Logic

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Exercise 1
Give LK_0 derivations for each of the following formulas :

1. $\phi \vee (\psi \wedge \rho) \equiv (\phi \vee \psi) \wedge (\phi \vee \rho)$
2. $\neg\neg\phi \Rightarrow \phi$
3. $\phi \vee (\phi \Rightarrow \psi)$

Exercise 2
We introduce in Figure 1 the natural deduction system NK_0. We will show that it is another sound and complete deduction system for classical propositional logic. It deals with the same kind of sequents as LJ_0, *i.e.*, they have a single formula on the right. However it features the raa (*reductio ad absurdum*) for building proofs by contradiction. In sequent calculus, we introduce connectives in the conclusion of inference rules, on either side of the sequent — we talk of left and right (introduction) rules. In natural deduction, we either introduce a connective on the right side of the conclusion sequent, or eliminate a connective on the right side of the first premise sequent — we talk of introduction and elimination rules.

1. Show that $\Gamma \vdash_{NK_0} P$ implies $\Gamma \vdash_{LK_0} P$.
2. Prove that $\Gamma \vdash_{LK_0} \phi_1, \ldots, \phi_n$ entails $\Gamma, \neg\phi_1, \ldots, \neg\phi_n \vdash_{NK_0} \bot$.
3. Establish that $\Gamma \vdash_{LK_0} \phi_1, \ldots, \phi_n$ yields $\Gamma \vdash_{NK_0} \phi_1 \vee \ldots \vee \phi_n$.

Exercise 3
The intuitionistic natural deduction system NJ_0 is obtained from NK_0 by removing the raa rule. Show that this system is sound and complete for intuitionistic logic.

Exercise 4
We consider the rule

$$\frac{\Gamma \vdash Q}{\Gamma \vdash P} \text{ (restart)}$$

with the proviso that it only applies when Q is the conclusion of a sequent previously encountered in the proof, *i.e.*, there is a sequent $\Delta \vdash Q$ on the path from the final conclusion of the derivation and the conclusion of the *restart* rule.

1. Show that NJ_0 + restart allows to derive the excluded middle.

$$\frac{}{\Gamma, \phi \vdash \phi}\ \text{ax} \qquad \frac{\Gamma \vdash \neg\neg\phi}{\Gamma \vdash \phi}\ \text{raa}$$

$$\frac{}{\Gamma \vdash \top}\ \top_i \qquad \frac{\Gamma \vdash \bot}{\Gamma \vdash \psi}\ \bot_e$$

$$\frac{\Gamma \vdash \phi_1 \quad \Gamma \vdash \phi_2}{\Gamma \vdash \phi_1 \wedge \phi_2}\ \wedge_i \qquad \frac{\Gamma \vdash \phi_1 \wedge \phi_2}{\Gamma \vdash \phi_i}\ \wedge_e$$

$$\frac{\Gamma \vdash \phi_i}{\Gamma \vdash \phi_1 \wedge \phi_2}\ \vee_i \qquad \frac{\Gamma \vdash \phi_1 \vee \phi_2 \quad \Gamma, \phi_1 \vdash \psi \quad \Gamma, \phi_2 \vdash \psi}{\Gamma \vdash \psi}\ \vee_e$$

$$\frac{\Gamma, \phi \vdash \psi}{\Gamma \vdash \phi \Rightarrow \psi}\ \Rightarrow_i \qquad \frac{\Gamma \vdash \phi \Rightarrow \psi \quad \Gamma \vdash \phi}{\Gamma \vdash \psi}\ \Rightarrow_e$$

FIGURE 1 – Rules of NK_0

2. Show that NJ_0 + restart is sound and complete for classical logic.

Exercise 5
The calculus LJ_0^- is given in Figure 2. Note that it does not contain cut and structural rules, and that its axiom rule is restricted to propositional variables.

1. Show that if $\Gamma, \phi \wedge \psi \vdash \rho$ is derivable in LJ_0^- then so does $\Gamma, \phi, \psi \vdash \rho$.
2. Show that if $\Gamma, \phi \vee \psi \vdash \rho$ is derivable in LJ_0^- then so does $\Gamma, \phi \vdash \rho$.
3. Show that if $\Gamma, \phi \Rightarrow \psi \vdash \rho$ is derivable in LJ_0^- then so does $\Gamma, \psi \vdash \rho$.
4. Show that contraction is admissible in LJ_0^-.
5. Show that LJ_0^- and LJ_0 derive the same sequents.
6. Show that the calculus would not be complete for intuitionistic logic if the implication formula would not be kept in the left premise of the $\Rightarrow_L$ rule. (Hint : consider $\neg\neg(A \vee \neg A)$.)

$$\frac{}{\Gamma, \bot \vdash \phi}\ \bot_L \qquad \frac{}{\Gamma, P \vdash P}\ \text{axiom} \qquad \frac{}{\Gamma \vdash \top}\ \bot_R$$

$$\frac{\Gamma, \phi_1, \phi_2 \vdash \psi}{\Gamma, \phi_1 \wedge \phi_2 \vdash \psi}\ \wedge_L \qquad \frac{\Gamma, \phi_1 \vdash \psi \quad \Gamma, \phi_2 \vdash \psi}{\Gamma, \phi_1 \vee \phi_2 \vdash \psi}\ \vee_L \qquad \frac{\Gamma, \phi_1 \Rightarrow \phi_2 \vdash \phi_1 \quad \Gamma, \phi_2 \vdash \psi}{\Gamma, \phi_1 \Rightarrow \phi_2 \vdash \psi}\ \Rightarrow_L$$

$$\frac{\Gamma \vdash \phi_1 \quad \Gamma \vdash \phi_2}{\Gamma \vdash \phi_1 \wedge \phi_2}\ \wedge_R \qquad \frac{\Gamma \vdash \phi_i}{\Gamma \vdash \phi_1 \vee \phi_2}\ \vee_R \qquad \frac{\Gamma, \phi \vdash \psi}{\Gamma \vdash \phi \Rightarrow \psi}\ \Rightarrow_R$$

FIGURE 2 – The LJ_0^- intuitionistic sequent calculus