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Presburger Functions Are Piecewise Linear

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Abstract

In this paper we geometrically characterize sets and functions definable in the first order additive theory of the reals and the integers, a decidable extension of the Presburger arithmetic combining both integral and real variables. We introduce the notion of polinear sets, an extension of the linear sets that characterizes these sets and we prove that a function is definable in this logic if and only if it is piecewise rational linear.

Key words: Presburger set, Presburger function

1 Introduction

Presburger arithmetic is the decidable first order additive logic $\text{FO}(\mathbb{Z}, \leq, +)$ over the integers [Pre29]. From [GS66], Presburger-definable sets are exactly finite unions of linear sets. Some results about Presburger arithmetic allow to use Presburger formulas for representing infinite sets of states in an efficient way. Any Presburger formula can be encoded into a finite automaton whose language is the binary encoding of the positive instances of the formula. This gives the possibility to efficiently manipulate Presburger formulas by means of (binary) finite automata [WB00, BC96, BHMV94]. Even if a finite automaton can represent sets that are not definable in Presburger arithmetic (for instance the powers of 2), it is possible to decide (in polynomial time) whether a finite automaton represent a Presburger set [Ler05]. But models and programs do

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not only contain integer variables: hybrid systems and particularly timed automata use clocks whose values are in the reals. It is also well-known that the first order logics on reals with addition $\mathrm{FO}(\mathbb{R}, \leq, +)$ (and in fact with multiplication $\mathrm{FO}(\mathbb{R}, \leq, +, *)$) is decidable; moreover a subset of real vectors is definable in $\mathrm{FO}(\mathbb{R}, +, \leq)$ if and only if it is a finite union of rational polyhedral convex sets.

Presburger sets: It is then natural to try to understand what are the sets definable in $\mathrm{FO}(\mathbb{R}, \mathbb{Z}, \leq, +)$, this means to obtain some algebraic, logics, geometrical characterizations of these sets. It is well-known that this logic is decidable; this fact is recalled in [BRW98] and the seminal ideas of Buchi may be used. It is also interesting to know whether there exists some efficient way for manipulating these sets. Surprisingly, it seems that very few results exist: in [BRW98] the logic was proved decidable by using Buchi automata; in [BJW01] the obtained Buchi automata are proved weak (allowing more efficient algorithmic boolean operations); note that a variant version of weak Buchi automata was proved more efficient in [EK06]; in [Wei99] the logic was proved decidable by using a quantification elimination algorithm.

Presburger functions: In the early development of the Presburger theory, characterizations of the class of functions which are definable in the Presburger arithmetic were given in different terms. Meyer and Ritchie in [MR67] proved that Presburger functions are computable by flat (without nested loops) counter automata using the following primitives: incrementation, reset and transfer; note that there is no decrementation neither zero test. In [Che76], Cherniavsky proved that one may add the zero test at the set of primitives without leaving the class of Presburger functions. This gives another characterization of Presburger functions: they are computable by flat counter automata using the following primitives: incrementation, reset, transfer and zero test. Gurari in [Gur85] showed that the class of Presburger functions is exactly the class of functions computable by flat counter automata where the following primitives are allowed: incrementation, reset, positive decrementation and zero test. Gurari and Ibarra in [GI79] proved that Presburger relations and Presburger functions can be recognized by reversal-bounded counters automata. [IL81] gave an algebraic view in showing that Presburger functions is exactly the closure by composition of the six following elementary functions: projection, successor, addition, positive subtraction, partial multiplication, integer part. Chan and Ibarra in [CI83] showed that Presburger functions are computable in linear space and quadratic time. Hence, the different characterizations of the class of Presburger functions say that this class is recognized by reversal-bounded counters automata or by flat counter automata, is the closure by composition of 6 elementary functions and that they are computable in linear space and quadratic time. Surprisingly, to the best of our knowledge, Presburger functions have not been geometrically characterized.

Our contribution: We present two results. First, we extend the well-known characterization of Presburger-definable sets (i.e. FO $(\mathbb{Z}, \leq, +)$-definable sets) by semi-linear sets to FO $(\mathbb{R}, \mathbb{Z}, \leq, +)$-definable sets. We show that a set is FO $(\mathbb{R}, \mathbb{Z}, \leq, +)$-definable if and only if it is a finite union of *polinear sets* where a polinear set is the sum $L + C$ of a linear set L and of a non-empty rational polyhedral convex set C. Second, we characterize functions definable in FO $(\mathbb{R}, \mathbb{Z}, \leq, +)$. If it is well-known that linear functions (with a Presburger domain) are Presburger-definable, there exist some Presburger functions which are not linear. So the problem we address here is to obtain a geometrical characterization of functions definable in FO $(\mathbb{R}, \mathbb{Z}, \leq, +)$. We show that these functions are exactly the piecewise linear functions (with rational matrices) with a FO $(\mathbb{R}, \mathbb{Z}, \leq, +)$-definable partition as domain. More generally, we show that FO $(\mathbb{R}, \mathbb{Z}, \leq, +)$-definable functions are exactly piecewise rational linear functions.

2 Presburger sets

In this section, we introduce the class of so-called *polinear sets* equal to the sum of a *polyhedral convex set* and a *linear set*. We prove that sets definable in FO $(\mathbb{R}, \mathbb{Z}, \leq, +)$ are exactly the finite unions of polinear sets.

Given a finite set $P \subseteq \mathbb{Z}^n$, we denote by $P^* = \{\vec{v} \in \mathbb{Z}^n \mid \vec{v} = \sum_{i=1}^{k} \alpha_i \vec{p_i},\ k \in \mathbb{N},\ \alpha_i \in \mathbb{N}\}$. A set $L \subseteq \mathbb{Z}^n$ is said *linear* if there exist a vector $\vec{b} \in \mathbb{Z}^n$ and a finite set $P \subseteq \mathbb{Z}^n$ such that $L = \vec{b} + P^*$. Since [GS66], we know that a subset of $\mathbb{Z}^n$ is definable in the Presburger logic FO $(\mathbb{Z}, \leq, +)$ if and only if it is a finite union of linear sets. A set $H \subseteq \mathbb{R}^n$ is called a *rational half-space* if there exists $\vec{\alpha} \in \mathbb{Q}^n$, $\# \in \{\geq, >\}$ and $\beta \in \mathbb{Q}$ such that $H = \{\vec{v} \in \mathbb{R}^n \mid \sum_{i=1}^{n} \alpha_i . v_i \# \beta\}$. A set $C \subseteq \mathbb{R}^n$ is said *rational polyhedral convex* if it is equal to a finite intersection of rational half-spaces. Recall (the Fourier-Motskin elimination) that a subset of $\mathbb{R}^n$ is definable in the additive logic of the reals FO $(\mathbb{R}, \leq, +)$ if and only if it is a finite union of *rational polyhedral convex sets*. In the sequel, the polyhedral convex sets considered are rational.

Definition 1 *A set $S \subseteq \mathbb{R}^n$ is said* polinear *if $S = C + L$ where $C \subseteq \mathbb{R}^n$ is a non-empty polyhedral convex set and $L \subseteq \mathbb{Z}^n$ is a linear set.*

Remark 2 *Observe that a polinear set S included in $\mathbb{Z}^n$ is a linear set. Let $C \subseteq \mathbb{R}^n$ be a non-empty polyhedral convex set and let $L \subseteq \mathbb{Z}^n$ be a linear set such that $S = C + L$. There exists $\vec{b} \in \mathbb{Z}^n$ and a finite subset $P \subseteq \mathbb{Z}^n$ such that $L = \vec{b} + P^*$. From $C + L \subseteq \mathbb{Z}^n$ we deduce that $C + \vec{b} \subseteq \mathbb{Z}^n$ and in particular $C \subseteq \mathbb{Z}^n$. Note that if C contains two distinct vectors then it contains some vectors that are not in $\mathbb{Z}^n$. As C is non empty and included in $\mathbb{Z}^n$, we deduce that C is reduced to $C = \{\vec{c}\}$ where $\vec{c} \in \mathbb{Z}^n$. Observe that $S = (\vec{c} + \vec{b}) + P^*$ is* *a linear set.*

A polinear set $S = C + L$ is said *canonical* if $C \subseteq [0,1[^n$. The projection of a set $S \subseteq \mathbb{R}^n$ is a set of the form $\Pi_{n,i}(S) \subseteq \mathbb{R}^{n-1}$ where $\Pi_{n,i} : \mathbb{R}^n \to \mathbb{R}^{n-1}$ is the *projection function* defined by $\Pi_{n,i}(\vec{x}) = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$.

Proposition 3 *The class of finite union of canonical polinear sets is stable by the boolean operations (intersection, union and complement), by projection and inverse projection.*

PROOF. Observe that class of finite union of canonical polinear sets is stable by union. Moreover, since the intersection of two canonical polinear sets C_1+L_1 and C_2+L_2 is $(C_1 \cap C_2)+(L_1 \cap L_2)$, we deduce that this class is also stable by intersection by recalling that the intersection of two linear sets is a finite union of linear sets. This class is also stable by complements since the complement of $C+L$ is equal to $(C+(\mathbb{Z}^n \backslash L)) \cup (([0,1[^n \backslash C)+L)$ and by recalling that the complement of a linear set is a finite union of linear sets and the complement of a polyhedral convex set is a finite union of polyhedral convex sets. The stability by projection is obtained by remarking that $\Pi_{n,i}(C+L) = \Pi_{n,i}(C) + \Pi_{n,i}(L)$ and by remarking that $\Pi_{n,i}(C)$ is a polyhedral convex set included in $[0,1[^{n-1}$ and $\Pi_{n,i}(L)$ of a linear set. The stability by inverse projection is obtained symmetrically by observing that $\Pi_{n,i}^{-1}(C+L) = ([0,1[^n \cap \Pi_{n,i}^{-1}(C)) + (\mathbb{Z}^n \cap \Pi_{n,i}^{-1}(L))$ and by remarking that $[0,1[^n \cap \Pi_{n,i}^{-1}(C)$ is a polyhedral convex set included in $[0,1[^n$ and $\mathbb{Z}^n \cap \Pi_{n,i}^{-1}(L)$ is a linear set. □

Theorem 4 *A set is definable in* FO $(\mathbb{R}, \mathbb{Z}, \leq, +)$ *iff it is a finite union of polinear sets.*

PROOF. Naturally a finite union of polinear sets is definable in FO $(\mathbb{R}, \mathbb{Z}, \leq, +)$. For the converse, note that the predicates $\mathbb{R}$, $\mathbb{Z}$, $\leq$ are canonical polinear. The set of vectors $\{\vec{s} \in \mathbb{R}^3 \mid s_1 + s_2 = s_3\}$ is equal to $S_1 \cup S_2$ where S_1 and S_2 are the following two canonical polinear sets:

$$S_1 = \{\vec{c} \in [0,1[^3 \ \mid \ c_1 + c_2 = c_3\} + P^*$$
$$S_2 = \{\vec{c} \in [0,1[^3 \ \mid \ c_1 + c_2 = c_3 + 1\} + (0,0,1) + P^*$$
$$P = \{(1,0,1), -(1,0,1), (0,1,1), -(0,1,1)\}$$

An induction over the the grammar of FO $(\mathbb{R}, \mathbb{Z}, \leq, +)$ proves that any set definable in this logic is a finite union of canonical polinear sets. In particular such a set is a finite union of polinear sets. □

Corollary 5 *A set $S \subseteq \mathbb{Z}^n$ definable in* FO $(\mathbb{R}, \mathbb{Z}, \leq, +)$ *is definable in* FO $(\mathbb{Z}, \leq, +)$.

PROOF. As S is definable in $\text{FO}(\mathbb{R}, \mathbb{Z}, \leq, +)$ it is equal to a finite union of polinear sets. As a polinear set included in $\mathbb{Z}^n$ is a linear set we deduce that S is a finite union of linear sets. Thus S is definable in $\text{FO}(\mathbb{Z}, \leq, +)$. □

3 Presburger-functions

In this section, a function $f : \mathbb{R}^n \to \mathbb{R}^q$ is proved definable in $\text{FO}(\mathbb{R}, \mathbb{Z}, \leq, +)$ if and only if it is *piecewise rational linear*.

The *graph* of a function $f : \mathbb{R}^n \to \mathbb{R}^q$ defined on a domain $\text{dom}(f) \subseteq \mathbb{R}^n$ is the set $G_f = \{(\vec{a}, \vec{b}) \in \text{dom}(f) \times \mathbb{R}^q \mid \vec{b} = f(\vec{a})\}$. A subset $S \subseteq \mathbb{R}^n \times \mathbb{R}^q$ is said *functional* if it is equal to the graph of such a function.

A function $f : \mathbb{R}^n \to \mathbb{R}^q$ is said *definable* in $\text{FO}(\mathbb{R}, \mathbb{Z}, \leq, +)$ if there exists a formula $\psi(\vec{x}, \vec{y})$ in this logic that encodes G_f. Observe that we can effectively decide if a formula $\psi(\vec{x}, \vec{y})$ encodes the graph of a function since this problem reduces to the satisfiability of the following formula:

$$\forall \vec{x} \forall \vec{y}_1 \forall \vec{y}_2 \; (\psi(\vec{x}, \vec{y}_1) \wedge \psi(\vec{x}, \vec{y}_2)) \Longrightarrow \vec{y}_1 = \vec{y}_2$$

Definition 6 *A function $f : \mathbb{R}^n \to \mathbb{R}^q$ is said* rational linear *if dom(f) is polinear and there exist $M \in \mathcal{M}_{n,q}(\mathbb{Q})$ and $\vec{v} \in \mathbb{Q}^q$ such that $f(\vec{a}) = M.\vec{a} + \vec{v}$ for any $\vec{a} \in$ dom(f). A function f is said* piecewise rational linear *if there exists a sequence $D_1, \dots, D_k$ such that dom$(f) = \bigcup_{j=1}^{k} D_j$ such that the restriction of f to each D_j is a rational linear function.*

Lemma 7 *A function $f : \mathbb{Q}^n \to \mathbb{Q}^q$ such that dom(f) + dom$(f) \subseteq$ dom(f) satisfies $G_f + G_f \subseteq G_f$ if and only if there exists $M \in \mathcal{M}_{n,q}(\mathbb{Q})$ such that $f(\vec{a}) = M\vec{a}$ for any $\vec{a} \in$ dom(f).*

PROOF. Naturally if there exists $M \in \mathcal{M}_{n,q}(\mathbb{Q})$ such that $f(\vec{a}) = M\vec{a}$ for any $\vec{a} \in \text{dom}(f)$ then $G_f + G_f \subseteq G_f$. Conversely, assume that $G_f + G_f \subseteq G_f$ and let us prove that there exists $M \in \mathcal{M}_{n,q}(\mathbb{Q})$ such that $f(\vec{a}) = M\vec{a}$ for any $\vec{a} \in \text{dom}(f)$. Let V be the $\mathbb{Q}$-*vector space* generated by G_f. Recall that V is the set of finite sums $\sum_{i=1}^{k} \lambda_i(\vec{a}_i, f(\vec{a}_i))$ where $k \geq 0$, $(\vec{a}_i)_{1 \leq i \leq k}$ is a sequence of vectors in $\text{dom}(f)$ and $\lambda_i \in \mathbb{Q}$. Classical linear algebra results show that it is sufficient to prove that V is functional to get the existence of M. Consider $(\vec{a}, \vec{b}_1)$ and $(\vec{a}, \vec{b}_2)$ in V. There exists $k \geq 0$, a sequence $((\vec{a}_i, f(\vec{a}_i)))_{1 \leq i \leq k}$ of vectors in G_f, and two sequences $(\lambda_{i,1})_{1 \leq i \leq k}$ and $(\lambda_{i,2})_{1 \leq i \leq k}$ in $\mathbb{Q}$ such that $(\vec{a}, \vec{b}_j) = \sum_{i=1}^{k} \lambda_{i,j}(\vec{a}_i, f(\vec{a}_i))$ where $j \in \{1, 2\}$. Hence $(\vec{0}, \vec{b}_1 - \vec{b}_2) = \sum_{i=1}^{k}(\lambda_{i,1} -$

$\lambda_{i,2})(\vec{a}_i, f(\vec{a}_i))$. Let us consider $d \in \mathbb{N}\backslash\{0\}$ larger enough such that $d.(\lambda_{i,1} - \lambda_{i,2}) \in \mathbb{Z}$ for any $1 \leq i \leq k$. As any integer in $\mathbb{Z}$ is equal to the difference of two integers in $\mathbb{N}$, there exist $\mu_{i,1}, \mu_{i,2} \in \mathbb{N}$ such that $\mu_{i,2} - \mu_{i,1} = d.(\lambda_{i,1} - \lambda_{i,2})$ for any $1 \leq i \leq k$. Let $(\vec{v}_j, \vec{w}_j) = \sum_{i=1}^{k} \mu_{i,j}(\vec{a}_i, f(\vec{a}_i))$ where $j \in \{1, 2\}$. From $(\vec{0}, d.\vec{b}_1 - d.\vec{b}_2) = \sum_{i=1}^{k}(\mu_{i,2} - \mu_{i,1})(\vec{a}_i, f(\vec{a}_i))$ we deduce that $\vec{v}_1 = \vec{v}_2$. Moreover as $G_f + G_f \subseteq G_f$ an induction proves that $\sum_{i=1}^{k} \mu_{i,j} f(\vec{a}_i) = f(\sum_{i=1}^{k} \mu_{i,j}\vec{a}_i)$. Thus $\vec{w}_j = f(\vec{v}_j)$ for any $j \in \{1, 2\}$. From $\vec{v}_1 = \vec{v}_2$ we deduce that $\vec{w}_1 = \vec{w}_2$. We have proved that $d.\vec{b}_1 - d.\vec{b}_2 = \vec{0}$. Thus $\vec{b}_1 = \vec{b}_2$ and we deduce that V is functional. □

Proposition 8 *A function is rational linear if and only if its graph is polinear.*

PROOF. Assume first that $f : \mathbb{R}^n \to \mathbb{R}^q$ is a rational linear function and let us prove that G_f is polinear. There exists $M \in \mathcal{M}_{n,q}(\mathbb{Q})$ and $\vec{v} \in \mathbb{Q}^n$ such that $f(\vec{a}) = M\vec{a} + \vec{v}$ for any $\vec{a} \in \mathrm{dom}(f)$. Moreover, as $\mathrm{dom}(f)$ is polinear there exists a non-empty polyhedral convex set C and a finite set $P \subseteq \mathbb{Z}^n$ such that $\mathrm{dom}(f) = C + P^*$. Just observe that $G_f = \{(\vec{c}, f(\vec{c})) \mid \vec{c} \in C\} + \{(\vec{p}, M\vec{p}) \mid \vec{p} \in P\}^*$. Thus G_f is polinear. Conversely, let us consider a function $f : \mathbb{R}^n \to \mathbb{R}^q$ such that G_f is polinear and let us prove that f is a rational linear function. Since G_f is polinear, there exists a non-empty polyhedral convex set $C \subseteq \mathbb{R}^n \times \mathbb{R}^q$ and a finite subsets $P \subseteq \mathbb{Z}^n \times \mathbb{Z}^q$ such that $S = C + P^*$. Observe that $\mathrm{dom}(f)$ is polinear since this set is equal to $C_0 + P_0^*$ where $C_0 = \{\vec{c} \in \mathbb{R}^n \mid \exists \vec{d} \in \mathbb{R}^q\ (\vec{c}, \vec{d}) \in C\}$ and $P_0 = \{\vec{p} \in \mathbb{Z}^n \mid \exists \vec{q} \in \mathbb{Z}^q\ (\vec{p}, \vec{q}) \in P\}^*$. Moreover, recall [Sch87] that any non-empty rational polyhedral convex set C contains at least one rational vector. Thus there exists $(\vec{c}, \vec{d}) \in C \cap (\mathbb{Q}^n \times \mathbb{Q}^q)$. Note that $\vec{d} = f(\vec{c})$. Let $S = G_f \cap (\mathbb{Q}^n \times \mathbb{Q}^q) - (\vec{c}, \vec{d})$ and observe that S is functional and $S + S \subseteq S$. Lemma 7 proves that there exists $M \in \mathcal{M}_{n,q}(\mathbb{Q})$ such that $(\vec{a}, \vec{b}) \in S$ if and only if $\vec{b} = M.\vec{a}$. Let us consider the rational vector $\vec{v} = \vec{d} - M.\vec{c}$. As $S \cap (\mathbb{Q}^n \times \mathbb{Q}^q)$ is equal to $(C \cap (\mathbb{Q}^n \times \mathbb{Q}^q)) + P^*$, we deduce that $f(\vec{a}) = M\vec{a} + \vec{v}$ for any $\vec{a} \in (C_0 \cap \mathbb{Q}^n) + P_0^*$. Recall [Sch87] that any vector in a non-empty rational polyhedral convex set C_0 is the limit of a sequence of rational vectors in C_0. Thus $f(\vec{a}) = M\vec{a} + \vec{v}$ for any $\vec{a} \in C_0 + P_0^*$. We have proved that f is a rational linear function. □

Theorem 9 *A function is definable in* $\mathrm{FO}(\mathbb{R}, \mathbb{Z}, \leq, +)$ *if and only if it is piecewise rational linear.*

PROOF. Let $f : \mathbb{R}^n \to \mathbb{R}^q$ be a function. Naturally, if f is piecewise rational linear then f is definable in $\mathrm{FO}(\mathbb{R}, \mathbb{Z}, \leq, +)$. Conversely, assume that f is definable in $\mathrm{FO}(\mathbb{R}, \mathbb{Z}, \leq, +)$. There exists a finite sequence $S_1, \ldots, S_k$ of polinear sets such that $G_f = \bigcup_{j=1}^{k} S_j$. Observe that S_j is functional since it is included in the functional set G_f. From proposition 8 we deduce that there exists a polinear set $D_j \subseteq \mathbb{R}^n$ such that $f|_{D_j}$ is rational linear and such that S_j is the graph of this function. Observe that $\text{dom}(f) = \bigcup_{j=1}^{k} D_j$. Thus f is piecewise linear. $\square$

We deduce the following corollary when considering functions over the integer vectors.

Corollary 10 *A function $f : \mathbb{Z}^n \to \mathbb{Z}^q$ is definable in* $\text{FO}(\mathbb{Z}, \leq, +)$ *if and only if there exists a sequence $D_1, \ldots, D_k \subseteq \mathbb{Z}^n$ of linear sets such that $dom(f) = \bigcup_{j=1}^{k} D_j$, a sequence $M_1, \ldots, M_k \in \mathcal{M}_{n,q}(\mathbb{Q})$ and a sequence $\vec{v_1}, \ldots, \vec{v_k} \in \mathbb{Q}^q$ such that $f(\vec{x}) = M_j\vec{x} + \vec{v_j}$ for any $\vec{x} \in D_j$ and for any $1 \leq j \leq k$.*

PROOF. Let $f : \mathbb{Z}^n \to \mathbb{Z}^q$ be definable in $\text{FO}(\mathbb{Z}, \leq, +)$. Theorem 9 shows that there exists a sequence of polinear sets $D_1, \ldots, D_k$ such that $\text{dom}(f) = \bigcup_{j=1}^{k} D_j$, a sequence $M_1, \ldots, M_k \in \mathcal{M}_{n,q}(\mathbb{Q})$ and a sequence $\vec{v_1}, \ldots, \vec{v_k} \in \mathbb{Q}^q$ such that $f(\vec{x}) = M_j\vec{x} + \vec{v_j}$ for any $\vec{x} \in D_j$ and for any $1 \leq j \leq k$. As D_j is a polinear set included in $\text{dom}(f) \subseteq \mathbb{Z}^n$, remark 2 proves that D_j is a linear set. We are done since the converse is immediate. $\square$

Finally, we observe that neither $M_j \in \mathcal{M}_{n,q}(\mathbb{Q})$ nor $\vec{v_j} \in \mathbb{Q}^q$ can be replaced by $M_j \in \mathcal{M}_{n,q}(\mathbb{Z})$ or $\vec{v_j} \in \mathbb{Z}^q$.

Lemma 11 *Let $f : \mathbb{Z} \to \mathbb{Z}$ be the function defined by the Presburger formula $\psi(x, y) := (x = 2y + 1)$, let $D_1, \ldots, D_k$ be a sequence of linear sets such that $dom(f) = \bigcup_{j=1}^{k} D_j$ and let $M_1, \ldots, M_k \in \mathbb{Q}$ and $v_1, \ldots, v_k \in \mathbb{Q}$ be sequences such that $f(x) = M_j x + v_j$ for any $x \in D_j$ and for any $1 \leq j \leq k$. There exists $1 \leq j \leq k$ such that $M_j \notin \mathbb{Z}$ and $\vec{v_j} \notin \mathbb{Z}$.*

PROOF. As $\text{dom}(f)$ is infinite, there exists $1 \leq j \leq k$ such that D_j is infinite. Thus there exists in D_j two distinct integers x_1, x_2. On one hand $f(x_1) = \frac{x_1 - 1}{2}$ and $f(x_2) = \frac{x_2 - 1}{2}$. On the other hand $f(x_1) = M_j x_1 + v_j$ and $f(x_2) = M_j x_2 + v_j$. As $x_1 \neq x_2$ we deduce that $M_j = \frac{1}{2}$ and $v_j = -\frac{1}{2}$. $\square$

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