The reachability problem for Vector Addition Systems with one zero-test

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Abstract. We consider here a variation of Vector Addition Systems where one counter can be tested for zero. We extend the reachability proof for Vector Addition System recently published by Leroux to this model. This provides an alternate, more conceptual proof of the reachability problem that was originally proved by Reinhardt.

1 Introduction

Context Petri Nets, Vector Addition Systems (VAS) and Vector Addition System with control states (VASS) are equivalent well known classes of counter systems for which the reachability problem is decidable ([9], [5], [8]). If we add to VAS the ability to test at least two counters to zero, one obtains a model equivalent to Minsky machines, for which all non-trivial properties are undecidable. The study of VAS with a *single* zero-test transition (VAS_0) began recently, and already a reasonable number of results are known for this model. Reinhardt [11] has shown that the reachability problem is decidable for VAS_0 (as well as for hierarchical zero-tests). Abdulla and Mayr have shown that the coverability problem is decidable in [1] by using the backward procedure of Well Structured Transition Systems. The boundedness problem (whether the reachability set is finite), the termination and the reversal-boundedness problem (whether the counters can alternate infinitely often between the increasing and the decreasing modes) are all decidable by using a forward procedure, a finite but *non-complete* Karp and Miller tree provided by Finkel and Sangnier in [4]. The decidability of the place-boundedness problem (whether one specific counter is unbounded), and more generally the possibility to compute a finite representation of the downward closure of the reachability set have been shown by Bonnet, Finkel, Leroux and Zeitoun in [3] using the notion of *productive sequences*.

The reachability problem The decidability of reachability for VAS has been originally solved by Mayr (1981, [9]) and Kosaraju (1982, [5]). Lambert later simplified these proofs (1992, [6]) while still using the same proof techniques. Recently, Leroux gave another way to prove of

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this problem, by using Presburger invariants and productive sequences ([7], [8]).
The history of the reachability problem for VAS_0 is shorter. The only proofs are the different versions of Reinhardt proof (original unpublished manuscript in 1995 [10], then published in 2008, [11]), which is based on showing that any expression representing a reachability problem can be put under a "normal form" for which satisfiability is easy to solve. However, the definition of the normal form is complex, and the proof of termination of the algorithm reducing any expression to the normal form is difficult to understand. Since this publication, some new results were found by reduction to reachability in VAS_0, for example decidability of minimal cost reachability in the Priced Timed Petri Nets of Abdulla and Mayr [1], and the decidability of reachability in a restricited class of pushdown counter automatas by Atig and Ganty [2].

Our contribution We propose here an alternate proof of decidability of reachability in VAS_0, using the principles Leroux introduced in [8]. The similarity between our proof with Leroux proof hopefully makes it easier to understand.

2 Preliminaries

Sets: $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{Q}_{\geq 0}$ refers respectively to non-negative integers, integers, rationnals and non-negative rationals. We define addition for $X, Y \subseteq \mathbb{Q}^d$ by $X + Y = \{x + y \mid x \in X, y \in Y\}$ and multiplication for $X \subseteq \mathbb{Q}^d$, $K \subseteq \mathbb{Q}$ by $K * X = \{k * x \mid x \in X, k \in K\}$. We also define $k \star X$ ($k \in \mathbb{N}$) by $0 \star X = \{0\}$ and $(k+1) \star X = X + (k \star X)$ and we generalize this notation to $K \subseteq \mathbb{N}$ by $K \star X = \bigcup_{k \in K}(k \star X)$. A function f from $\mathbb{N}^d$ (resp. $\mathbb{Q}^d_{\geq 0}$) to $\mathbb{N}^{d'}$ (resp. $\mathbb{Q}^{d'}_{\geq 0}$) is *linear* if $f(x + y) = f(x) + f(y)$ and for $k \in \mathbb{N}$ (resp. $k \in \mathbb{Q}_{\geq 0}$), $f(k * x) = k * f(x)$ We will also allow ourselves to shorten the singleton $\{x\}$ as x when the risk of confusion is low. $X \subseteq \mathbb{Q}^d$ is a *vector space* if $\mathbb{Q}X \subseteq X$ and $X + X \subseteq X$. Finally, we define $\mathbb{N}^d_0 = \{0\} \times \mathbb{N}^{d-1}$.

Relations: A relation on X is a set $R \subseteq X \times X$. We will write $x\ R\ y$ to mean $(x, y) \in R$. Composition of relations on X is defined by $R \circ R' = \{(x, y) \in X \times X \mid \exists z \in X,\ (x, z) \in R \wedge (z, y) \in R'\}$. We shorten $R \circ R'$ as RR' when there is no ambiguousity. R^* is the transitive closure of R. For R a relation on X and $X' \subseteq X$, we define $R(X') = \{y \in X \mid \exists x \in X', (x, y) \in R\}$. A set $X' \subseteq X$ is a *R-forward invariant* if $R(X') \subseteq X'$. We define R^{-1} by $R^{-1} = \{(x, y) \in X \times X \mid (y, x) \in R\}$. A set $X' \subseteq X$ is a *R-backward invariant* if it is a R^{-1}-forward invariant. Similarly, for f a function from X to Y, we define $f(X') = \{y \in Y \mid \exists x \in X', y = f(x)\}$.

Words, Parikh Images: Given X a set, the set of words on X is written X^*. A word $w \in X^*$ is written $a_1 a_2 \ldots a_n$ with $a_i \in X$ or optionally $\prod_{1 \leq i \leq n} a_i$. A language L is a subset of X^*. The concatenation of two words $w_1, w_2 \in X^*$ is written $w_1 w_2$ and we extend this notation to languages by $LL' = \{uv \mid u \in L, v \in L'\}$. $\mathbb{N}^X$ is the set of functions from X to $\mathbb{N}$. For $u \in X^*$, the *Parikh image* $|u| \in \mathbb{N}^X$ is defined by $|u|(x) =$ 'number of x's in u'.

Orders, Well-orders: An order $\preceq$ on a set X is a transitive, reflexive and antisymmetric relation on X. The relation $\prec$ is defined by $x \prec y$ iff $x \preceq y$ and $x \neq y$. An element $x \in X$ is *minimal* if there exists no $x' \in X$, $x' \prec x$. The order $\preceq$ is a well-order on X if for all sequences $(x_i)_{i \in \mathbb{N}}$ with $x_i \in X$, there exists $i < j$ with $x_i \preceq x_j$. If X is well-ordered by $\preceq$, then all subsets of X admit a finite number of minimal elements. Common well-orders are $\leq$ on $\mathbb{N}$ and $\leq$ on $X \times Y$ when X is well-ordered by $\leq_X$, Y is well-ordered by $\leq_Y$ and $(x,y) \leq (x',y') \iff x \leq_X x' \wedge y \leq_Y y'$.

Word embedding, Higman lemma: If X is ordered by $\leq$, we define $\leq^{emb}$ (the *word embedding* order) on X^* is ordered by $a_i \dots a_n \leq^{emb} b_1 \dots b_p$ if there exists a strictly increasing function φ from $\{1,\dots,n\}$ to $\{1,\dots,p\}$ such that $\forall i \in \{1,\dots,n\},\ a_i \leq b_{\varphi(i)}$. If $\leq$ is a well-order on X, then $\leq^{emb}$ is a well-order on X^* (Higman's lemma)

3 Vector Addition Systems with one zero-test

3.1 Transition systems

Definition 1. *A* Labelled Transition System *(LTS) is a tuple $\langle X, A, \rightarrow \rangle$ where X is the set of states, A is a finite set of transition labels and $\rightarrow \subseteq X \times A \times X$ is the transition relation.*

We write $x \xrightarrow{a} x'$ if $(x,a,x') \in \rightarrow$, and we extend this notation to words of A^* by $x \xrightarrow{\epsilon} x$ and $x \xrightarrow{uv} x'$ if there exists $x'' \in X$, $x \xrightarrow{u} x'' \xrightarrow{v} x'$. If $L \subseteq A^*$, we define $x \xrightarrow{L} y \iff \exists u \in L, x \xrightarrow{u} y$ and we shorten $x \xrightarrow{A^*} y$ as $x \xrightarrow{*} y$. A transition sequence $u \in A^*$ is said *fireable* from $x \in X$ if there exists $y \in X$ such that $x \xrightarrow{u} y$.

3.2 Vector Addition Sytems

Definition 2. *A Vector Addition System (shortly: VAS) is a pair $\langle A, \delta \rangle$ where A is a finite set of transition labels and δ a function from A to $\mathbb{Z}^d$. The integer d is called the dimension of the VAS.*

A Vector Addition System $\mathcal{V} = \langle A, \delta \rangle$ induces a transition system $TS(\mathcal{V}) = \langle \mathbb{N}^d, A, \rightarrow \rangle$ where $\rightarrow$ is defined by:

$$x \xrightarrow{a} y \iff y = x + \delta(a)$$

Reachability is known to be decidable for VAS:

Theorem 1. *([9], [5], [8]) If X and Y are Presburger sets and $\mathcal{V}$ a VAS, one can decide whether $\{(x,y) \in X \times Y \mid x \xrightarrow{*}_{\mathcal{V}} y\}$ is empty.*

Definition 3. *A Vector Addition System with one zero-test (shortly: VAS_0) is a tuple $\langle A_z, \delta, a_z \rangle$ where (A_z, δ) is a VAS and $a_z \in A_z$ is the special zero-test transition.*

A VAS_0 $\mathcal{V}_z = \langle A_z, \delta, a_z \rangle$ induces a transition system $TS(\mathcal{V}_z) = \langle \mathbb{N}^d, A_z, \rightarrow \rangle$ where $\rightarrow$ is defined by:

$$x \xrightarrow{a} y \iff y = x + \delta(a) \qquad a \neq a_z$$
$$x \xrightarrow{a_z} y \iff \begin{cases} y = x + \delta(a_z) \\ x(1) = 0 \end{cases}$$

The function δ is extended to Parikh images and words: for $v \in \mathbb{N}^{A_z}$, $\delta(v) = \sum_{a \in A_z} \delta(v(a))$ and for $u \in A_z^*$, $\delta(u) = \delta(|u|)$. This means that $x \xrightarrow{u} y \implies y = x + \delta(u)$.

The following statement shows a VAS_0 is partially monotonic (the proof is by an easy induction):

Proposition 1. *Let $\mathcal{V}_z$ be a VAS_0 of dimension d. Let $x, y \in \mathbb{N}^d$ with $x \leq y$ and $x(1) = y(1)$. If a transition sequence $u \in A_z^*$ is fireable from x, then u is fireable from y.*

4 Structure of the proof

Let us try to summarize the proof structure of [8], that we will mimic. The main idea is that if a relation has some properties, one can find a witness of non-reachability. These required properties are given by the notion of Petri set, which itself relies on the notions of polytope sets and Lambert sets, that generalizes linear and semilinear sets. After having given in section 4.1 the definitions of polytope, Lambert and Petri sets, we will recall in section 4.2 some tools from [8], and especially the result that if a relation is Petri, one can find a witness of non-reachability which is a Presburger forward invariant.

Now, to prove that our reachability relation is Petri, we have to show that to each transition sequence (a run) can be associated a production relation, such that (1) the runs ordered by inclusion of their production relations is well-ordered and (2) these production relations are polytope. With a few additionnal assumptions, this means that the reachability relation can be written as a finite sum and union of production relations (the relations associated to the minimal elements of the previously defined well-order) and can be shown to be Petri. We will introduce our version of these production relations in section 5 and prove the well-ordering in section 6. Then, section 7 will show that these production relations are polytopes and we will conclude in section 8.

Given the similarity between VAS and VAS_0, we will reuse a lot of Leroux results. The later sections will focus on the changes between the two proofs, with non-critical proofs being moved to A. Parts that are left mostly unchanged from Leroux paper are moved to the appendix B.

4.1 Polytope, Lambert and Petri sets

A set $P \subseteq \mathbb{Q}^d$ is *periodic* if $P + P \subseteq P$. A set $X \subseteq \mathbb{N}^d$ is a *finitely generated* periodic set if there exists $\{x_1, \ldots, x_n\} \subseteq X$, $X = \mathbb{N}x_1 + \mathbb{N}x_2 + \cdots + \mathbb{N}x_n$.

A *semilinear* set (also called *Presburger* set) is a finite union of sets $b_i + X_i$ where $b_i \in \mathbb{N}$ and $X_i \subseteq \mathbb{N}^d$ is a finitely generated periodic set.

A set $C \subseteq \mathbb{Q}^d$ is *conic* if it is periodic and $\mathbb{Q}_{\geq 0}C = C$. A conic set is finitely generated if there exists a finite set $\{c_1, \ldots, c_n\} \subseteq \mathbb{Q}$ such that $C = \mathbb{Q}_{\geq 0}c_1 + \ldots + \mathbb{Q}_{\geq 0}c_n$.

Definition 4. *([8], Definitions 4.1 and 4.6)*
A periodic set $P \subseteq \mathbb{N}^d$ is polytope *if $\mathbb{Q}_{\geq 0}P$ is definable in $FO(\mathbb{Q}, +, \leq, 0, 1)$ (the first order logic on the specified symbols). A set $L \subseteq \mathbb{N}^d$ is* Lambert *if it is a finite union of sets $b_i + P_i$ where $b_i \in \mathbb{N}^d$ and $P_i \subseteq \mathbb{N}^d$ is a polytope periodic set.*

The stability of Lambert sets will be of importance in the sequel. We have the following properties: (proofs of these statements are reasonably direct, and available in the appendix A):

Proposition 2. *Given $L \subseteq \mathbb{N}^{d_1}, L' \subseteq \mathbb{N}^{d_2}$ Lambert sets and $k \in \mathbb{N}$:*
1. *For $d_1 = d_2$, $L \cup L'$ is Lambert.*
2. *$L \times L'$ is Lambert.*
3. *For $d_1' < d_1$, $\{x \in \mathbb{N}^{d_1'} \mid \exists y \in \mathbb{N}^{d_1 - d_1'},\ (x, y) \in L\}$ is Lambert.*
4. *For $d_1 = d_2$, $L + L'$ is Lambert.*
5. *$k \star L$ is Lambert.*
6. *$\mathbb{N} \star L$ is polytope.*
7. *If δ is a linear function, then $\delta(L)$ is Lambert.*

Definition 5. *([8], Definition 4.7)*
A set $X \subseteq \mathbb{N}^d$ is Petri *if for all Presburger sets S, $S \cap X$ is Lambert.*

4.2 Important results from Leroux

We recall in this section a few important results from [8].

For a set $X \subseteq \mathbb{Q}^d$, the adherence of X, written $\overline{X}$ is defined by:

$$\overline{X} = \{l \mid \forall \tau > 0,\ \exists x \in X,\ max_i(x - l)(i) < \tau \wedge max_i(l - x)(i) < \tau\}$$

We will use the following useful characterization to show that our production relation is polytope:

Theorem 2. *([8], Theorem 3.5)*
A periodic set $P \subseteq \mathbb{N}^d$ is polytope if and only if the conic set $\overline{(\mathbb{Q}_{\geq 0}P) \cap V}$ is finitely generated for every vector space $V \subseteq \mathbb{Q}^d$

The second theorem needed is the one motivating Petri sets. A Petri relation admits witnesses of non-reachability:

Theorem 3. *([8], Theorem 6.1)*
Let R be a reflexive relation over $\mathbb{N}^d$ such that R^* is Petri. Let $X, Y \subseteq \mathbb{N}^d$ be two Presburger sets such that $R^* \cap (X \times Y)$ is empty. There exists a partition of $\mathbb{N}^d$ into a Presburger R-forward invariant that contains X and a Presburger R-backward invariant that contains Y.

And finally, we will also use that the reachability relation of a VAS is known to be Petri:

Theorem 4. *([8], Theorem 9.1)*
The reachability relation of a Vector Addition System is a Petri relation.

Since we can add counters that contain how many times each transition has been fired, we can extend this result to include the Parikh image of transition sequences:

Corollary 1. *Let* $\mathcal{V} = \langle A, \delta \rangle$ *be a VAS. Then,* $\{(x, v, y) \in \mathbb{N}^d \times \mathbb{N}^A \times \mathbb{N}^d \mid \exists u,\ x \xrightarrow{u}_{\mathcal{V}} y \wedge |u| = v\}$ *is a Petri set.*

5 Production relations

For all the remaining sections, we will fix a VAS_0 $\mathcal{V}_z = \langle A_z, \delta, a_z \rangle$ of dimension d. We consider the set $A = A_z \backslash \{a_z\}$ and $\mathcal{V} = \langle A, \delta_{|A} \rangle$ the restriction of $\mathcal{V}_z$ to its non-a_z transitions. We have $\xrightarrow{*}$ (or $\xrightarrow{A_z^*}$) the transition relation of $\mathcal{V}_z$ and $\xrightarrow{A^*}$ the transition relation of $\mathcal{V}$.

A run μ of $\mathcal{V}_z$ is a sequence $m_0.a_1.m_1.a_2 \ldots a_n.m_n$ alternating markings $m_i \in \mathbb{N}^d$ and actions $a_i \in A$ such that for all $1 \leq i \leq n$, $m_{i-1} \xrightarrow{a_i} m_i$. m_0 is called the *source* of μ, written $src(\mu)$ and m_n is called the *target* of μ, written $tgt(\mu)$. A run ρ of $\mathcal{V}_z$ is also a run of $\mathcal{V}$ if a_z doesn't appear in ρ.

We recall the definitions of the production relations for a VAS of [8], adapted to our case by restricting the relation to runs that don't use the zero-test.

- For a marking $m \in \mathbb{N}^d$, $\xrightarrow[\mathcal{V},m]{} \subseteq \mathbb{N}^d \times \mathbb{N}^d$ is defined by:

$$x \xrightarrow[\mathcal{V},m]{} y \iff \exists u \in A^*,\ m + x \xrightarrow{u} m + y$$

- For a run $\rho = m_0.a_1.m_1 \ldots a_n.m_n$ of $\mathcal{V}$, $\xrightarrow[\rho]{}$ is defined by:

$$\xrightarrow[\rho]{} = \xrightarrow[\mathcal{V},m_0]{} \circ \xrightarrow[\mathcal{V},m_1]{} \circ \cdots \xrightarrow[\mathcal{V},m_n]{}$$

We also define the production relation $\xrightarrow[\mathcal{V}_z,m]{} \subseteq \mathbb{N}^d \times \mathbb{N}^d$ of a marking $m \in \mathbb{N}_0^d$ inside $\mathcal{V}_z$ by:

$$x \xrightarrow[\mathcal{V}_z,m]{} y \iff \begin{cases} \exists u \in A_z^*,\ m + x \xrightarrow{u} m + y \\ x(1) = y(1) = 0 \end{cases}$$

To extend the definition of a production relation to a run μ of $\mathcal{V}_z$, we consider the decomposition of $\mu = \rho_0.a_z.\rho_1 \ldots a_z.\rho_p$ such that forall $1 \leq i \leq p$, ρ_i is a run of $\mathcal{V}$. In that case, we define the production relation of μ by:

$$\xrightarrow[\mu]{} = \xrightarrow[\rho_0]{} \circ \xrightarrow[\mathcal{V}_z,tgt(\rho_0)]{} \circ \xrightarrow[\rho_1]{} \circ \cdots \circ \xrightarrow[\mathcal{V}_z,tgt(\rho_{p-1})]{} \circ \xrightarrow[\rho_p]{}$$

Proposition 3. *For* $m \in \mathbb{N}^d$, $m' \in \mathbb{N}_0^d$ *and* μ *a run of* $\mathcal{V}_z$ *(a run* $\mathcal{V}$ *being a special case), the relations* $\xrightarrow[\mathcal{V},m]{}$, $\xrightarrow[\mathcal{V}_z,m']{}$ *and* $\xrightarrow[\mu]{}$ *are periodic.*

Proof: The result is easy for $\xrightarrow[\mathcal{V},m]{}$ and $\xrightarrow[\mathcal{V}_z,m']{}$. The periodicity of $\xrightarrow[\mu]{}$ comes from the fact the composition of periodic relations is periodic.

□

One can prove by a simple induction on μ (available in the appendix A) the following statement:

Proposition 4. *For a run μ of $\mathcal{V}_z$, we have:*

$$(src(\mu), tgt(\mu)) + \xrightarrow[\mu]{} \subseteq \xrightarrow{*}$$

6 Well-orderings of production relations

For two runs μ, μ', let us define $\preceq$ by:

$$\mu \preceq \mu' \iff (src(\mu'), tgt(\mu')) + \xrightarrow[\mu']{} \subseteq (src(\mu), tgt(\mu)) + \xrightarrow[\mu]{}$$

Our aim is to show that $\preceq$ is a well-order. To do that, we define the order $\trianglelefteq$ on runs of $\mathcal{V}_z$ in the following way:

- For $\rho = m_0.a_1.m_1 \dots a_p.m_p$ and $\rho' = m'_0.a'_1.m'_1 \dots a'_q.m'_q$ runs of $\mathcal{V}$ ($a_i, a'_i \in A$), we get the same definitions as in [8]:

$$m_0.a_1.m_1 \dots a_p.m_p \trianglelefteq m'_0.a'_1.m'_1 \dots a'_q.m'_q \iff \begin{cases} m_0 \leq m'_0 \\ m_p \leq m'_q \\ \prod_{1\leq i\leq p}(a_i, m_i) \leq^{emb} \prod_{1\leq i\leq q}(a'_i, m'_i) \end{cases}$$

  with $(a, m) \leq (a', m') \iff a = a' \wedge m \leq m'$
- For $\mu = \rho_0.a_z.\rho_1 \dots a_z.\rho_p$ and $\mu' = \rho'_0.a_z.\rho'_1 \dots a_z.\rho'_q$ runs of $\mathcal{V}_z$ (with ρ_i, ρ'_i runs of $\mathcal{V}$), we have:

$$\rho_0.a_z.\rho_1 \dots a_z.\rho_p \trianglelefteq \rho'_0.a_z.\rho'_1 \dots a_z.\rho'_q \iff \begin{cases} \rho_0 \leq \rho'_0 \\ \rho_p \leq \rho'_q \\ \prod_{1\leq i\leq p} \rho_i \trianglelefteq^{emb} \prod_{1\leq i\leq q} \rho'_i \end{cases}$$

Two applications of Higman's lemma gives us the following result:

Proposition 5. *The order $\trianglelefteq$ is well.*

Now, we only need to prove the following:

Proposition 6. *For μ, μ' runs of $\mathcal{V}_z$, we have:*

$$\mu \trianglelefteq \mu' \implies \mu \preceq \mu'$$

Proof Sketch: The full proof is available in the appendix A. [8] already contains the result for runs without the zero-test.
The idea is that our run can be decomposed in the following way, where $\varphi_{i,j}$ refers to "suppressed" sequences, and ρ''_i are greater than ρ_i for $\trianglelefteq$.

$$\prod_{1\leq k\leq q} \rho'_k = \rho''_0 \left(\prod_{1\leq j\leq n_0} \varphi_{0,j}\right) \rho''_1 \left(\prod_{1\leq j\leq n_1} \varphi_{1,j}\right) \rho''_2 \cdots \left(\prod_{1\leq j\leq n_{p-1}} \varphi_{p-1,j}\right) \rho''_p$$

Now, the outline of the proof is to base ourselves on Leroux result for runs without zero-tests, and to show that the productions of suppressed sequences are included in $\xrightarrow[\mathcal{V}_{z,tgt(\rho_i)}]{}$ where ρ_i is the part of the run before the suppressed sequence.

□

We can now combine propositions 5 and 6 to get:

Theorem 5. *$\preceq$ is a well-order on runs of $\mathcal{V}_z$.*

7 Polytopie of the production relation

Note that the relation $\xrightarrow[\mu]{}$ is a finite composition of relations $\xrightarrow[\mathcal{V},m]{}$ (for $m \in \mathbb{N}^d$) and $\xrightarrow[\mathcal{V}_{z},m]{}$ (for $m \in \mathbb{N}_0^d$). To show that $\xrightarrow[\mu]{}$ is polytope, we first recall two results from [8] regarding production relations:

Lemma 1. *([8], Lemma 8.2)*
If R and R' are two polytope periodic relations, then $R \circ R'$ is a polytope periodic relation.

Theorem 6. *([8], Theorem 8.1)*
For $m \in \mathbb{N}^d$, $\xrightarrow[\mathcal{V},m]{}$ is polytope.

These two results mean we only need to prove that $\xrightarrow[\mathcal{V}_z,m]{}$ is a polytope periodic relation for $m \in \mathbb{N}_0^d$.

Proposition 7. *For $m \in \mathbb{N}_0^d$, $\xrightarrow[\mathcal{V}_z,m]{}$ is polytope.*

Proof: Theorem 2 shows that $\xrightarrow[\mathcal{V}_z,m]{}$ is polytope if and only if the following conic space is finitely generated for every vector space $V \subseteq \mathbb{Q}^d \times \mathbb{Q}^d$:

$$\overline{(\mathbb{Q}_{\geq 0} \xrightarrow[\mathcal{V}_z,m]{}) \cap V} = \overline{\mathbb{Q}_{\geq 0}(\xrightarrow[\mathcal{V}_z,m]{} \cap V)}$$

Let us define $V_0 = (\mathbb{N}_0^d \times \mathbb{N}_0^d) \cap V$. We will re-use the idea of Leroux' intraproductions, but by restricting them to $\mathbb{N}_0^d$. Let $Q_{m,V} = \{y \in \mathbb{N}_0^d \mid \exists (x,z) \in (m,m) + V_0,\ x \xrightarrow{*} y \xrightarrow{*} z\}$ and $I_{m,V} \subseteq \{1,\ldots,d\}$ by $i \in I_{m,V} \iff \{q(i) \mid q \in Q_{m,V}\}$ is infinite. Please note that $1 \notin I_{m,V}$, as for all $q \in Q_{m,V}$, $q(1) = 0$. An *intraproduction* for (m, V_0) is a triple (r, x, s) such that $x \in \mathbb{N}_0^d$ and $(r, s) \in V_0$ with:

$$r \xrightarrow[\mathcal{V}_z,m]{} x \xrightarrow[\mathcal{V}_z,m]{} s$$

An intraproduction is *total* if $x(i) > 0$ for every $i \in I_{m,V}$. The following lemma can be proved exactly as Lemma 8.3 of [8] (a precise proof is available in the appendix B):

Lemma 2. *There exists a total intraproduction for (m, V_0).*

Now we define $\mathbb{N}_\infty = \mathbb{N} \cup \{\infty\}$, ordered by $x < \infty$ for every $x \in \mathbb{N}$. Given a finite set $I \subseteq \{1, \ldots, d\}$ and a marking $m \in \mathbb{N}^d$, we denote by m^I the vector of $\mathbb{N}_\infty^d$ defined by $m^I(i) = \infty$ if $i \in I$ and $m^I(i) = m(i)$ otherwise. We also define the order $\leq_\infty$ by $x \leq_\infty y$ if for all i, $y(i) = \infty$ or $x(i) = y(i)$ (equivalently there exists $I \subseteq \{1, \ldots, d\}$, $x^I = y$). For a relation $\to$, and $(x, y) \in \mathbb{N}_\infty^d$. We define $x \to x'$ if there exists $(m, m') \in \mathbb{N}^d$, $m \leq_\infty x$ and $m' \leq_\infty x'$ with $m \to m'$.
Let $Q = \{q^{I_{m,V}} \mid q \in Q_{m,V}\}$ and $\mathcal{G}$ the complete directed graph with nodes Q whose edges from q to q' are labeled by (q, q'). For $w \in (Q \times Q)^*$, we define $TProd(w) \subseteq \mathbb{N}^{A_z}$ by:

$$TProd(\varepsilon) = \{0^{A_z}\}$$
$$TProd((q, q')) = \left\{|u| \mid \exists (x, x') \in \mathbb{N}_0^d \times \mathbb{N}_0^d,\ x \leq_\infty q,\ x' \leq_\infty q',\ u \in a_z A^* \cup A^*,\ x \xrightarrow{u} x'\right\}$$
$$TProd(uv) = TProd(u) + TProd(v)$$

We define the periodic relation $R_{m,V}$ on V_0 by $r\ R_{m,V}\ s$ if:

1. $r(i) = s(i) = 0$ for every $i \notin I_{m,V}$
2. there exists a cycle labelled by w in $\mathcal{G}$ on the state $m^{I_{m,V}}$ and $v \in TProd(w)$ such that $r + \delta(v) = s$.

Lemma 3. *The periodic relation $R_{m,V}$ is polytope.*

Proof: First, let's show that $TProd((q, q'))$ is Lambert for every $(q, q') \in Q \times Q$. We define $X_1 = \{(x', y) \in \mathbb{N}_0^d \times \mathbb{N}_0^d \mid \exists x \leq_\infty q,\ x \xrightarrow{a_z} x' \wedge y \leq_\infty q'\}$ and $X_2 = \{(x, y) \in \mathbb{N}_0^d \times \mathbb{N}_0^d \mid x \leq_\infty q \wedge y \leq_\infty q'\}$ which are Presburger sets. Because, $Y = \{(x', v, y) \in \mathbb{N}^d \times \mathbb{N}^A \times \mathbb{N}^d \mid \exists u \in A^*,\ x' \xrightarrow{u} y \wedge |u| = v\}$ is a Petri set (corollary 1), $Y_1 = Y \cap (X_1 \times \mathbb{N}^A \times \mathbb{N}^d)$ and $Y_2 = Y \cap (X_2 \times \mathbb{N}^A \times \mathbb{N}^d)$ are Lambert sets, and by projection (proposition 2), $TProd((q, q')) = \big(|a_z| + \{u \mid \exists (x, y) \in \mathbb{N}^d \times \mathbb{N}^d,\ (x, u, y) \in Y_1\}\big) \cup \{u \mid \exists (x, y) \in \mathbb{N}^d,\ (x, u, y) \in Y_2\}$ is Lambert.

Let $P \subseteq \mathbb{N}^{Q \times Q}$ be the Parikh image of the language L made of words labelling cycles in $\mathcal{G}$ on the state $m^{I_{m,V}}$. L is a language recognized by a finite automata, hence P is a Presburger set.
Now, let's show that $R'_{m,V} = \{TProd(w) \mid w \in L\}$ is a Lambert set. We have:

$$R'_{m,V} = \left\{ \sum_{a \in Q \times Q} v(a) \star TProd(a) \mid v \in P \right\}$$

P is Presburger, hence there exists $(d_i)_{1 \leq i \leq p}$, $(e_{i,j})_{1 \leq i \leq p, 1 \leq j \leq n_i}$ with $d_i, e_{i,j} \in \mathbb{N}^{Q \times Q}$ and $P = \bigcup_i d_i + \Sigma_j \mathbb{N} e_{i,j}$. This gives:

$$\begin{aligned} R'_{m,V} &= \bigcup_{1 \leq i \leq p} \bigcup_{v \in \mathbb{N}^p} \sum_{1 \leq j \leq n_i} \sum_{a \in Q \times Q} (d_i + v(j) * e_{i,j})(a) \star TProd(a) \\ &= \bigcup_{1 \leq i \leq p} \sum_{a \in Q \times Q} d_i(a) \star TProd(a) + \bigcup_{1 \leq i \leq p} \sum_{1 \leq j \leq n_i} \bigcup_{k \in \mathbb{N}} \sum_{a \in Q \times Q} (k * e_{i,j})(a) \star TProd(a) \\ &= \bigcup_{1 \leq i \leq p} \sum_{a \in Q \times Q} d_i(a) \star TProd(a) + \bigcup_{1 \leq i \leq p} \sum_{1 \leq j \leq n_i} \mathbb{N} \star \left(\sum_{a \in Q \times Q} e_{i,j}(a) \star TProd(a) \right) \end{aligned}$$

For all $a \in Q \times Q$, we have seen that $TProd(a)$ is Lambert. So because Lambert sets are stable by addition, union and $\mathbb{N}\star$, (proposition 2), $R'_{m,V}$ is Lambert.

We define $V_{I_{m,V}} = \{x \in \mathbb{N}^d \mid \forall i \notin I_{m,V},\ x(i) = 0\}$ and $R''_{m,V} = \{(r, r + \delta(x)) \mid r \in V_{I_{m,V}} \wedge x \in R'_{m,V}\} = \{(r,r) \mid r \in V_{I_{m,V}}\} + \{0\}^d \times \delta(R'_{m,V})$. By proposition 2, we have $R''_{m,V}$ built from $R'_{m,V}$ by the image through a linear function and the sum with a Presburger set, which means $R''_{m,V}$ is Lambert. But, $R''_{m,V}$ is periodic, which means $R''_{m,V} = \mathbb{N} \star R''_{m,V}$ is polytope. Finally, as proposition 2, gives us that polytope sets are stable by intersection with vector spaces, $R_{m,V} = R''_{m,V} \cap V$ is polytope.

□

We will now show that our graph $\mathcal{G}$ is an acurate representation of the reachability relation:

Lemma 4. *Let w be the label of a path in $\mathcal{G}$ from $m_1^{I_{m,V}}$ to $m_2^{I_{m,V}}$ and $v \in TProd(w)$. Then, there exists $u \in A_z^*$ with $|u| = v$ and $(x, y) \in \mathbb{N}_0^d \times \mathbb{N}_0^d$, $x \leq_\infty m_1^{I_{m,V}}$ and $y \leq_\infty m_2^{I_{m,V}}$ such that $x \xrightarrow{u} y$.*

Proof: We show this by induction on the length of w. Let $w = w_0(q, q')$ where w_0 is a path from $m_1^{I_{m,V}}$ to $m_3^{I_{m,V}}$ and (q, q') is an edge from $m_3^{I_{m,V}}$ to $m_2^{I_{m,V}}$ and $v \in TProd(w_0(q, q'))$. This means there exists $v_1 \in TProd(w_0)$, $v_2 \in TProd(q, q')$ such that $v = v_1 + v_2$. By induction hypothesis, there exists $u_1 \in \mathbb{N}_0^d \times \mathbb{N}_0^d$, $x'_0 \leq_\infty m_1^{I_{m,V}}$ and $y'_0 \leq_\infty m_3^{I_{m,V}}$ such that $x'_0 \xrightarrow{u_1} y'_0$ and $|u_1| = v_1$.
By definition of $TProd((q, q'))$, as $v_2 \in TProd((q, q'))$, there exists $x'_1 \leq m_3^{I_{m,V}}$, $y'_1 \leq_\infty m_2^{I_{m,V}}$ and $u_2 \in a_z A^* \cup A^*$ such that $x'_1 \xrightarrow{u_2} y'_1$ and $|u_2| = v_2$. Let $z = max(y'_0, x'_1)$. We have $z(1) = y'_0(1) = x'_1(1) = m_3(1) = 0$, which gives us:

$$x'_0 + (z - y'_0) \xrightarrow{u_1} z \xrightarrow{u_2} y'_1 + (z - x'_1)$$

As $z^{I_{m,V}} = {y'_0}^{I_{m,V}} = {x'_1}^{I_{m,V}} = m_3^{I_{m,V}}$, we have $(z - y'_0) \leq_\infty 0^{I_{m,V}}$ and $(z - x'_1) \leq_\infty 0^{I_{m,V}}$, which allows us to define $x = x'_0 + (z - y'_0) \leq_\infty m_1^{I_{m,V}}$ and $y = y'_1 + (z - x'_1) \leq_\infty m_2^{I_{m,V}}$. $u = u_1 u_2$ completes the result.

□

We now show a lemma for the other direction:

Lemma 5. *Let $(m_1, m_2) \in Q_{m,V} \times Q_{m,V}$ with $u \in A_z^*$ such that $m_1 \xrightarrow{u} m_2$. There exists $w \in (Q \times Q)^*$ label of a path from $m_1^{I_{m,V}}$ to $m_2^{I_{m,V}}$ such that $|u| \in TProd(w)$.*

Proof: Let $u = u_1 a_z u_2 \dots a_z u_n$ with $u_i \in A^*$. We define $(x_i)_{1 \leq i \leq n}$, $x_i \in \mathbb{N}_0^d$ by:

$$m \xrightarrow{u_1} x_1 \xrightarrow{a_z u_2} x_2 \xrightarrow{a_z u_3} x_3 \cdots \xrightarrow{a_z u_n} x_n = m_2$$

We have for all i, $x_i \in \mathbb{N}_0^d$, which leads that $|u_1| \in TProd((m_1^{I_{m,V}}, x_1^{I_{m,V}}))$ and for all $i \in \{1, \dots, n-1\}$, $|a_z u_n| \in TProd((x_i^{I_{m,V}}, x_{i+1}^{I_{m,V}}))$. Hence, we can define $w = (m_1^{I_{m,V}}, x_1^{I_{m,V}})(x_1^{I_{m,V}}, x_2^{I_{m,V}}) \dots (x_{n-1}^{I_{m,V}}, m_2^{I_{m,V}})$ and we have $|u| \in TProd(w)$.

□

Thanks to lemmas 4 and 5, we can now prove the following lemma exactly in the same way as Lemma 8.5 of [8] (full proof in the appendix B)

Lemma 6. $\overline{\mathbb{Q}_{\geq 0} R_{m,V}} = \overline{\mathbb{Q}_{\geq 0}(\xrightarrow{\ \ \mathcal{V}_{z,m}\ \ } \cap V_0)}$

By lemma 3, $R_{m,V}$ is polytope, hence $\overline{\mathbb{Q}_{\geq 0} R_{m,V}}$ is finitely generated. Finally, we have proven proposition 7.

□

Finally, as $\xrightarrow{\ \mu\ }$ is a finite composition of elements of the form $\xrightarrow{\ \mathcal{V},m\ }$ and $\xrightarrow{\ \mathcal{V}_{z,m}\ }$, we have proven the following result:

Theorem 7. *If μ is a run of $\mathcal{V}_z$, then $\xrightarrow{\ \mu\ }$ is polytope.*

8 Decidability of Reachability

We have now all the results necessary to show the following:

Theorem 8. *$\xrightarrow{*}$ is a Petri relation.*

Proof Sketch: Similarly as in Theorem 9.1 of [8], one can show thanks to proposition 4 and theorem 5 that for any $(m, n) \in \mathbb{N}^d \times \mathbb{N}^d$ and $P \subseteq \mathbb{N}^d$ finitely generated periodic set, there exists a finite set B of runs of $\mathcal{V}_z$ such that:

$$\xrightarrow{*} \cap ((m, n) + P) = \bigcup_{\mu \in B} (src(\mu), tgt(\mu)) + (\xrightarrow{\ \mu\ } \cap P)$$

Then, proposition 5 allows to conclude that $\xrightarrow{*}$ is Petri. The full proof is available in the appendix B.

□

Because $\left(\xrightarrow{a_z A^* \cup A^*}\right)^* = \xrightarrow{A_z^*}$, we can now apply theorem 3 and get:

Proposition 8. *If X and Y are two Presburger sets such that $\xrightarrow{A_z^*} \cap (X \times Y) = \emptyset$, then there exists a Presburger $\xrightarrow{a_z A^* \cup A^*}$-forward invariant X' with $X' \cap Y = \emptyset$.*

Now that we have shown the existence of such an invariant, we only need to show that we are able to test whether a given set is an invariant:

Proposition 9. *Whether a Presburger set X is a $\xrightarrow{a_z A^* \cup A^*}$-forward invariant is decidable.*

Proof: X is a forward invariant for $\xrightarrow{a_z A^* \cup A^*}$ if and only if $\xrightarrow{a_z} (X) \subseteq X$ and $\xrightarrow{A^*} (X) \subseteq X$. Because $\xrightarrow{a_z} (X)$ is a Presburger set, the first condition is decidable as the inclusion of Presburger sets, and the second reduces to deciding whether $\xrightarrow{A^*} \cap (X \times \mathbb{N}^d \backslash X)$ is empty, which is a reachability problem in a VAS (Theorem 1).

□

This allows us to conclude:

Theorem 9. *Reachability in VAS_0 is decidable.*

Proof: By the propositions 8 and 9, reachability is co-semidecidable by enumerating Presburger forward invariants, and semidecidability is clear.

□

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A Additionnal proofs

Proof of proposition 2
Given $L \subseteq \mathbb{N}^{d_1}, L' \subseteq \mathbb{N}^{d_2}$ Lambert sets and $k \in \mathbb{N}$, we have:

1. *For $d_1 = d_2$, $L \cup L'$ is Lambert.*
2. *$L \times L'$ is Lambert.*
3. *For $d'_1 < d_1$, $\{x \in \mathbb{N}^{d'_1} \mid \exists y \in \mathbb{N}^{d_1 - d'_1},\ (x, y) \in L\}$ is Lambert.*
4. *For $d_1 = d_2$, $L + L'$ is Lambert.*
5. *$k \star L$ is Lambert.*
6. *$\mathbb{N} \star L$ is polytope (more generally Lambert).*
7. *If δ is a linear function, then $\delta(L)$ is Lambert.*

Proof: We have $L = \bigcup_{1 \le i \le p} b_i + P_i$ and $L' = \bigcup_{1 \le i \le q} b'_i + P'_i$ with $b_i \in \mathbb{N}^{d_1}$, $b'_i \in \mathbb{N}^{d_2}$ and $P_i \subseteq \mathbb{N}^d$, $P'_i \subseteq \mathbb{N}^d$ polytope periodic sets.
(1) is by definition of a Lambert set.

For (2), we have:

$$\begin{aligned} L \times L' &= \bigcup_{1 \le i \le p} \bigcup_{1 \le j \le q} (b_i + P_i) \times (b'_i + P'_i) \\ &= \bigcup_{1 \le i \le p} \bigcup_{1 \le j \le q} (b_i, b'_j) + P_i \times P'_j. \end{aligned}$$

Because P_i and P'_i are polytope periodic, $P_i \times P'_i$ is polytope periodic, which makes $L \times L'$ Lambert.

To show (3), we first show the property for polytope sets. Let's take P a polytope periodic set and $P' = \{x \in \mathbb{Q}^{d'_1} \mid \exists y \in \mathbb{Q}^{d_1 - d'_1},\ (x, y) \in P\}$. Then if $x \in P'$ and $x' \in P'$, we have $y, y' \in \mathbb{Q}^{d_1 - d'_1}$ such that $(x, y) \in P$ and $(x', y') \in P$, which gives $(x + x', y + y') \in P$ and $x + x' \in P'$. Moreover, we have:

$$\begin{aligned} \mathbb{Q}_{\ge 0} P' &= \{x \in \mathbb{Q}^{d'_1} \mid \exists y \in \mathbb{Q}^{d_1 - d'_1},\ \exists k \in \mathbb{Q}_{\ge 0},\ (kx, y) \in P\} \\ &= \{x \in \mathbb{Q}^{d'_1} \mid \exists y \in \mathbb{Q}^{d_1 - d'_1},\ (x, y) \in \mathbb{Q}_{\ge 0} P\} \end{aligned}$$

which means that from a definition of $\mathbb{Q}_{\ge 0} P$ in $FO(\mathbb{Q}, +, \le, 0, 1)$, we easily get the definition of $\mathbb{Q}_{\ge 0} P'$. And if $b_i = (c_i, c'_i)$ with $c_i \in \mathbb{Q}^{d'_1}$, we have:

$$\{x \in \mathbb{Q}^{d'_1} \mid \exists y \in \mathbb{Q}^{d_1 - d'_1},\ (x, y) \in L\} = \bigcup_i c_i + \{x \in \mathbb{Q}^{d'_1} \mid \exists y \in \mathbb{Q}^{d_1 - d'_1},\ (x, y) \in P_i\}$$

which gives us the result.

To show (4), we note that $L + L' = \bigcup_{1 \le i \le p} \bigcup_{1 \le j \le q} (b_i + b'_j) + (P_i + P'_j)$. Because the sum of periodic sets is periodic, $L + L'$ is periodic. Moreover, we get easily get the definition of $\mathbb{Q}_{\ge 0}(P + P') = \mathbb{Q}_{\ge 0} P + \mathbb{Q}_{\ge 0} P'$ from the definition of $\mathbb{Q}_{\ge 0} P$ and $\mathbb{Q}_{\ge 0} P'$ in $FO(\mathbb{Q}, +, \le, 0, 1)$. Hence, $L + L'$ is Lambert.

(5) is a direct consequence of (4).

To show (6), we notice that $\mathbb{N} \star L$ is periodic, and we have $\mathbb{Q}_{\geq 0}(\mathbb{N} \star L) = \sum_i \mathbb{Q}_{\geq 0} b_i + \sum_i \mathbb{Q}_{\geq 0} P_i$. As $\mathbb{Q}_{\geq 0} P_i$ is definable in $FO(\mathbb{Q}, +, \leq, 0, 1)$, so is $\mathbb{Q}_{\geq 0}(\mathbb{N} \star L)$. This makes $\mathbb{N} \star L$ polytope.

Let's finally show (7). We have $\delta(L) = \bigcup_{1 \leq i \leq p} \delta(b_i) + \delta(P_i)$. As δ is linear, we have $\delta(P_i)$ periodic and $\mathbb{Q}_{\geq 0}\delta(P_i) = \delta(\mathbb{Q}_{\geq 0} P_i)$, which makes $\mathbb{Q}_{\geq 0}\delta(P_i)$ easily definable from the definition of $\mathbb{Q}_{\geq 0} P_i$ in $FO(\mathbb{Q}, +, \leq, 0, 1)$.

□

Proof of proposition 4
For a run μ of $\mathcal{V}_z$, we have:

$$(src(\mu), tgt(\mu)) + \xrightarrow[\mu]{} \subseteq \xrightarrow{*}$$

Proof: We show this result by induction on μ. We have to consider three cases:

1. $\mu = m$ is immediate by definition of $\xrightarrow[\mu]{} = \xrightarrow[\mathcal{V},m]{}$ given $src(\mu) = tgt(\mu) = m$.
2. $\mu = m.a.\mu'$ with $a \neq a_z$. Let $(x, z) \in \xrightarrow[\mu]{}$. Then, as $\xrightarrow[\mu]{} = \xrightarrow[\mathcal{V},m]{} \circ \xrightarrow[\mu']{}$, there exists $y \in \mathbb{N}^d$ such that $x \xrightarrow[\mathcal{V},m]{} y \xrightarrow[\mu']{} z$. By definition of $\xrightarrow[\mathcal{V},m]{}$, there exists $u_1 \in A^*$ such that $m+x \xrightarrow{u_1} m+y$. Similarly, by induction hypothesis, there exists $u_2 \in A_z^*$, such that $src(\mu')+y \xrightarrow{u_2} tgt(\mu') + z$. As μ is a run, we have $m \xrightarrow{a} src(\mu')$, which gives us:

   $$m + x \xrightarrow{u_1} m + y \xrightarrow{a} src(\mu') + y \xrightarrow{u_2} tgt(\mu') + z$$

   And by noticing that $m = src(\mu)$ and $tgt(\mu') = tgt(\mu)$, we have $(src(\mu), tgt(\mu)) + (x, z) \subseteq \xrightarrow{*}$
3. $\mu = m.a_z.\mu'$. Let $(x, z) \in \xrightarrow[\mu]{}$. Then, as $\xrightarrow[\mu]{} = \xrightarrow[\mathcal{V},m]{} \circ \xrightarrow[\mathcal{V}_z,m]{} \circ \xrightarrow[\mu']{}$, there exists $y, y' \in \mathbb{N}_0^d$, such that $x \xrightarrow[\mathcal{V},m]{} y \xrightarrow[\mathcal{V}_z,m]{} y' \xrightarrow[\mu']{} z$. By definition of $\xrightarrow[\mathcal{V},m]{}$ and $\xrightarrow[\mathcal{V}_z,m]{}$, there exists $u_1 \in A^*, u_2 \in A_z^*$ such that $m + x \xrightarrow{u_1} m + y \xrightarrow{u_2} m + y'$. Again, by induction hypothesis, there exists $u_3 \in A_z^*$ such that $src(\mu') + y' \xrightarrow{u_3} tgt(\mu') + z$. As μ is a run, we have $m \xrightarrow{a_z} src(\mu')$ and because $y' \in \mathbb{N}_0^d$, we have $m + y' \xrightarrow{a_z} src(\mu') + y'$, and combining all these statements, we get:

   $$m + x \xrightarrow{u_1} m + y \xrightarrow{u_2} m + y' \xrightarrow{a_z} src(\mu') + y' \xrightarrow{u_3} tgt(\mu') + z$$

   Finally, we have $m = src(\mu)$ and $tgt(\mu') = tgt(\mu)$, which gives $(src(\mu), tgt(\mu)) + (x, z) \subseteq \xrightarrow{*}$

□

Proof of proposition 6
For μ, μ' runs of $\mathcal{V}_z$, we have:

$$\mu \trianglelefteq \mu' \implies \mu \preceq \mu'$$

Proof: We first recall the following result (a proof is available in the appendix B):

Lemma 7. *([8], Lemma 7.6) For ρ, ρ' runs of $\mathcal{V}$, we have:*

$$\rho \trianglelefteq \rho' \implies \rho \preceq \rho'$$

Let's take $(r, s) \in \xrightarrow[\mu']{}$. As $\mu \trianglelefteq \mu'$, we have $\mu = \rho_0.a_z.\rho_1.a_z \dots a_z.\rho_p$ and $\mu' = \rho'_0.a_z.\rho'_1.a_z \dots a_z.\rho'_q$, with $\rho_0 \trianglelefteq \rho'_0$, $\rho_p \trianglelefteq \rho'_q$ and $\prod \rho_i \trianglelefteq^{emb} \prod \rho'_i$. Let $(\rho''_i)_{0 \le i \le p}$ and $(\varphi_{i,j})_{0 \le i \le p-1, 1 \le j \le n_i}$ be sequences of runs of $\mathcal{V}$ such that for all $i \in \{0, \dots, p\}$, $\rho_i \le \rho''_i$ and:

$$\prod_{1 \le k \le q} \rho'_k = \rho''_0 \left(\prod_{1 \le j \le n_0} \varphi_{0,j} \right) \rho''_1 \left(\prod_{1 \le j \le n_1} \varphi_{1,j} \right) \rho''_2 \cdots \left(\prod_{1 \le j \le n_{p-1}} \varphi_{p-1,j} \right) \rho''_p$$

For $1 \le i \le p$, we define $\varphi_i = \varphi_{i,1}.a_z \varphi_{i,2}.a_z \dots a_z.\varphi_{i,n_i}$ (with the possibility that $\varphi_i = \varepsilon$ if $n_i = 0$).
By definition of $\xrightarrow[\mu']{}$, there exists for $0 \le i \le p$, $r_i \in \mathbb{N}^d$ and for $0 \le i \le p-1$, $r'_i, s_i, s'_i \in \mathbb{N}^d$ with $r_0 = r$, and:

- For $i \in \{0, \dots, p-1\}$, if $n_i \neq 0$ ($\varphi_i \neq \varepsilon$):

$$r_i \xrightarrow[\rho''_i]{} r'_i \xrightarrow[\mathcal{V}_z, tgt(\rho''_i)]{} s_i \xrightarrow[\varphi_i]{} s'_i \xrightarrow[\mathcal{V}_z, tgt(\varphi_i)]{} r_{i+1}$$

- For $i \in \{0, \dots, p-1\}$, if $n_i = 0$ ($\varphi_i = \varepsilon$):

$$r_i \xrightarrow[\rho''_i]{} r'_i \xrightarrow[\mathcal{V}_z, tgt(\rho''_i)]{} r_{i+1}$$

- For $i = p$:

$$r_p \xrightarrow[\rho''_p]{} s$$

Note that because of the definition of $\xrightarrow[\mathcal{V}_z, m]{}$, we have for all $i \in \{0, \dots, p-1\}$, $r'_i(1) = s(1) = s'(1) = tgt(\varphi_i)(1) = tgt(\rho_i)(1) = tgt(\rho''_i)(1) = 0$. Let's show that for all $i \in \{0, \dots, p-1\}$, $r_i + src(\rho''_i) - src(\rho_i) \xrightarrow[\rho_i]{} \xrightarrow[\mathcal{V}_z, tgt(\rho_i)]{} r'_i + src(\rho''_{i+1}) - src(\rho_{i+1})$
We have two cases to consider:

- $\underline{n_i = 0}$
  We have $r_i \xrightarrow[\rho''_i]{} r'_i$, so by lemma 7, as $\rho_i \trianglelefteq \rho''_i$, we get $r_i + src(\rho''_i) - src(\rho_i) \xrightarrow[\rho_i]{} r'_i + tgt(\rho''_i) - tgt(\rho_i)$.
  We have $r'_i \xrightarrow[\mathcal{V}_z, tgt(\rho''_i)]{} s_i$, which gives $tgt(\rho''_i) + r'_i \xrightarrow{*} tgt(\rho''_i) + s_i$.
  As $tgt(\rho_i)(1) = 0$ and $tgt(\rho_i) \le tgt(\rho''_i)$, we get $r'_i + tgt(\rho''_i) - tgt(\rho_i) \xrightarrow[\mathcal{V}_z, tgt(\rho_i)]{} s_i + tgt(\rho''_i) - tgt(\rho_i)$.

Finally as we have $tgt(\rho_i'') \xrightarrow{a_z} src(\rho_{i+1}'')$ (because μ' is a run) and $tgt(\rho) \xrightarrow{a_z} src(\rho_{i+1})$ (because μ is a run), we get $tgt(\rho_i'') - tgt(\rho) = src(\rho_{i+1}'') - src(\rho_{i+1})$, which gives by combination of the previous results:

$$\begin{aligned} r_i + src(\rho_i'') - src(\rho_i) &\xrightarrow[\rho_i]{} r_i' + tgt(\rho_i'') - tgt(\rho_i) \\ &\xrightarrow[\mathcal{V}_{z,tgt(\rho_i)}]{} r_{i+1} + tgt(\rho_i'') - tgt(\rho_i) \\ &= r_{i+1} + src(\rho_{i+1}'') - src(\rho_{i+1}) \end{aligned}$$

This gives $r_i + src(\rho_i'') - src(\rho_i) \xrightarrow[\rho_i]{}\xrightarrow[\mathcal{V}_{z,tgt(\rho_i)}]{} r_{i+1} + src(\rho_{i+1}'') - src(\rho_{i+1})$.

- $\underline{n_i \neq 0}$
  We have $r_i \xrightarrow[\rho_i'']{} r_i'$, so by lemma 7, as $\rho_i \trianglelefteq \rho_i''$, we get $r_i + src(\rho_i'') - src(\rho_i) \xrightarrow[\rho_i]{} r_{i+1} + tgt(\rho_i'') - tgt(\rho_i)$.
  We have $r_i' \xrightarrow[\mathcal{V}_{z,tgt(\rho_i'')}]{} s_i$, which gives $tgt(\rho_i'') + r_i' \xrightarrow{*} tgt(\rho_i'') + s_i$.
  As $tgt(\rho_i)(1) = 0$ and $tgt(\rho_i) \leq tgt(\rho_i'')$, we get $r_i' + tgt(\rho_i'') - tgt(\rho_i) \xrightarrow[\mathcal{V}_{z,tgt(\rho_i)}]{} s_i + tgt(\rho_i'') - tgt(\rho_i)$.
  We have $s_i \xrightarrow[\varphi_i]{} r_i' \xrightarrow[\mathcal{V}_{z,tgt(\varphi_i)}]{} r_{i+1}$, which by proposition 4, gives $s_i + src(\varphi_i) \xrightarrow{*} r_i' + tgt(\varphi_i) \xrightarrow{*} r_{i+1} + tgt(\varphi_i)$. Moreover, because μ' is a run, we have $tgt(\rho_i'') \xrightarrow{a_z} src(\varphi_i)$ and $tgt(\varphi_i) \xrightarrow{a_z} src(\rho_{i+1}'')$. Because $s_i(1) \in \mathbb{N}_0^d$, $tgt(\rho_i'') \xrightarrow{a_z} src(\varphi_i)$ implies $s_i + tgt(\rho_i'') \xrightarrow{a_z} s_i + src(\varphi_i)$. This gives $s_i + tgt(\rho_i'') \xrightarrow{*} r_{i+1} + tgt(\varphi_i)$. Now, we note that $tgt(\varphi_i) \xrightarrow{a_z} src(\rho_{i+1}'')$ (because μ' is a run) and $tgt(\rho_i) \xrightarrow{a_z} src(\rho_{i+1})$ (because μ is a run). This leads to $tgt(\varphi_i) - tgt(\rho_i) = src(\rho_{i+1}'') - src(\rho_{i+1})$ and in particular $tgt(\rho_i) \leq tgt(\varphi_i)$ (because $\rho_{i+1} \trianglelefteq \rho_{i+1}''$). Hence, $s_i + tgt(\rho_i'') \xrightarrow{a_z} r_{i+1} + tgt(\varphi_i)$ implies $s_i + tgt(\rho_i'') - tgt(\rho_i) \xrightarrow[\mathcal{V}_{z,tgt(\rho_i)}]{} r_{i+1} + tgt(\varphi_i) - tgt(\rho_i)$, and because $tgt(\varphi_i) - tgt(\rho_i) = src(\rho_{i+1}'') - src(\rho_{i+1})$, we get $s_i + tgt(\rho_i'') - tgt(\rho_i) \xrightarrow[\mathcal{V}_{z,tgt(\rho_i)}]{} r_{i+1} + src(\rho_{i+1}'') - src(\rho_{i+1})$.
  We can now combine the previous results to get:

$$\begin{aligned} r_i + src(\rho_i'') - src(\rho_i) &\xrightarrow[\rho_i]{} r_i' + tgt(\rho_i'') - tgt(\rho_i) \\ &\xrightarrow[\mathcal{V}_{z,tgt(\rho_i)}]{} s_i + tgt(\rho_i'') - tgt(\rho_i) \\ &\xrightarrow[\mathcal{V}_{z,tgt(\rho_i)}]{} r_{i+1} + src(\rho_{i+1}'') - src(\rho_{i+1}) \end{aligned}$$

  This gives $r_i + src(\rho_i'') - src(\rho_i) \xrightarrow[\rho_i]{}\xrightarrow[\mathcal{V}_{z,tgt(\rho_i)}]{} r_{i+1} + src(\rho_{i+1}'') - src(\rho_{i+1})$.

We have shown that for all $i \in \{0, \ldots, p-1\}$, $r_i + src(\rho_i'') - src(\rho_i) \xrightarrow[\rho_i]{}\xrightarrow[\mathcal{V}_{z,tgt(\rho_i)}]{} r_{i+1} + src(\rho_{i+1}'') - src(\rho_{i+1})$. Moreover, by lemma 7, $\rho_p \trianglelefteq \rho_p''$ and $r_p \xrightarrow[\rho_p'']{} s$ implies $r_p + src(\rho_p'') - src(\rho_p) \xrightarrow[\rho_p]{} s + tgt(\rho_p'') - tgt(\rho_p)$. By combining these results, we get:

$$r+src(\rho_0'')-src(\rho_0) \xrightarrow[\rho_0]{} \xrightarrow[\mathcal{V}_z,tgt(\rho_0)]{} \xrightarrow[\rho_1]{} \xrightarrow[\mathcal{V}_z,tgt(\rho_1)]{} \cdots \xrightarrow[\rho_{p-1}]{} \xrightarrow[\mathcal{V}_z,tgt(\rho_{p-1})]{} \xrightarrow[\rho_p]{} s+tgt(\rho_p'')-tgt(\rho_p)$$

And because $\xrightarrow[\mu]{} = \xrightarrow[\rho_0]{} \xrightarrow[\mathcal{V}_z,tgt(\rho_0)]{} \xrightarrow[\rho_1]{} \xrightarrow[\mathcal{V}_z,tgt(\rho_1)]{} \cdots \xrightarrow[\rho_{p-1}]{} \xrightarrow[\mathcal{V}_z,tgt(\rho_{p-1})]{} \xrightarrow[\rho_p]{}$, $src(\mu) = src(\rho_0)$, $src(\mu') = src(\rho_0'')$, $tgt(\mu') = tgt(\rho_p'')$ and $tgt(\mu) = tgt(\rho_p)$, we get:

$$r + src(\mu') - src(\mu) \xrightarrow[\mu]{} s + tgt(\mu') - tgt(\mu)$$

We have shown that for any $(r, s) \in \xrightarrow[\mu']{}$, $(src(\mu'), tgt(\mu')) + (r, s) \subseteq (src(\mu), tgt(\mu)) + \xrightarrow[\mu]{}$. This implies $\mu \preceq \mu'$.

□

B Extensions of proofs by Leroux

All proofs of this section have been taken from [8] with some minor adaptations to work within our setup.

Proof of lemma 7
For ρ, ρ' runs of $\mathcal{V}$, we have:

$$\rho \trianglelefteq \rho' \implies \rho \preceq \rho'$$

Proof: Let $\rho = m_0.a_1.m_1 \dots a_k.m_k$. We first show that there exists a sequence $(v_j)_{0\leq j\leq k+1}$ of vectors in $\mathbb{N}^d$ and a sequence of runs $(\rho_j')_{0\leq j\leq k}$ such that $\rho' = \rho_0'.a_1.\rho_1' \dots a_k.\rho_k'$ with $src(\rho_j') = m_j + v_j$ and $tgt(\rho_j') = m_j + v_{j+1}$.

As we have $\rho \trianglelefteq \rho'$, we deduce that there exists $(\rho_j')_{0\leq j\leq k}$ such that $\rho' = \rho_0'.a_1.\rho_1' \dots a_k.\rho_k'$ and for all $j \in \{0, \dots, k\}$, $m_j \leq src(\rho_j')$. We define $v_0 = src(\rho') - src(\rho)$, $v_{k+1} = tgt(\rho') - tgt(\rho)$ and $v_j = src(\rho_j') - m_j$. Observe that $v_j \in \mathbb{N}^d$ for every $j \in \{0, \dots, k+1\}$. Because for $j \in \{1, \dots, k\}$, $m_{j-1} \xrightarrow{a_j} m_j$ and $tgt(\rho_{j-1}') \xrightarrow{a_j} src(\rho_j')$, we have $tgt(\rho_{j-1}) = m_{j-1} + v_j$. Our decomposition fulfills the required conditions.

Now, by lemma 4, we have $(src(\rho_j'), tgt(\rho_j')) + \xrightarrow[\rho_j']{} \subseteq \xrightarrow{A^*}$. Hence, $(v_j, v_{j+1}) + \xrightarrow[\rho_j']{} \subseteq \xrightarrow[\mathcal{V},m_j]{}$. We deduce that $(v_0, v_{k+1}) + \xrightarrow[\rho']{} \subseteq \xrightarrow[\rho]{}$ by composition. Since $(src(\rho'), tgt(\rho')) = (src(\rho), tgt(\rho)) + (v_0, v_{k+1})$, we get $\rho \preceq \rho'$.

□

Proof of lemma 2
There exists a total intraproduction for (m, V_0).

Proof: Since finite sums of intraproductions are intraproductions, it is sufficient to prove that for every $i \in I_{m,V}$, there exists an intraproduction (r, x, s) for (m, V_0) such that $x(i) > 0$. We fix $i \in I_{m,V}$.

Let us first prove that there $q \leq q' \in Q_{m,V}$ such that $q(i) < q'(i)$. Since $i \in I_{m,V}$, there exists a sequence $(q_n)_{n\in\mathbb{N}}$ of markings $q_n \in Q_{m,V}$ such that $(q_n(i))_{n\in\mathbb{N}}$ is strictly increasing. Since $(\mathbb{N}^d, \leq)$ is well ordered, we can find $q \leq q'$ in $Q_{m,V}$ such that $q(i) < q'(i)$.
As $q \in Q_{m,V}$ then there exists $(r, s) \in V_0$, such that $m+r \xrightarrow{*} q \xrightarrow{*} m+s$. Symmetrically, there exists $(r', s') \in V_0$ such that $m+r' \xrightarrow{*} q' \xrightarrow{*} m+s'$. Let us introduce $\delta = q' - q$. We deduce:

- $(m+r')+r \xrightarrow{*} q'+r$ from $m+r' \xrightarrow{*} q'$.
- $q+(\delta+r) \xrightarrow{*} (m+s)+(\delta+r)$ from $q \xrightarrow{*} m+s$.
- $(m+r)+(\delta+s) \xrightarrow{*} q+(\delta+s)$ from $m+r \xrightarrow{*} q$.
- $q'+s \xrightarrow{*} (m+s')+s$ from $q' \xrightarrow{*} m+s'$.

Since $q'+r = q+\delta+r$ and $q+\delta+s = q'+s$ and $r, r', \delta \in \mathbb{N}_0^d$, we have shown, for $x = s+r+\delta$:

$$r + r' \xrightarrow[\mathcal{V}_{z,m}]{} x \xrightarrow[\mathcal{V}_{z,m}]{} s + s'$$

As $(r+r', s+s') \in V_0$, we have shown that $(r+r', x, s+s')$ is an intraproduction for (m, V_0), with $x(i) > 0$.

□

Proof of lemma 6
We have:

$$\overline{\mathbb{Q}_{\geq 0} R_{m,V}} = \overline{\mathbb{Q}_{\geq 0}(\overrightarrow{\mathcal{V}_{z,m}} \cap V_0)}$$

Proof: Let us first prove the inclusion $\supseteq$. Let $(r, s) \in V_0$ be such that $r \xrightarrow[\mathcal{V}_{z,m}]{} s$. In this case, there exists a word $u \in A_z^*$ such that $m+r \xrightarrow{w} m+s$. Observe that $m+n*r$ and $m+n*s$ are in $Q_{m,V}$ for every $n \in \mathbb{N}$. Hence, $r(i) > 0$ or $s(i) > 0$ implies $i \in I_{m,V}$ and we deduce that $(m+r)^{I_{m,V}} = (m+s)^{I_{m,V}} = m^{I_{m,V}}$. By lemma 5, because $m+r \xrightarrow{u} m+s$, there exists w label of a cycle on $m^{I_{m,V}}$ and such that $|u| \in TProd(w)$. As $r + \delta(|u|) = s$, we have proved that $(r, s) \in R_{m,V}$.

Now, let us prove the inclusion $\subseteq$. Let $(r, s) \in R_{m,V}$. In this case, $(r, s) \in V_0$ satisfies $r(i) = s(i) = 0$ for every $i \notin I_{m,V}$ and there exists a word $w = a_1 \ldots a_k$ with $a_i \in Q \times Q$, $v \in TProd(w)$ such that $r + \delta(v) = s$. By lemma 4, there exists $u \in A_z^*$ with $|u| = v$, $r' \leq_\infty 0^{I_{m,V}}$, and $s' \leq_\infty 0^{I_{m,V}}$ such that $m+r' \xrightarrow{u} m+s'$. We consider a total intraproduction (r'', x, s'') for (m, V_0). Because $r' \leq_\infty 0^{I_{m,V}}$, there exists $p \in \mathbb{N}$ such that $r' \leq p*x$. Because $r'(1) = x(1) = 0$, from $m+r' \xrightarrow{u} m+s'$, we get $m+p*x \xrightarrow{u} m+p*x+\delta(u)$. And as we also have $r(1) = 0$, we get:

$$m+p*x+r \xrightarrow{w'} m+p*x+r+\delta(w') = m+p*x+s$$

This means $(r, s) \in \overrightarrow{\mathcal{V}_{z,m'}}$ where $m' = m+p*x$. Since a production relation is periodic, we get for all $n \in \mathbb{N}$, $(n*r, n*s) \in \overrightarrow{\mathcal{V}_{z,m'}}$. As $(p*r'', p*x, p*s'')$ is an intraproduction for (m, V_0), we have $m+p*r'' \xrightarrow{*} m' \xrightarrow{*} m+s''$. We deduce the relation $(m+p*r'')+n*r \xrightarrow{*} m'+n*r$

from $(m+p*r'') \xrightarrow{*} m'$ and the relation $m'+n*s \xrightarrow{*} (m+p*s'')+n*s$ from $m' \xrightarrow{*} (m+p*s'')$. We deduce that the following relation holds for every $n \in \mathbb{N}$:

$$m + p * r'' + n * r \xrightarrow{*} m + p * s'' + n * s$$

And as we have $(r'', s'') \in V_0$ and $(r, s) \in V_0$, we have $p * (r', s') + \mathbb{N} * (r, s) \subseteq \xrightarrow[\nu_{z,m}]{} \cap V_0$. Thus $(r, s) \in \overline{\mathbb{Q}_{\geq 0}(\xrightarrow[\nu_{z,m}]{} \cap V_0)}$. From the inclusion $R_{m,V} \subseteq \overline{\mathbb{Q}_{\geq 0}(\xrightarrow[\nu_{z,m}]{} \cap V_0)}$ we get the inclusion $\overline{\mathbb{Q}_{\geq 0} R_{m,V}} = \overline{\mathbb{Q}_{\geq 0}(\xrightarrow[\nu_{z,m}]{} \cap V_0)}$.

□

Proof of theorem 8
$\xrightarrow{*}$ is a Petri relation.

Proof: We are interested in proving that $\xrightarrow{*}$ is a Petri relation. This problem is equivalent to prove that $\xrightarrow{*} \cap((m,n)+P)$ is a Lambert relation for every $(m,n) \in \mathbb{N}^d \times \mathbb{N}^d$ and for every finitely generated periodic relation $P \subseteq \mathbb{N}^d \times \mathbb{N}^d$. We introduce the order $\leq_P$ over P defined by $p \leq_P p'$ if $p' \in p+P$. Because P is finitely generated, there exists $a_1, \dots, a_q \in P$ such that $P = \mathbb{N}a_1 + \mathbb{N}a_2 + \dots \mathbb{N}a_q$. Hence, if we define the surjective function f from $\mathbb{N}^q$ to P defined by $f(x) = \Sigma_i x(i) a_i$, we have $x \leq x' \implies f(x) \leq_P f(x')$, and because $\leq$ is a well-order on $\mathbb{N}^q$, $\leq_P$ is a well-order on P. We introduce the set $\Omega_{m,P,n}$ of runs μ such that $(src(\mu), tgt(\mu)) \in (m,n)+P$. Thanks to theorem 5, this set is well-ordered by the relation $\preceq_P$ defined by $\mu \preceq \mu'$ and $(src(\mu), tgt(\mu)) - (m,n) \leq_P (src(\mu'), tgt(\mu')) - (m,n)$. We deduce that $B = min_{\preceq_P}(\Omega_{m,P,n})$ is finite.
We now show the following equality:

$$\xrightarrow{*} \cap((m,n)+P) = \bigcup_{\mu \in B} (src(\mu), tgt(\mu)) + (\xrightarrow[\mu]{} \cap P)$$

Let us first prove $\supseteq$. Let $\mu \in \Omega_{m,P,n}$. Proposition 4 shows that $(src(\mu), tgt(\mu)) + \xrightarrow[\rho]{} \in \xrightarrow{*}$. Since $(src(\mu), tgt(\mu)) \in (m,n)+P$ and P is periodic we deduce the inclusion $\subseteq$.
Now, let us prove $\subseteq$. Let (x', y') in the intersection $\xrightarrow{*} \cap((m,n)+P)$. There exists a run $\mu' \in \Omega_{m,P,n}$ such that $x' = src(\mu')$ and $y' = tgt(\mu')$. There exists $\mu \in min_{\preceq_P}(\Omega_{m,P,n})$ such that $\mu \preceq_P \mu'$. We deduce that $(x', y') \in (src(\mu), tgt(\mu)) + (\xrightarrow[\rho]{} \cap P)$ and we have proved the inclusion $\subseteq$.

Theorem 7 shows that $\xrightarrow[\mu]{}$ is a polytope periodic relation. As P is a finitely generated relation, it is a polytope periodic relation. Polytope periodic relations are stable by finite intersections ([8], Lemma 4.5) and we deduce that $\xrightarrow{\mu} \cap P$ is a polytope periodic relation. This induces that $\xrightarrow{*} \cap((m,n)+P)$ is a Lambert relation for every $(m,n) \in \mathbb{N}^d \times \mathbb{N}^d$ and for every finitely generated periodic relation $P \subseteq \mathbb{N}^d \times \mathbb{N}^d$. Therefore, $\xrightarrow{*}$ is a Petri relation.

□