: given a weighted timed automaton $\mathcal{A}$, can we compute the optimal cost that allows to reach the set of target states. The optimal cost is defined as

$$\mathsf{optcost}_{\mathcal{A}} = \inf\{\mathsf{cost}(\varrho) \mid \varrho \text{ run from } (\ell_0, \mathbf{0}) \text{ to } L_F\}$$

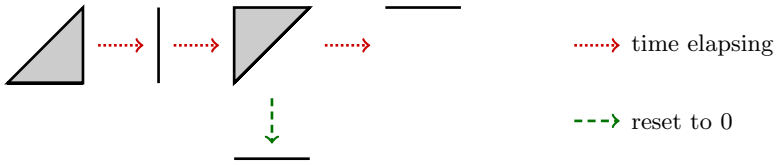
Example 25. If we consider the weighted timed automaton of Example 21, this optimal cost can be computed as:

$$\inf_{0\leq t\leq 2}\Big(5t+\min(10(2-t)+1,(2-t)+7)\Big)=9$$

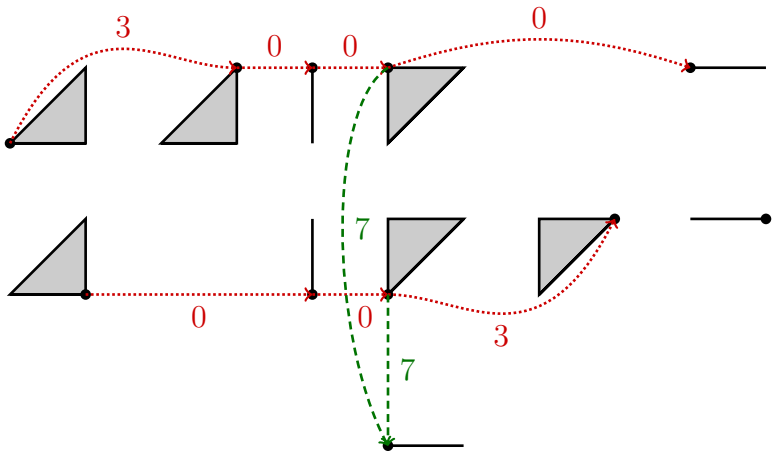
and the "strategy" is to take the first transition when $x=0$.

Theorem 16. *We can compute the optimal cost for reaching the target location in weighted timed automata in polynomial space.*

First notice that the region abstraction is too rough to correctly take into account the cost information:



Therefore the proof of this theorem relies on a refinement of the region automaton construction, called the *corner-point abstraction*, which is a finite weighted graph:



We fix a weighted timed automaton $\mathcal{A}$, and w.l.o.g. we assume all clocks are bounded by maximal constant M. We fix its set of diagonal-free regions R. A corner-point is a pair (r,α) where $r\in R$, and $\alpha\in\overline{r}$ with integral coordinates. Note that if r is given by the following order on fractional part $X_0<X_1<\cdots<X_p$ where (X_i) is a partition of the set of clocks, X_0 contains all clocks with integral values, for every i, the fractional parts of all clocks in X_i are equal, and all clocks in X_i have fractional part strictly larger than clocks in X_{i-1}, then there are $p+1$ corner-points to r, each one is characterized by $0\leq i\leq p$ such that the value of clocks in $X_0\cup X_1\cup\cdots\cup X_i$ is the integral part whereas the value of clocks in $X_{i+1}\cup\cdots\cup X_p$ is the integral part plus one.

We write R_{cp} for the set $\{(r,\alpha)\mid r\in R \text{ and } \alpha \text{ is a corner-point of } r\}$. We write r_0 for the region where all clocks have value 0, and α_0 for its unique corner-point.

Example 26. Region $0 < x < 1 \wedge 1 < y < 2 \wedge y - x < 1$ has three corner-points:

$$(x = 0, y = 1) \qquad (x = 1, y = 1) \qquad (x = 0, y = 2)$$

The *corner-point abstraction* $\mathcal{R}_{cp}(\mathcal{A})$ of $\mathcal{A}$ is a finite weighted graph $(Q, q_0, Q_F, \rightarrow)$ where $Q = L \times R_{cp}$, $q_0 = (r_0, \alpha_0)$, $Q_F = L \times R_{cp}$, and $\rightarrow \subseteq Q \times \mathbb{Z} \times Q$ is defined as follows:

- there is a transition $(\ell, r, \alpha) \xrightarrow{\mathsf{weight}(\ell)} (\ell, r, \alpha')$ if α and α' are corner-points of r (this is only possible if $\alpha' = \alpha + 1$)
- there is a transition $(\ell, r, \alpha) \xrightarrow{0} (\ell, r', \alpha)$ if r' is the immediate successor of r, and α is a corner-point of both r and r'
- there is a transition $(\ell, r, \alpha) \xrightarrow{\mathsf{weight}(e)} (\ell', r', \alpha')$ if edge $e = (\ell, g, Y, \ell')$ is such that $r \subseteq [\![g]\!]$, $r' = [Y \leftarrow 0]r$ and $\alpha' = [Y \leftarrow 0]\alpha$.[17]

We will show that this abstraction is sound and complete w.r.t. optimal reachability, that is $\mathsf{optcost}_{\mathcal{A}} = \mathsf{optcost}_{\mathcal{R}_{cp}(\mathcal{A})}$. More precisely we will prove the following proposition:

Proposition 11. *For every accepted finite run ϱ in $\mathcal{A}$, there is an accepted finite path π in $\mathcal{R}_{cp}(\mathcal{A})$ such that $\mathsf{cost}(\pi) \leq \mathsf{cost}(\varrho)$.*

For every accepted finite path π in $\mathcal{R}_{cp}(\mathcal{A})$, for every $\varepsilon > 0$, there is an accepted finite run ϱ in $\mathcal{A}$ such that $\mathsf{cost}(\varrho) \leq \mathsf{cost}(\pi) + \varepsilon$.

We first give some technical results.

This section contains technical results that will be useful in the following. Let A be a closed set of $\mathbb{R}^n$ (with $n \geq 1$). The *border* of A is denoted by $\mathsf{Border}_n(A)$ and is defined as $A \setminus \mathring{A}$ where $\mathring{A}$ denotes the interior of A. Let A be a closed set and x a point in $\mathbb{R}^n$. The following statements are equivalent and characterize the border of A:

- $x \in \mathsf{Border}_n(A)$
- $x \in A$ and for every $\varepsilon > 0$, there exists $y \notin A$ such that $\|x - y\|_\infty < \varepsilon$. [18]

Lemma 12. *Let $n \geq 1$ and Z be a bounded zone. Let f be a function*

$$f : (x_1, ..., x_n) \mapsto \sum_{i=1}^{n} c_i x_i + c$$

Then $\inf_Z f$ is obtained on the border of $\overline{Z}$ at integer coordinates.

Proof. The result is trivial when $n = 1$.

Assume now $n > 1$, and fix a compact set A. The minimum of f on A is obtained for some value α for x_1. Now, assume that $g_\alpha(x_2, ..., x_n) = f(\alpha, x_2, ..., x_n)$ (α is viewed as a parameter). Fix that α, and notice that $A \cap (x_1 = \alpha) \neq \varnothing$. The function g_α is defined on a subset of $\mathbb{R}^{n-1}$. We denote by B_α the projection of $A \cap (x_1 = \alpha)$ onto the last $n - 1$ coordinates, B_α is a compact set of $\mathbb{R}^{n-1}$. Note also that g_α is also affine. We know from

[17] Note that $[Y \leftarrow 0]\alpha$ is a corner-point of $[Y \leftarrow 0]r$ if α is a corner-point of r.

[18] $\|.\|_\infty$ represents the usual infinite norm defined as $\|(x_i)_{i=1...n}\|_\infty = \max\{|x_i| \mid i = 1...n\}$.

induction hypothesis that $\min_{B_\alpha} g_\alpha$ is obtained on $\mathsf{Border}_{n-1}(B_\alpha)$. To get the induction step, it is then sufficient to prove that if $(x_2, ..., x_n)$ is in $\mathsf{Border}_{n-1}(B_\alpha)$, then $(\alpha, x_2, ..., x_n)$ is in $\mathsf{Border}_n(A)$.

Pick $(x_2, ...x_n)$ in $\mathsf{Border}_{n-1}(B_\alpha)$ and $\varepsilon > 0$. Then $(x_2, ..., x_n) \in B_\alpha$ and there exists $(y_2, ..., y_n) \notin B_\alpha$ such that $\|(x_2, ..., x_n) - (y_2, ..., y_n)\|_\infty < \varepsilon$. B_α is the projection of $A \cap (x_1 = \alpha)$ onto the $n-1$ last coordinates, thus $(\alpha, x_2, ..., x_n) \in A \cap (x_1 = \alpha)$ and $(\alpha, y_2, ..., y_n) \notin A \cap (x_1 = \alpha)$. However, $\|(\alpha, x_2, ..., x_n) - (\alpha, y_2, ..., y_n)\|_\infty = \|(x_2, ..., x_n) - (y_2, ..., y_n)\|_\infty < \varepsilon$. We conclude that $(\alpha, x_2, ..., x_n)$ is in $\mathsf{Border}_n(A)$.

Consider now the case where f is defined on a bounded zone Z. Applying the previous result, we get that $\min_{\overline{Z}} f$ is obtained on $\mathsf{Border}_n(\overline{Z})$. Recall that $\overline{Z}$ can be obtained from Z by replacing the constraints $x - y < c$ by $x - y \leq c$ and that $\mathsf{Border}_n(\overline{Z})$ corresponds to the union of all the facets of $\overline{Z}$.[19]

The infimum of f on Z is thus on a facet whose equation is $x - y = c$ (resp. $x = c$). We eliminate variable x in f by replacing x with $y + c$ (resp. c) and we get a function g which has $n-1$ variables. Without loss of generality we can assume that $x = x_1$. We then use the property that $\min_{\overline{Z}} f = \min_{Z'} g$ where $Z' = \mathsf{proj}_{(x_2,...,x_n)}(\overline{Z} \cap x_1 = y + c)$ (resp. $Z' = \mathsf{proj}_{(x_2,...,x_n)}(\overline{Z} \cap x_1 = c)$).

We know by induction hypothesis that the minimum of g is obtained with each x_i ($i > 1$) having integer values. Thus, the minimum of f is obtained with $x_1 = y + c$ (resp. $x_1 = c$) which is an integer and all other clocks also have integer values. □

This lemma implies that the infimum of such a function f on a bounded zone Z is obtained in one of the extremal points of the zone.

Proof (of Proposition 11). We first prove the correctness, and then the soundness.

Correctness. Let $\varrho = (\ell_0, u_0) \to (\ell_0, u_0 + d_0) \to (\ell_1, u_1) \to (\ell_1, u_1 + d_1) \cdots \to (\ell_n, u_n)$ be a finite run in $\mathcal{A}$ (with alternating delay and discrete transitions). We set $t_0 = 0$, and for every $1 \leq i \leq n$, $t_i = \sum_{0 \leq j < i} d_j$. We moreover assume that this execution is read on the sequence of transitions $\ell_0 \xrightarrow{g_1, Y_1} \ell_1 \ldots \xrightarrow{g_n, Y_n} \ell_n$ in $\mathcal{A}$. Writing $t_0 = 0$, the cost of ϱ is given by:

$$f(t_1, ..., t_n) = \sum_{i=1}^{n} c_i(t_i - t_{i-1}) + c$$

where c_i's are the weights of the locations ℓ_i's, and c is the sum of all the discrete weights of transitions along ϱ.

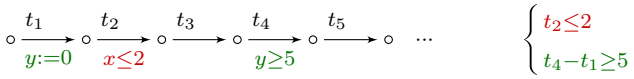
We want to minimize this function with the constraints that we have a run which is region-equivalent to ϱ:

- if $v_i(x) = t_i - t_j$ where $j = \max\{k \leq i \mid x \in Y_k\}$ and $v'_i(x) = t_{i+1} - t_j$ where $j = \max\{k \leq i \mid x \in Y_k\}$
- then $v_i \in r_i$, $v'_i \in r'_i$ where r_i (resp. r'_i) is the region of u_i (resp. $u_i + d_i$).

[19] A *facet* of a closed zone $\overline{Z}$ is an intersection $Z \cap (x = c)$ (or $Z \cap (x - y = c)$) where $x\{\leq, \geq\}c$ (or $x - y\{\leq, \geq\}c$) is a constraint, as tight as possible, defining Z.

This set of constraints on variables $(t_i)_{1\leq i\leq n}$ defines a zone Z. We can apply Lemma 12 and we get that the infimum of f on Z is obtained in (at least) a point with integer coordinates, say $(\alpha_i)_{i=1\dots n}$. Note that this point is in the closure of Z, and thus that it satisfies in particular the set of constraints $\{v_i \models \overline{r}_i,\ v'_i \models \overline{r}'_i\}$.

Example 27 (Optimal reachability as a linear programming problem). We illustrate the above construction:



We define the valuations $(\sigma_i)_{i=1\dots n}$ by $\sigma_i(x) = \alpha_i - \alpha_j$ where $j = \max\{k \leq i \mid x \in Y_k\}$. Each valuation σ_i is in $\overline{r_i}$ and has integer coordinates. It is thus a corner-point of r_i. Moreover, the sequence of valuations $(\sigma_i)_i$ would be an accepted sequence if we replace the constraints r_i by $\overline{r_i}$. In addition, the time elapsed in each state ℓ_i would then be $\alpha_{i+1} - \alpha_i$. It is technical but easy to build a corresponding path π in the corner-point abstraction $\mathcal{R}_{cp}(\mathcal{A})$.

As $(\alpha_i)_{i=1\dots n}$ minimizes f over $\overline{Z}$, we get that $\mathsf{cost}(\pi) \leq \mathsf{cost}(\varrho)$ and we are done.

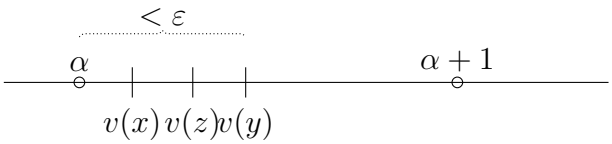
Completeness. Note that if all guards are closed, then this is straightforward. In the general case, it relies on the following lemma:

Lemma 13. *Let $\pi = (\ell_0, r_0, \alpha_0) \to (\ell_1, r_1, \alpha_1) \to \dots$ be a path in $\mathcal{A}_{cp}$. For every $\varepsilon > 0$, there exists a real run $\varrho_\varepsilon = (\ell_0, v_0) \to (\ell_1, v_1) \to \dots$ such that for every i, $v_i \in r_i$ and $\|\alpha_i - v_i\|_\infty < \varepsilon$.*

To prove this lemma we show the following: for every $(\ell, r, \alpha) \to (\ell', r', \alpha')$, for every $\varepsilon > 0$, for every $v \in r$ such that $\delta_\alpha(v) < \varepsilon$, there exists $(\ell, v) \to (\ell', v')$ in $\mathcal{A}$ such that $v' \in r'$ and $\delta_{\alpha'}(v') < \varepsilon$, where the *diameter of v w.r.t. α* is defined as $\delta_\alpha(v) = \max\Big(\max_x(|v(x) - \alpha_x|), \max_{x,y}(|(v(x) - \alpha_x) - (v(y) - \alpha_y)|)\Big)$.

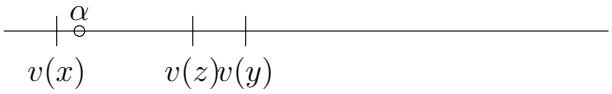
To prove this we distinguish between all cases for the transition $(\ell, r, \alpha) \to (\ell', r', \alpha')$.

- Assume $(\ell, r, \alpha) \to (\ell', r', \alpha')$ is a discrete transition and $\delta_\alpha(v) < \varepsilon$ with $v \in r$. Then let v' be such that $(\ell, v) \to (\ell', v')$. Let Y be the set of clocks which are reset along that move. Then $v'(y) = 0$ and $\alpha'_y = 0$ for every $y \in Y$, and $v'(y) = v(y)$ and $\alpha'_y = \alpha_y$ for every $y \notin Y$. Then obviously $\delta_{\alpha'}(v') \leq \delta_\alpha(v) < \varepsilon$.
- Assume $(\ell, r, \alpha) \to (\ell', r', \alpha')$ is a delay transition. There are several cases:
  - $(\ell, r, \alpha) \to (\ell, r, \alpha + 1)$



    In that case, write $\tau = \min\{v(x) - \alpha_x \mid x \in X\}$ and $\tau' = \max\{v(x) - \alpha_x \mid x \in X\}$, and define $v' = v + 1 - \tau' - \tau$. With this value we easily get $\delta_{\alpha+1}(v') = \delta_\alpha(v) < \varepsilon$.

- $(\ell, r, \alpha) \to (\ell, r', \alpha)$ where r' is the immediate successor of r. There are several cases, which depend on the position of v relative to α. We give only one case: not finished



This concludes the proof. □

Remark 20. The corner-point abstraction can also be used to compute the optimal mean-cost [BBL08] and the optimal time-discounted cost [FL08].

11.3 Optimal reachability in weighted timed games

The optimal cost is defined as

$$\mathsf{optcost}_{\mathcal{G}} = \inf\{\mathsf{cost}(f) \mid f \text{ winning strategy in } \mathcal{G}\}$$

Example 28. In the game of Example 24, the optimal cost can be computed as:

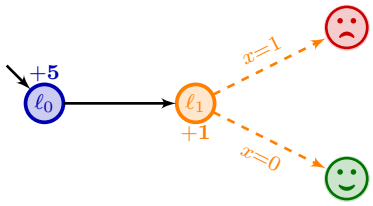
$$\inf_{0\le t\le 2}\Big(5t + \max(10(2-t)+1, (2-t)+7)\Big) = \inf_{0\le t\le 2} \max(21-5t, 9+4t) = 14+\frac{1}{3}$$

The optimal strategy is therefore to take the first transition when $t = \frac{4}{3}$.

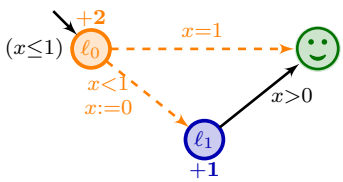
Corollary 3 (of the above example). *Neither the region abstraction nor the corner-point abstraction can be used to compute the optimal cost in weighted timed games.*

Remark 21. Even for one-clock games,

– there might no be optimal strategies



– optimal strategies might require memory

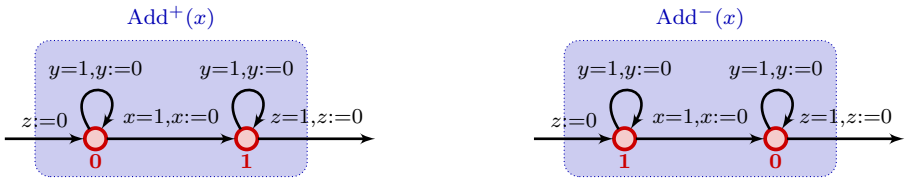


We consider the following decision problem, called the *threshold problem*: given a weighted timed game $\mathcal{G}$, a threshold c, does there exist a strategy f such that $\mathsf{cost}(f) \le c$?

Theorem 17. *For weighted timed games with three clocks or more, the threshold problem is undecidable.*

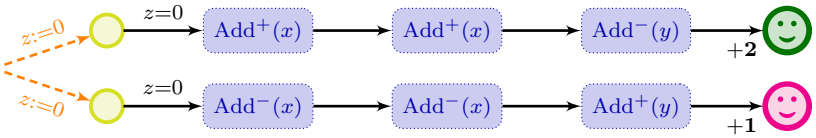
Proof. We first prove the theorem with four clocks. It relies on the following constructions:

– Given two clock values x_0 (for x) and y_0 (for y), we can check whether $y_0 = 2x_0$:



The cost is increased by x_0 The cost is increased by $1-x_0$

And then:



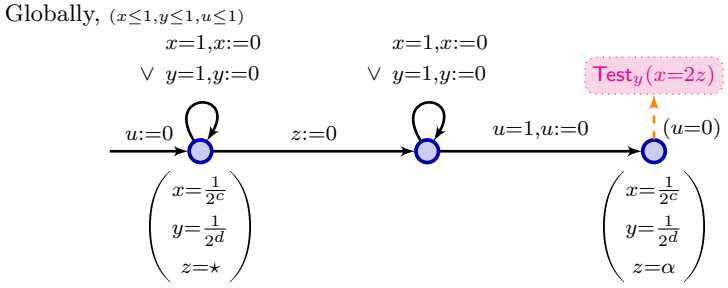
- In , cost $= 2x_0 + (1 - y_0) + 2$
  In , cost $= 2(1 - x_0) + y_0 + 1$
- if $y_0 < 2x_0$, player 2 chooses the first branch: cost > 3
  if $y_0 > 2x_0$, player 2 chooses the second branch: cost > 3
  if $y_0 = 2x_0$, in both branches, cost $= 3$

Lemma 14. *Player 1 has a winning strategy with cost* ≤ 3 *iff* $y_0 = 2x_0$

– Player 1 will simulate a two-counter machine:
- each instruction is encoded as a module;
- the values c_1 and c_2 of the counters are encoded by the values of two clocks:

$$x = \frac{1}{2^{c_1}} \qquad \text{and} \qquad y = \frac{1}{3^{c_2}}$$

when entering the corresponding module.
- The module for incrementing the first counter is depicted below:



Lemma 15. *The two-counter machine has an halting computation iff player 1 has a winning strategy to ensure a cost no more than* 3*.*

The number of clocks used can be reduced to three by encoding the two counters with a single clock $x = \frac{1}{2^c 3^d}$. The modules for decrementing and incrementing can be inferred from the previous construction. The test-to-zero is more involved, and, for instance for counter c, requires to increase the counter d until the value of the clock is 1. This is only possible if the initial value of the clock is of the form $\frac{1}{3^d}$. □

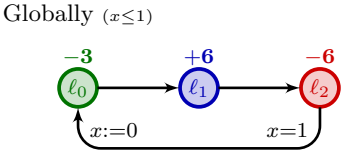
On the other hand we can prove the following theorem:

Theorem 18 ([BLMR06]). *Optimal cost in turn-based one-clock weighted games can be computed.*

11.4 Managing resources in weighted timed automata and games

We only give some examples of behaviours and energy constraints.

Example 29. Consider the following weighted timed automaton:



- L-problem: lower-bound on the cost

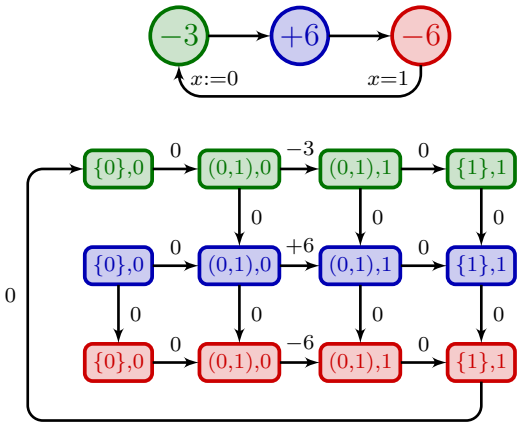
  (take example with $L = 0$)
- LU-problem: lower and upper bounds on the cost

  (take example with $L = 0$ and $U \in \{3, 2, 1\}$)
- LW-problem: lower and weak-upper bounds on the cost

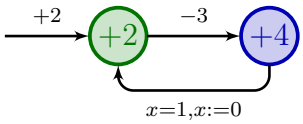
  (take example with $L = 0$ and $U = 1$)

Some results

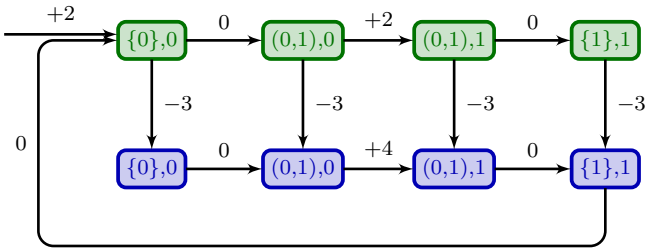
- Mean-payoff games (and hence parity games) and L-games (untimed case) have the same complexity (there log-space reducible).
- The corner-point abstraction can be used to solve the L-problem and the LW-problem in one-clock weighted automata with no discrete costs.



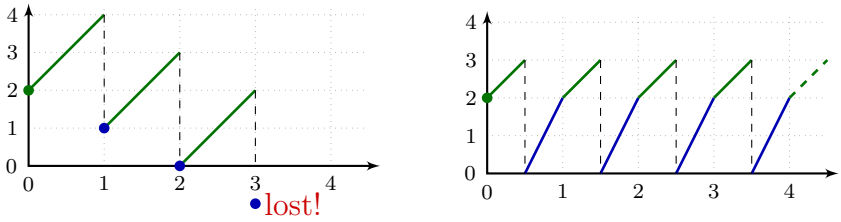
– The corner-point abstraction cannot be used to solve the L-problem in one-clock weighted automata



Its corner-point abstracion:



This corresponds to the behavour on the left, whereas on the right there is a feasible schedule which satisfies the lower-bound constraint.



– The L-problem is undecidable in one-clock weighted games.
– ...

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