Classical Sequent Calculus

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1 Classical sequent calculus

We introduce classical sequent calculus, in a way that enables a very simple proof of completeness, with cut elimination as a corollary. This motivates a slightly different style from the one used for LJ_0 concerning the handling of contexts.

1.1 The calculus LK_0

Definition 1.1. A classical sequent $\Gamma \vdash \Delta$ is built from two multisets of formulas of $\mathcal{F}(\mathcal{P})$. The sequent $\phi_1, \ldots, \phi_n \vdash \psi_1, \ldots, \psi_m$ should be read as "the *conjunction* of the ϕ_i implies the *disjunction* of the ψ_j".

Definition 1.2. The rules of LK_0 are given in Figure 1.

Note that the rules for negation can be derived from those for implication, following the of usual reading $\neg\phi$ as $\phi \Rightarrow \bot$. We also note that structural rules and cut are missing; we will comment on this after the next theorem.

1.2 Soundness and completeness

Definition 1.3. A sequent $\Gamma \vdash \Delta$ is said to be *valid*, noted $\Gamma \models \Delta$ when, for all interpretation $I : \mathcal{P} \to \{0,1\}$ such that $I \models \Gamma$, there is some $\phi \in \Delta$ such that $I \models \phi$.

Theorem 1.4. The calculus LK_0 is sound and complete for propositional classical logic: $\Gamma \vdash_{\mathrm{LK}} \Delta$ iff $\Gamma \models \Delta$.

Proof. Soundness amounts to check that each rule of LK_0 preserves validity: if the premises are valid then the conclusion is valid. In fact, in LK_0, the converse holds, which is rare: if the conclusion is valid, then so are the premises. Together with the fact that premises of logical rules contain less logical connectives than their conclusions, this allows to easily prove that each valid sequent has a derivation in LK_0, by induction on the number of connectives of the sequent. □

Obviously, the above proof of completeness still holds if we add more rules. We could add the cut rule, expressed as follows:

$$\frac{\Gamma \vdash \Delta, \phi \quad \phi, \Gamma \vdash \Delta}{\Gamma \vdash \Delta}$$

$$\overline{\Gamma, \phi \vdash \phi, \Delta}$$

$$\overline{\Gamma, \bot \vdash \Delta} \qquad \overline{\Gamma \vdash \top, \Delta}$$

$$\frac{\Gamma, \phi_1, \phi_2 \vdash \Delta}{\Gamma, \phi_1 \wedge \phi_2 \vdash \Delta} \qquad \frac{\Gamma \vdash \phi_1, \Delta \quad \Gamma \vdash \phi_2, \Delta}{\Gamma \vdash \phi_1 \wedge \phi_2, \Delta}$$

$$\frac{\Gamma, \phi_1 \vdash \Delta \quad \Gamma, \phi_2 \vdash \Delta}{\Gamma, \phi_1 \vee \phi_2 \vdash \Delta} \qquad \frac{\Gamma \vdash \phi_1, \phi_2, \Delta}{\Gamma \vdash \phi_1 \vee \phi_2, \Delta}$$

$$\frac{\Gamma \vdash \phi_1, \Delta \quad \Gamma, \phi_2 \vdash \Delta}{\Gamma, \phi_1 \Rightarrow \phi_2 \vdash \Delta} \qquad \frac{\Gamma, \phi_1 \vdash \phi_2, \Delta}{\Gamma \vdash \phi_1 \Rightarrow \phi_2, \Delta}$$

$$\frac{\Gamma \vdash \phi, \Delta}{\Gamma, \neg\phi \vdash \Delta} \qquad \frac{\Gamma, \phi \vdash \Delta}{\Gamma \vdash \neg\phi, \Delta}$$

Figure 1: Sequent calculus LK_0 for propositional classical logic

It is easy to check that it is sound. The cut rule provides a form of indirect, intelligent reasoning that is hard to exploit in automated proofs but can lead to shorter proofs built by humans with insight.

Structural rules would be expressed as follows:

$$\frac{\Gamma \vdash \Delta}{\Gamma, \phi \vdash \Delta} \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \phi, \Delta} \qquad \frac{\Gamma, \phi, \phi \vdash \Delta}{\Gamma, \phi \vdash \Delta} \qquad \frac{\Gamma \vdash \phi, \phi, \Delta}{\Gamma \vdash \phi, \Delta}$$

Weakening can be useful to simplify proofs by removing unnecessary formulas — again, this only applies to proof built by smart humans, and a priori not to automatically constructed proofs. Contraction is quite useless in LK_0 as defined here. Both rules would make more sense (and may be necessary) in variants of the calculus where the contexts Γ and Δ would be split rather than shared between the several premises of some rules, in a similar way as we did for LJ_0.

1.3 Symmetries of LK_0

The rules of LK_0 exhibit some striking symmetries. The $\top_R$ rule and the $\bot_L$ rules are similar: $\top$ on the right of sequents is treated in the same way as $\bot$ on the left. The same goes for conjunction and disjunction: the rules $\vee_R$ and $\wedge_L$ are similar, taking place on opposite sides of the sequent; the same goes for $\wedge_R$ and $\vee_L$ which both branch.

This symmetry is strongly related to the dualities exhibited by de Morgan laws, and more generally by negation elimination rules:

$$\begin{array}{lll@{\qquad\qquad}lll}
\neg(\phi \wedge \psi) & \equiv_\perp & \neg\phi \vee \neg\psi & \neg(\phi \vee \psi) & \equiv_\perp & \neg\phi \wedge \neg\psi \\
\neg\top & \equiv_\perp & \perp & \neg\perp & \equiv_\perp & \top \\
\phi \Rightarrow \psi & \equiv_\perp & \neg\phi \vee \psi & \neg\neg\phi & \equiv_\perp & \phi
\end{array}$$

We recall that these rules allow to turn any formula into its negation normal form, where negation can only occur directly above propositional variables.

We can now observe our symmetries more concretely, by showing that if we replace one formula in a sequent according to one of the above laws, the structure of the proof can be adapted in a straightforward local fashion. For instance, see how we can adapt the proof structure following the law $\phi \wedge \psi \equiv_\perp \neg(\neg\phi \vee \neg\psi)$ applied on the left of a sequent:

$$\dfrac{\Gamma, \phi, \psi \vdash \Delta}{\Gamma, \phi \wedge \psi \vdash \Delta}\ \wedge_L \qquad \longleftrightarrow \qquad \dfrac{\dfrac{\dfrac{\Gamma, \phi, \psi \vdash \Delta}{\Gamma \vdash \neg\phi, \neg\psi, \Delta}\ \neg_R \times 2}{\Gamma \vdash \neg\phi \vee \neg\psi, \Delta}\ \vee_R}{\Gamma, \neg(\neg\phi \vee \neg\psi) \vdash \Delta}\ \neg_L$$

On the right of a sequent:

$$\dfrac{\Gamma \vdash \phi, \Delta \quad \Gamma \vdash \psi, \Delta}{\Gamma \vdash \phi \wedge \psi, \Delta}\ \wedge_R \qquad \longleftrightarrow \qquad \dfrac{\dfrac{\dfrac{\Gamma \vdash \phi, \Delta}{\Gamma, \neg\phi \vdash \Delta}\ \neg_L \quad \dfrac{\Gamma \vdash \psi, \Delta}{\Gamma, \neg\psi \vdash \Delta}\ \neg_L}{\Gamma, \neg\phi \vee \neg\psi \vdash \Delta}\ \vee_L}{\Gamma \vdash \neg(\neg\phi \vee \neg\psi), \Delta}\ \neg_R$$

Finally, we illustrate the effect of $\phi \Rightarrow \psi \equiv_\perp \neg\phi \vee \psi$ on the left of a sequent:

$$\dfrac{\Gamma \vdash \phi, \Delta \quad \Gamma, \psi \vdash \Delta}{\Gamma, \phi \Rightarrow \psi \vdash \Delta}\ \Rightarrow_L \qquad \longleftrightarrow \qquad \dfrac{\dfrac{\Gamma \vdash \phi, \Delta}{\Gamma, \neg\phi \vdash \Delta}\ \neg_L \quad \Gamma, \psi \vdash \Delta}{\Gamma, \neg\phi \vee \psi \vdash \Delta}\ \vee_L$$

These remarks are very useful to understand the design of LK_0, or memorize it: once you have half of the rules, the other half can be automatically derived by symmetry. We can in fact make this more formal, by designing a *one-sided* variant of LK_0 in which $\equiv_\perp$ is internalized and only one of the two symmetric halves of LK_0 is kept.

Definition 1.5. The one-sided classical sequent calculus has sequents of the form $\vdash \Delta$ in which formulas are implicitly identified modulo $\equiv_\perp$. It has only four rules:

$$\dfrac{}{\vdash \top, \Delta}\ \top \qquad \dfrac{}{\vdash \phi, \neg\phi, \Delta}\ \text{axiom}$$

$$\dfrac{\vdash \phi, \psi, \Delta}{\vdash \phi \vee \psi, \Delta}\ \vee \qquad \dfrac{\vdash \phi, \Delta \quad \vdash \psi, \Delta}{\vdash \phi \wedge \psi, \Delta}\ \wedge$$

Proposition 1.6. The sequent $\phi_1, \ldots, \phi_n \vdash \Delta$ is derivable in LK_0 iff $\vdash \Delta, \neg\phi_1, \ldots, \neg\phi_n$ is derivable in the one-sided variant.

Proof. Simple induction on the derivations, building on the above observations. □

Note here that the simplication of LK_0 into its one-sided variant comes at the cost of considering formulas up to the equivalence relation $\equiv_\perp$. This is acceptable because the equivalence is simple and can be computed very easily. Doing the same with the full logical equivalence would not be acceptable: it would yield to a trivial deduction system with only one rule for deriving $\top$, but checking proofs would take exponential time.