Table 6.2. Closure and expressiveness properties of asynchronous automata

	$\cup$	$\cap$	$\overline{\cdot}$	det	EMSO	MSO	Empt.
$\mathcal{AA}^+$	✓	✓	✓	$=$	$=$	$=$	✓
$\mathcal{AA}^-$	✓	✓	✓	$=$	$=$	$=$	✓
$\mathcal{PA}^+$	✓	✓	✓	$=$	$\subsetneq$	$\subsetneq$	✓
$\mathcal{PA}^-$	✓	✓	✓	$=$	$\subsetneq$	$\subsetneq$	✓

6.7 Bibliographic Notes

A comprehensive overview of the theory of Mazurkiewicz traces and related automata models is provided by [27], which covers many important research lines of trace theory, including an algebraic view thereof. Recognizability of trace languages is the subject of [79]. In [96], recognizability and definability in terms of MSO logic are considered in terms of traces, MSCs, graphs, and infinite structures. Comparisons of the models of ACAs and asynchronous automata relative to traces can be found in [28] and [82]. Product automata and product trace languages are studied in [88].

$\{1, \ldots, B\}$, which leads to a finite automaton over $\Gamma \times \{1, \ldots, B\}$ recognizing $Lin(Tr_B^\alpha(L))$.

"If": As shown in [45, 56], any asynchronous automaton over $\widetilde{\Gamma}_B$ has an equivalent counterpart in the form of a (strongly) $\forall$-bounded communicating finite-state machine (cf. Chap. 8). Together with Theorems 6.22 and 8.22, this proves the lemma. □

Proposition 7.32. *For any $\alpha \in \{+,-\}$, $B \geq 1$, and $L \subseteq \mathbb{MSC}_{\forall B}$,*

$$L \in \mathcal{RP}^0_{\mathrm{MSC}} \quad \textit{iff} \quad Tr_B^\alpha(L) \in \mathcal{RP}^0_{\mathbb{TR}^\alpha(\widetilde{\Gamma}_B)} \wedge \{\mathcal{M} \mid L \vdash_{Ag} \mathcal{M}\} \subseteq \mathbb{MSC}_{\forall B}$$

Proof. According to Proposition 7.31, the operator Tr_B^α and its converse both preserve regularity.

"Only if": Suppose $L \subseteq \mathbb{MSC}_{\forall B}$ to be a weak regular product MSC language. Recall that $Tr_B^\alpha(L)$ is a regular trace language over $\widetilde{\Gamma}_B = (\overline{\Gamma}_\gamma)_{\gamma \in Ag \cup Co}$. Moreover, let $\mathcal{T} \in \mathbb{TR}^\alpha(\widetilde{\Gamma}_B)$ such that, for any $\gamma \in Ag \cup Co$, there is a trace $\mathcal{T}_\gamma \in Tr_B^\alpha(L)$ satisfying $\mathcal{T}_\gamma \restriction \gamma = \mathcal{T} \restriction \gamma$. Then, $\mathcal{T} \in Tr_B^\alpha(\mathbb{MSC}_{\forall B})$ and, in particular, we have $\mathcal{T}_i \restriction i = \mathcal{T} \restriction i$ and, thus, $(Tr_B^\alpha)^{-1}(\mathcal{T}_i) \restriction i = (Tr_B^\alpha)^{-1}(\mathcal{T}) \restriction i$ for any $i \in Ag$, which implies $(Tr_B^\alpha)^{-1}(\mathcal{T}) \in L$ and $\mathcal{T} \in Tr_B^\alpha(L)$.

"If": Suppose $L \subseteq \mathbb{MSC}_{\forall B}$ to generate a weak regular trace language over $\widetilde{\Gamma}_B$, i.e., $Tr_B^\alpha(L) \in \mathcal{RP}^0_{\mathbb{TR}^\alpha(\widetilde{\Gamma}_B)}$, and let $\mathcal{M} \in \mathbb{MSC}_{\forall B}$ such that, for any $i \in Ag$, there is $\mathcal{M}_i \in L$ with $\mathcal{M}_i \restriction i = \mathcal{M} \restriction i$. Trivially, we have that, for any $i \in Ag$, $Tr_B^\alpha(\mathcal{M}_i) \restriction i = Tr_B^\alpha(\mathcal{M}) \restriction i$. Moreover, for any $\gamma = (i!j, j?i, n) \in Co$, $Tr_B^\alpha(\mathcal{M}_i) \restriction \gamma = Tr_B^\alpha(\mathcal{M}) \restriction \gamma$ (note that also $Tr_B^\alpha(\mathcal{M}_j) \restriction \gamma = Tr_B^\alpha(\mathcal{M}) \restriction \gamma$). This is because, in the trace of a $\forall B$-bounded MSC, the n-th receipt of a message through (i,j) is ordered before sending a message from i to j for the $(n+B)$-th time. Altogether, we have $Tr_B^\alpha(\mathcal{M}) \in Tr_B^\alpha(L)$ and, consequently, $\mathcal{M} \in L$. □

The proof of the following extension towards finite unions of weak regular product languages is left to the reader as an easy exercise.

Corollary 7.33. *For any $\alpha \in \{+,-\}$, $B \geq 1$, and $L \subseteq \mathbb{MSC}_{\forall B}$,*

$$L \in \mathcal{RP}_{\mathrm{MSC}} \quad \textit{iff} \quad L = L_1 \cup \ldots \cup L_n \textit{ with}$$
$$Tr_B^\alpha(L_i) \in \mathcal{RP}^0_{\mathbb{TR}^\alpha(\widetilde{\Gamma}_B)} \wedge \{\mathcal{M} \mid L_i \vdash_{Ag} \mathcal{M}\} \subseteq \mathbb{MSC}_{\forall B} \ \forall i$$

A nonempty partial MSC $\mathcal{M}$ is called *prime* if $\mathcal{M} = \mathcal{M}_1 \cdot \mathcal{M}_2$ implies $\mathcal{M}_1 = \mathbf{1}_{\mathrm{MSC}}$ or $\mathcal{M}_2 = \mathbf{1}_{\mathrm{MSC}}$. Consider Fig. 7.16, for example. The partial MSCs from parts (a) and (d) are prime, while the partial MSCs in between are not. For the rest of this section, let Π be a nonempty finite set of prime partial MSCs, which will be the universe of a trace alphabet. The notion of prime partial MSCs, which was first given in [43], gives rise to a natural dependence relation based on the distributed alphabet $\widetilde{\Sigma}_\Pi := (\Sigma_i)_{i \in Ag}$ where, for any $i \in Ag$, $\Sigma_i = \{\mathcal{M} \in \Pi \mid i \in Ag(\mathcal{M})\}$. We may accordingly declare prime partial MSCs $\mathcal{M}$ and $\mathcal{M}'$ independent if $Ag(\mathcal{M}) \cap Ag(\mathcal{M}') = \emptyset$.