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Abstract

Call a topology well *if and only if every open is compact. The starting point of this note is that this notion generalizes that of well-quasi order, in the sense that an Alexandroff topology is well if and only if its specialisation quasi-ordering is well. For more general topologies, this opens the way to verifying infinite transition systems based on non-well quasi ordered sets, but where the* Pre *operator satisfies an additional continuity assumption. The technical development rests heavily on techniques arising from topology and domain theory, including sobriety and the de Groot dual of a stably compact space. We show that the category* $\mathbf{Well}$ *of well topological spaces is finitely complete and finitely cocomplete. Finally, we note that if X is well topologized, then the set of all (even infinite) subsets of X is again well topologized, a result that fails for well-quasi orders.*

1. Introduction

Consider the following short, funny-looking definition: a topology $\mathcal{O}$ on a X is *well* if and only if every open subset of X is compact. Our purpose here is to explain how this (new) notion generalizes the theory of well quasi-orderings.

Recall that a well quasi-ordering is a quasi-ordering (a reflexive and transitive relation) that has no infinite antichain (a set of incomparable elements). One use of well quasi-orderings is in verifying *well-structured transition systems* [2, 4, 9, 11]. These are transition systems, usually infinite-state, with two ingredients.

First, a *well* quasi-ordering $\leq$ on the set X of states. Second, the transition relation δ commutes with the quasi-ordering, in the sense that if $x \ \delta \ y$ and $x \leq x'$, then there is a state y' such that $x' \ \delta \ y'$ and $y \leq y'$:

$$\begin{tikzcd} x \arrow[r, "\leq"] \arrow[d, "\delta"'] & x' \arrow[d, dotted, "\delta"] \\ y \arrow[r, dotted, "\leq"] & y' \end{tikzcd} \qquad (1)$$

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Examples include Petri nets, VASS [12], lossy channel systems [3], timed Petri nets [6] to cite a few.

For any subset A of X, let $\mathrm{Pre}^{\exists}\delta(A) = \delta^{-1}(A)$ be the preimage $\{x \in X | \exists y \in A \cdot x \ \delta \ y\}$. The commutation property ensures that the preimage $\mathrm{Pre}^{\exists}\delta(V)$ of any upward-closed subset V is again upward-closed (V is upward-closed iff whenever $x \in V$ and $x \leq x'$, then $x' \in V$). Standard arguments then show that one may compute $\mathrm{Pre}^{\exists *} \ \delta(V)$, the set of states in X from which we can reach some state in V in finitely many steps: Compute the set V_i of states from which we can reach some state in V in at most i steps, backwards, by letting $V_0 = V$, $V_{i+1} = V_i \cup \mathrm{Pre}^{\exists} \ \delta(V_i)$: this stabilizes at some stage i, where $V_i = \mathrm{Pre}^{\exists *} \ \delta(V)$.

This provides an algorithm for coverability: given two states $x, x' \in X$, is there a trace $x = x_0 \ \delta \ x_1 \ \delta \ \ldots \ \delta \ x_k$ such that $x' \leq x_k$? Just check $x \in \mathrm{Pre}^{\exists *}(\uparrow x')$, where $\uparrow x'$ is the upward-closed set $\{y \in X | x' \leq y\}$.

Outline. We generalize this by replacing quasi-orderings by topologies. We shall definitely rest on the rich relationship between theories of order and topology. We recapitulate what we need in two sections, Section 2 for basic notions, and Section 5 for more advanced concepts such as Stone duality, sobriety, and stable compactness which we don't need in earlier sections. Our contribution occupies the other sections. We first show the tight relationship between well-quasi orders and well topologies in Section 3, and show a few easy constructions of new well topologies from given well topologies in Section 4. This culminates in showing that the category $\mathbf{Well}$ of well topological spaces is finitely cocomplete. Section 6 is technically more challenging, and characterizes those well topological spaces that are also sober. This is the cornerstone of the theory. E.g., this is instrumental to show that $\mathbf{Well}$ is finitely complete, and that the Hoare space of a well topological space is again well. We show the latter in Section 7. We then prove the unexpected result that the set of *all* subsets of a well topological space X (even infinite ones) has a topology that makes it well. This would be wrong in a pure theory of orders; topology makes the difference. Our theory of well sober spaces also suggests an alternative algorithm for coverability based on computing downward-closed sets, which we describe in Section 8. We conclude in Section 9.

We stress that this paper is not specifically geared towards applications. Its aim is rather to lay the theoretical basis for well topologies; we hope to have done it... well.

Related Work. If $\leq$ is a quasi-ordering on X then let $\mathbb{P}_{fin}(X)$ be the set of finite subsets of X, and order it by $\leq^\sharp$, where $A \leq^\sharp B$ iff for every $y \in B$ there is an $x \in A$ such that $x \leq y$. It is well-known that $\leq^\sharp$ needs not be well even when $\leq$ is well. This is a shortcoming, among others, of the theory of well quasi-orderings. Such shortcomings led Nash-Williams [20] to invent better quasi-orderings (bqos). Bqos have a rather unintuitive definition but a wonderful theory, see [18]. The only application of bqos we know of to verification problems is by Abdulla and Nylén [5], where it is used to show the termination of the backward reachability iteration, using *disjunctive* constraints.

This paper is not on bqos, and in fact not specifically on well quasi-orderings. While bqos are *restrictions* of well quasi-orderings, well topologies *generalize* the latter. We hope that well topologies will be valuable in verification in the future. The fact that the powerset $\mathbb{P}_{fin}(X)$, with the upper topology of $\leq^\sharp$ (see above), and that $\mathbb{P}(X)$, with another topology, are well topologized whenever X is (Section 7) is a promising result.

One may legitimately say that our work is more connected to topology, and in particular to topology as it is practiced in domain theory. As we shall see later, the notions of specialisation quasi-ordering of a topological space, of upper, Scott and Alexandroff topologies, of sober space, of sobrification of a space, and of stably compact spaces are central to our work. Topology and domain theory form another wonderful piece of mathematics, and one may consult [10, 7, 15, 19]. This being said, we believe well topologies have never been defined or studied before. In particular, there is no connection whatsoever with the Ellentuck topology on $X^{(\omega)}$ (X an infinite subset of $\mathbb{N}$), a topology that is instrumental in the study of bqos [17].

2. Preliminaries I: Order and Topology

A *topology* $\mathcal{O}$ on a set X is a collection of subsets of X that is closed under arbitrary unions and finite intersections. We say that X itself is a topological space, leaving $\mathcal{O}$ implicit. The elements of $\mathcal{O}$ are the *opens*, their complements are *closed*. The largest open contained in A is its *interior*, the smallest closed subset $cl(A)$ containing it is its *closure*.

Every topology comes with a *specialisation quasi-ordering* $\leq$, defined as $x \leq y$ if and only if every open that contains x also contains y. Equivalently, $x \in cl\{y\}$. It is easy to see that every open is upward-closed with respect to $\leq$. The converse need not hold. A subset A of X is *saturated* if and only if A equals the intersection of all opens U containing A. A set is saturated if and only if it is upward-closed with respect to the specialisation quasi-ordering $\leq$.

A subset K of X is *compact* if and only if every open cover $(U_i)_{i\in I}$ (a family of opens U_i whose union contains K) contains a finite subcover. It is equivalent to say that K is compact if and only if, for every directed family $(U_i)_{i\in I}$ of opens such that $K \subseteq \bigcup_{i\in I} U_i$, then $K \subseteq U_i$ for some $i \in I$ already. Such a family is *directed* if and only if it is not empty, and whatever $i, j \in I$, then $U_i \cup U_j \subseteq U_k$ for some $k \in I$. (A family $(x_i)_{i\in I}$ of elements of a set quasi-ordered by $\leq$ is a non-empty family such that for every $i, j \in I$ there is $k \in I$ such that $x_i \leq x_k$ and $x_j \leq x_k$.)

Write $\uparrow E = \{x \in X | \exists y \in E \cdot y \leq x\}$, $\downarrow E = \{x \in X | \exists y \in E \cdot x \leq y\}$. The set $\uparrow E$ is also the smallest saturated set containing E. If K is compact, then $\uparrow K$ is, too, and is also saturated. We shall usually reserve the letter Q and variants for saturated compacts.

When E is finite, then $\uparrow E$ is compact saturated: call these the *finitary compacts*. Similarly, $\downarrow E$ is always closed; call such closed subsets the *finitary closed subsets*.

We have gone one direction, from topology to quasi-orderings. There are in general many return paths: there are in general several topologies having a given specialisation quasi-ordering $\leq$. The finest, i.e., the one that has the most opens, is the *Alexandroff topology* of $\leq$. Its opens are exactly the upward-closed subsets of X with respect to $\leq$. The coarsest (the one with the least opens) is the *upper topology*. It is generated by the complements of sets of the form $\downarrow \{x\}$, $x \in X$. That is, it is the coarsest topology making the sets $\downarrow \{x\}$ closed. Another way of describing it is to say that the closed sets in the upper topology are exactly the unions of subsets of the form $\downarrow E$, E finite. An intermediate topology is the *Scott topology*, whose opens are those upward-closed subsets U such that for every directed family $(x_i)_{i\in I}$ that has a least upper bound in U, then there is $i \in I$ such that $x_i \in U$. The latter crops up in domain theory, where a *cpo* is a partially ordered set where every directed family has a least upper bound.

A topological space is *Alexandroff-discrete* iff every intersection of opens is again open. Equivalently, iff its topology be the Alexandroff topology of its specialisation quasi-ordering. While every finitary compact is compact saturated, the converse holds in Alexandroff-discrete spaces.

A map f from X to Y is *continuous* iff $f^{-1}(V)$ is open in X for every open V of Y. Any continuous function is monotonic with respect to the specialisation quasi-orderings of X and Y: writing them both $\leq$, $x \leq y$ implies $f(x) \leq f(y)$. Not all monotonic functions are continuous in general. When both X and Y are Alexandroff-discrete, then every monotonic function is continuous. While we usually think of continuity as being a stronger requirement than monotonicity, we must be aware that continuity also *generalizes* monotonicity, in the sense that monotonicity is just continuity with respect to Alexandroff topologies.

For any continuous map $f : X \to Y$, the image $f(K)$ of any compact K of X by f is compact in Y.

When X and Y are equipped with Scott topologies, $f : X \to Y$ is continuous if and only if f is *Scott-continuous*, i.e., f is monotonic and, for every directed family $(x_i)_{i \in I}$ in X having a least upper bound x, the family $(f(x_i))_{i \in I}$ (which is directed in Y) admits $f(x)$ as least upper bound.

Continuity notions extend to binary relations. A relation R from X to Y is a subset of $X \times Y$. It is *lower semi-continuous* if and only if $\mathrm{Pre}^{\exists} R(V) = \{x \in X | \exists y \in V \cdot x \; R \; y\}$ is open whenever V is. It is *upper semi-continuous* if and only if $\mathrm{Pre}^{\forall} R(V) = \{x \in X | \forall y \cdot x \; R \; y \Rightarrow y \in V\}$ is open whenever V is.

3. Well-Quasi Orderings and Well Topologies

We first show the precise relationship between well-quasi orderings and well topologies.

Proposition 3.1 *Consider the following properties of a topological space X, with specialisation quasi-ordering $\leq$:*
1. X is well;
2. X is a space where every open is finitary compact;
3. $\leq$ is a well quasi-ordering.
Then 3 implies 2, 2 implies 1, and if X is Alexandroff-discrete then 1 implies 3.

Proof. $3 \Rightarrow 2$: Every open is upward-closed, and every upward-closed subset is finitary compact by assumption. $2 \Rightarrow 1$ is obvious. Let us show $1 \Rightarrow 3$, assuming X Alexandroff-discrete. Each upward-closed subset is open, hence compact saturated by 1, hence finitary compact since X is Alexandroff-discrete. □

I.e., in the subcategory of Alexandroff-discrete spaces, well topologies are exactly the topological counterpart of the order-theoretic notion of well quasi-ordering. In particular, there are many well topological spaces: equip any well quasi-ordered set X with its Alexandroff topology. As we shall see later, there are other well topologies.

The following characterization of well topologies will be useful. Let $\Omega(X)$ be the complete lattice of all opens of X, ordered by inclusion. A set Y with a quasi-ordering $\sqsubseteq$ has the *ascending chain condition* iff every infinite ascending chain $y_0 \sqsubseteq y_1 \sqsubseteq \ldots \sqsubseteq y_k \sqsubseteq \ldots$ *stabilizes*, i.e., there is an integer N such that $y_k \sqsubseteq y_N$ for every $k \geq N$.

Proposition 3.2 *Let X be a topological space. Then X is well if and only if $\Omega(X)$ has the ascending chain condition.*

Proof. Assume X well, and let $U_0 \subseteq U_1 \subseteq \ldots \subseteq U_k \subseteq \ldots$ be an infinite ascending chain. $U = \bigcup_{n \in \mathbb{N}} U_n$ is open, hence compact. Notice that the family $(U_n)_{n \in \mathbb{N}}$ is directed, so $U \subseteq U_N$ for some $N \in \mathbb{N}$. For every $k \geq N$, then, $U_k \subseteq \bigcup_{n \in \mathbb{N}} U_n \subseteq U_N \subseteq U_k$, so $U_k = U_N$.

Conversely, assume X is not well, and construct an infinite non-stabilizing ascending chain. Let U be a non-compact open subset of X. There is an open cover $(V_i)_{i \in I}$ of U that has no finite subcover. By induction on $k \in \mathbb{N}$, build an infinite sequence of opens W_k, as follows. At each step W_k will be a finite union of opens of the form V_i, $i \in I$. Moreover, the sequence $W_1, W_2, \ldots, W_k, \ldots$, will be strictly increasing. Assume $W_1, W_2, \ldots, W_{k-1}$ have been built. By assumption, W_{k-1} does not contain U, otherwise W_{k-1} would induce a finite subcover of U. So there is an element x_k of U outside W_{k-1}. This x_k must belong to some V_{i_k}, $i_k \in I$. Let $W_k = W_{k-1} \cup V_{i_k}$. Since $x_k \in W_k$ but $x_k \notin W_{k-1}$, W_{k-1} is strictly contained in W_k. □

One may extend the backward computation of $\mathrm{Pre}^{\exists *}$ described in the introduction easily, as follows.

Definition 3.3 *A* topological well-structured transition system *is a pair (X, δ), where X, the* state space*, is a well topological space, and δ, the* transition relation*, is lower semi-continuous.*

Indeed, the sequence of backward iterates V_i terminates, by Proposition 3.2:

Proposition 3.4 *Let (X, δ) be a topological well-structured transition system. For any open subset V, let $V_0 = V$, $V_{i+1} = V_i \cup \mathrm{Pre}^{\exists} \delta(V_i)$. The sequence $(V_i)_{i \in \mathbb{N}}$ is an ascending chain, which stabilizes on $\mathrm{Pre}^{\exists *}(V)$.*

We retrieve that backwards iterations terminate on well-structured transition systems (a well-known fact), because:

Proposition 3.5 *Each well-structured transition system (X, δ) is a topological well-structured transition system and conversely, where X is seen with its Alexandroff topology.*

Write $\overline{A}$ the complement of A, as a subset of X. Observe that $\mathrm{Pre}^{\forall}\delta(\overline{A}) = \overline{\mathrm{Pre}^{\exists}\delta(A)}$, that complements of upward-closed sets are downward-closed, and complements of downward-closed sets are upward-closed.

It follows that (again when X is Alexandroff-discrete) δ is upper semi-continuous if and only if the dual of Diagram (1) holds:

$$\begin{tikzcd} x \arrow[r, "\leq"] \arrow[d, dotted, "\delta"'] & x' \arrow[d, "\delta"] \\ y \arrow[r, dotted, "\leq"] & y' \end{tikzcd} \qquad (2)$$

One may then go a bit further than just reachability. Define the following negation-free fragment of the modal μ-calculus. Let $L = L_{must} \cup L_{may}$ be a finite set of *transition* *labels*, taken as the (not necessarily disjoint) union of two subsets of *must labels* and *may labels* respectively. Let $\mathcal{A}$ be a recursive set of so-called *atomic formulae*.

$$
\begin{array}{lcll}
F & ::= & A & \text{atomic formula } (A \in \mathcal{A}) \\
 & | & X & \text{variable} \\
 & | & \top & \text{true} \\
 & | & F \wedge F & \text{conjunction} \\
 & | & \perp & \text{false} \\
 & | & F \vee F & \text{disjonction} \\
 & | & [\ell]F & \text{box modality } (\ell \in L_{must}) \\
 & | & \langle \ell \rangle F & \text{diamond modality } (\ell \in L_{may}) \\
 & | & \mu X \cdot F & \text{least fixed point}
\end{array}
$$

Formulae are interpreted in a *Kripke structure* $I = (X, (\delta_\ell)_{\ell \in L}, (U_A)_{A \in \mathcal{A}})$, where X is a topological space, δ_ℓ is a binary relation on X, which is lower semi-continuous when $\ell \in L_{may}$ and upper semi-continuous when $\ell \in L_{must}$, and U_A is an open of X for every atomic formula A. An *environment* ρ maps variables X to opens of X. Define the satisfaction relation $x \models^I_\rho F$ ("F holds at state x") classically: $x \models^I_\rho A$ iff $x \in U_A$; $x \models^I_\rho X$ iff $x \in \rho(X)$; $x \models^I_\rho \top$ always; $x \models^I_\rho \perp$ never; $x \models^I_\rho F_1 \wedge F_2$ iff $x \models^I_\rho F_1$ and $x \models^I_\rho F_2$; $x \models^I_\rho F_1 \vee F_2$ iff $x \models^I_\rho F_1$ or $x \models^I_\rho F_2$; $x \models^I_\rho [\ell]F$ iff for every state y such that $x \; \delta_\ell \; y$, $y \models^I_\rho F$; $x \models^I_\rho \langle \ell \rangle F$ iff for some state y such that $x \; \delta_\ell \; y$, $y \models^I_\rho F$; and $x \models^I_\rho \mu X \cdot F$ iff $x \in \bigcup_{i=0}^{+\infty} U_i$, where $U_0 = \emptyset$, $U_{i+1} = \{z \in X | z \models^I_{\rho[X:=U_i]} F\}$; $\rho[X := U]$ is the environment mapping X to U, and every $Y \neq X$ to $\rho(Y)$.

It is easy to see that the semantics of formulae is monotonic in ρ, because our formulae are negation-free. In other words, if $x \models^I_\rho F$ and $\rho(X) \subseteq \rho'(X)$ for every variable X, then $x \models^I_{\rho'} F$. One may also show that it is (Scott-)continuous, in the sense that for every directed family $(U_i)_{i \in I}$ of opens, $x \models^I_{\rho[X:=\bigcup_{i\in I} U_i]} F$ if and only if $x \models^I_{\rho[X:=U_i]} F$ for some $i \in I$. This last statement in particular implies that the set of states x such that $x \models^I_\rho \mu X \cdot F$ is indeed a fixed point, and therefore the least fixed point of the Scott-continuous function mapping every open set U to $\{z \in X | z \models^I_{\rho[X:=U]} F\}$.

Let lfp be the least fixed point operator of Scott-continuous functions f: $\mathsf{lfp}(f) = \bigcup_{i=0}^{+\infty} f^i(\emptyset)$, and write $I \llbracket F \rrbracket_\delta \rho$ for the set of elements $z \in Z$ such that $z \models^I_\rho F$. The semantics of formulae is characterized by the clauses:

$$
\begin{array}{ll}
I \llbracket A \rrbracket_\delta \rho = U_A & I \llbracket X \rrbracket_\delta \rho = \rho(X) \\
I \llbracket \top \rrbracket_\delta \rho = X & I \llbracket F_1 \wedge F_2 \rrbracket_\delta \rho = I \llbracket F_1 \rrbracket_\delta \rho \cap I \llbracket F_2 \rrbracket_\delta \rho \\
I \llbracket \perp \rrbracket_\delta \rho = \emptyset & I \llbracket F_1 \vee F_2 \rrbracket_\delta \rho = I \llbracket F_1 \rrbracket_\delta \rho \cup I \llbracket F_2 \rrbracket_\delta \rho
\end{array}
$$

$$I \llbracket [\ell] F \rrbracket_\delta \rho = \mathrm{Pre}^\forall \delta_\ell (I \llbracket F \rrbracket_\delta \rho)$$

$$I \llbracket \langle \ell \rangle F \rrbracket_\delta \rho = \mathrm{Pre}^\exists \delta_\ell (I \llbracket F \rrbracket_\delta \rho)$$

$$I \llbracket \mu X \cdot F \rrbracket_\delta \rho = \mathsf{lfp}(\lambda U \in \Omega(X) \cdot I \llbracket F \rrbracket_\delta (\rho[X := U]))$$

An easy structural induction on F then shows that $I \llbracket F \rrbracket_\delta \rho$ is always open.

When X is well, the above formulae describe an obvious algorithm for computing $I \llbracket F \rrbracket_\delta \rho$. The only non-trivial case is for formulae of the form $\mu X \cdot F$. However, we may compute $\mathsf{lfp}(f)$ for any Scott-continuous function $f : \Omega(X) \to \Omega(X)$ (in fact for any monotonic f: when X is well, every monotonic $f : \Omega(X) \to \Omega(X)$ is Scott-continuous) by: $U_0 = \emptyset$, $U_{i+1} = f(U_i)$; this defines an ascending chain, which stabilizes by Proposition 3.2. We need to detect when this stabilizes, and so we require the inclusion relation to be decidable. Note that by Proposition 3.1, every open U can be represented as a finitary compact $\uparrow E$, that is, as a finite list of elements. Clearly, $\uparrow E \subseteq \uparrow E'$ if and only if $E' \leq^\sharp E$, i.e., for every $x \in E$, there is a $y \in E'$ such that $y \in x$. The quasi-ordering $\leq^\sharp$ is usually called the *Smyth quasi-ordering*, and is decidable as soon as $\leq$ is. Assume that U_A and $\rho(X)$ are specified by given finite sets E_A and E'_X, i.e., $U_A = \uparrow E_A$ and $\rho(X) = \uparrow E'_X$. We obtain:

Theorem 3.6 *Let X be a well topological space, and assume that its specialisation quasi-ordering $\leq$ is recursive. Assume that δ_ℓ is recursive, in the sense that for any finite subset E of X, we can compute a finite subset E' of X such that $\mathrm{Pre}^\exists \delta(\uparrow E) = \uparrow E'$ ($\ell \in L_{may}$) and $\mathrm{Pre}^\forall \delta(\uparrow E) = \uparrow E'$ ($\ell \in L_{must}$). Let U_A, $\rho(X)$ be specified by given finite sets.*

Then there is an algorithm which, given a formula F, computes a finite set E of elements such that $I \llbracket F \rrbracket_\delta \rho = \uparrow E$. In particular, checking whether $x \models^I_\rho F$ is decidable.

When $\ell \in L_{may}$, computing $\mathrm{Pre}^{\exists *} \delta_\ell(V)$ is a special case of the above evaluation scheme for formulae: $\mathrm{Pre}^{\exists *} \delta_\ell(V) = I \llbracket \mu X \cdot A \vee \langle \ell \rangle X \rrbracket_\delta \rho$, where ρ is arbitrary and $U_A = V$. One may also evaluate some forms of monotonic games [1, 8]: reading δ_{ℓ_1} as the transition relation for player 1, and δ_{ℓ_2} as that for player 2, the formula $\mu X \cdot A \vee \langle \ell_1 \rangle (B \wedge [\ell_2] X)$ is true exactly at those states x_0 such that either $x_0 \in U_A$ ("player 1 wins"), or player 1 may move to some state x_1 such that $x_1 \in U_B$ ("preventing player 2 from winning", where player 2 would win by reaching some state outside U_B) such that, whatever state x_2 player 2 moves to, either player 1 wins or player 1 may move to ... More succinctly, $\mu X \cdot A \vee \langle \ell_1 \rangle (B \wedge [\ell_2] X)$ holds at those states where player 1 has a strategy to win (reach the open U_A while preventing player 2 from reaching U_B), whatever player 2's strategy.

4. Easy Constructions of Well Topologies

We know that every well quasi-ordering yields a well topological space, through its Alexandroff topology. For example, $\mathbb{N}$, $\mathbb{N}^k$ with the componentwise ordering (Dickson's Lemma), the set of finite words over a well-quasi-ordered alphabet, ordered by embedding (Higman's Lemma), the set of finite labelled trees over a well-quasi-ordered signature, ordered by embedding (Kruskal's Theorem). However we need to know more constructions of new well topologies from old. (The easy proofs are relegated to the appendix.) In this section, we start with some easy ones. We shall turn to finite products in Section 6.2; this will turn out to be surprisingly challenging.

The first observation is similar to the fact that, for every well quasi-ordering $\leq$, any quasi-ordering $\leq'$ such that $x \leq y$ implies $x \leq' y$ is also a well quasi-ordering.

Lemma 4.1 *Every topology coarser than a well topology is well.*

So for example, if $\leq$ is a well quasi-ordering, then its Scott topology and its upper topology are well topologies. Every topological space with finitely many opens is also trivially well. This includes the case of finite spaces.

Recall that a *subspace* Y of a topological space X is a subset of X whose topology is given by the intersections of opens of X with Y—the *induced* topology.

Lemma 4.2 *Every subspace of a well topological space is well.*

Proof. Let $U_1 \cap Y \subseteq U_2 \cap Y \subseteq \ldots \subseteq U_k \cap Y \subseteq \ldots$ be an ascending chain of opens in Y. The open subset $U = \bigcup_{i=1}^{+\infty} U_i$ of X is compact, since X is well. So for some K, $U \subseteq \bigcup_{i=1}^{K} U_i$. It follows that $U \cap Y \subseteq \bigcup_{i=1}^{K} U_i \cap Y = U_K \cap Y$, hence $U_k \cap Y \subseteq U_K \cap Y$ for every $k \geq 1$. We conclude by Proposition 3.2. $\Box$

A *retract* Z of a space X is by definition such that there are two continuous maps $s : Z \to X$ (the *section*) and $r : X \to Z$ (the *retraction*) such that $r \circ s = \mathrm{id}_X$.

Corollary 4.3 *Any retract of a well topological space is well.*

The *coproduct* $X_1 + \ldots + X_k$ of k spaces is their disjoint union. Its opens are all disjoint unions of opens, one from each X_i, $1 \leq i \leq k$.

Lemma 4.4 *Let $X_1, \ldots, X_k$ be k well topological spaces. Their coproduct $X_1 + \ldots + X_k$ is well.*

Being well is also preserved under images of surjective continuous maps:

Lemma 4.5 *Let $q : X \to Z$ be a surjective continuous map. If X is well then so is Z.*

Given an equivalence relation $\equiv$ on a topological space X, the *quotient space* $X/\equiv$ is the set of equivalence classes of $\equiv$, topologized by taking the finest topology that makes the *quotient map* $q_\equiv : X \to X/\equiv$ continuous, where $q_\equiv$ maps $x \in X$ to its equivalence class.

Lemma 4.6 *Let X be a well topological space, $\equiv$ an equivalence relation on X. Then $X/\equiv$ is well.*

Let $\mathbf{Well}$ be the category of well topological spaces and continuous maps. In other words, $\mathbf{Well}$ is the full subcategory of $\mathbf{Top}$ (the category of topological spaces) consisting of well topological spaces.

Corollary 4.7 $\mathbf{Well}$ *is finitely cocomplete.*

Proof. It is enough to show that it has all finite coproducts (Lemma 4.4), and all coequalizers of parallel pairs $f, f' : X \to Y$. Such a coequalizer exists in $\mathbf{Top}$, and is given by $Y/\equiv$, where $\equiv$ is the smallest equivalence relation such that $f(x) \equiv f'(x)$ for every $x \in X$. Then apply Lemma 4.6. $\Box$

5. Preliminaries II: Sober Spaces

The material we shall now need is more involved, and can be found in [10, 7, 19] and in [15].

Stone Duality. For every topological space X, $\Omega(X)$ is a complete lattice. Every continuous map $f : X \to Y$ defines a function $\Omega(f) : \Omega(Y) \to \Omega(X)$, which maps every open subset V of Y to $\Omega(f)(V) = f^{-1}(V)$. The map $\Omega(f)$ preserves all least upper bounds (unions) and all finite greatest lower bounds (finite intersections), i.e. it is a *frame homomorphism*. Letting $\mathbf{CLat}$ be the category of complete lattices and frame homomorphisms, Ω defines a functor from $\mathbf{Top}$ to $\mathbf{CLat}^{op}$, the opposite category of $\mathbf{CLat}$.

A *frame* is any complete lattice that obeys the infinite distributivity law $x \wedge \bigvee_{i \in I} x_i = \bigvee_{i \in I}(x \wedge x_i)$. Let $\mathbf{Frm}$ be the category of frames. Its opposite category $\mathbf{Loc} = \mathbf{Frm}^{op}$ is the category of *locales*.

Going the other way around is known as *Stone duality*. A *filter* F on a complete lattice L is a non-empty upward-closed family of elements of L, such that whenever $x, y \in F$, the greatest lower bound $x \wedge y$ is also in F. A filter F is *completely prime* if and only if for every family $M \subseteq L$ whose least upper bound $\bigvee M$ is in F, then some element of M is already in F. A *point* of L is by definition a completely prime filter of L. Let $\mathsf{pt}(L)$ be the set of points of L. Topologize it by defining the opens of $\mathsf{pt}(L)$ as the sets $\mathcal{O}_x = \{F \in \mathsf{pt}(L) | x \in F\}$, $x \in L$. One may check that this is indeed a topology [7, Proposition 7.1.13]. Moreover, pt defines a functor from $\mathbf{CLat}^{op}$ to $\mathbf{Top}$, and by restriction, from $\mathbf{Loc}$ to $\mathbf{Top}$.

Then Ω is left adjoint to pt, in notation $\Omega \dashv \mathsf{pt}$. This means that there are natural transformations $\eta_X : X \to \mathsf{pt}(\Omega(X))$ (the *unit* of the adjunction) and $\epsilon_L : \Omega(\mathsf{pt}(L)) \to L$ (the *counit* of the adjunction) such that $\epsilon_{\Omega(X)} \circ \Omega(\eta_X) = \mathrm{id}_{\Omega(X)}$ and $\mathsf{pt}(\epsilon_L) \circ \eta_{\mathsf{pt}(L)} = \mathrm{id}_{\mathsf{pt}(L)}$. Explicitly, $\eta_X(x)$ is the completely prime filter of all open neighborhoods of x in X, and ϵ_L maps $z \in L$ to the open set $\mathcal{O}_z$.

Sober Spaces. The space $\mathsf{pt}(\Omega(X))$ is called the *sobrification* of X. One may understand this as noticing that (at least if X is a T_0 space, i.e., when its specialisation quasi-ordering is a partial ordering) η_X is an embedding of X into $\mathsf{pt}(\Omega(X))$, so that $\mathsf{pt}(\Omega(X))$ is obtained from X by adding elements, viz. those points of $\Omega(X)$ that are not of the form $\eta_X(x)$, $x \in X$. The space $\mathsf{pt}(\Omega(X))$ is then *sober*: a sober space is a T_0 space in which every irreducible closed set is the closure of a unique point. A closed set C is *irreducible* iff it is not empty, and if there are two closed sets C_1 and C_2 such that $C \subseteq C_1 \cup C_2$ then $C \subseteq C_1$ or $C \subseteq C_2$.

One shows that η_X is injective iff X is T_0, and surjective iff X is sober. Equivalently, X is sober iff X is homeomorphic to $\mathsf{pt}(L)$, for some complete lattice L, iff $X \cong \mathsf{pt}(\Omega(X))$, iff η_X is bijective (in which case it is automatically a homeomorphism).

The explicit description of $\mathsf{pt}(\Omega(X))$ of X is relatively uninteresting. We have already said that $\mathsf{pt}(\Omega(X))$ was a form of completion of X, where we add elements. A crucial point is that this completion adds elements *but no new opens*: then opens of $\mathsf{pt}(\Omega(X))$ are of the form $\mathcal{O}_U$, one for each open subset U of X. Alternatively, the specialisation quasi-ordering $\leq$ of a sober space turns it into a cpo. I.e., the $\mathsf{pt} \circ \Omega$ construction formally adds all missing least upper bounds. One may for example check that the sobrification of $\mathbb{N}$ (with the Alexandroff topology if its natural ordering) is, up to homeomorphism, $\mathbb{N} \cup \{+\infty\}$ with non-empty open subsets $\uparrow n$, $n \in \mathbb{N}$. (This is the Scott topology on $\mathbb{N} \cup \{+\infty\}$, minus the Scott-open $\{+\infty\}$. Check that this is the upper topology of $\leq$ on $\mathbb{N} \cup \{+\infty\}$.)

Up to homeomorphism, there is a simpler way of describing the sobrification of X [10, Chapter V, Exercise 4.9], as the space $\mathcal{S}(X)$ of all irreducible closed subsets of X, with open subsets given by $\Diamond U = \{F$ irreducible closed $| F \cap U \neq \emptyset\}$, for each open subset U of X. Its specialisation quasi-ordering is just inclusion. Up to this homeomorphism, the unit η_X can be seen as a function from X to $\mathcal{S}(X)$ that maps $x \in X$ to the irreducible closed set $\downarrow x$.

Stably Compact Spaces. Sober spaces are *well-filtered* [15, Definition 2.7]: for every open subset U, for every filtered family $(Q_i)_{i \in I}$ of saturated compacts such that $\bigcap_{i \in I} Q_i \subseteq U$, there is an $i \in I$ such that $Q_i \subseteq U$. (A family is *filtered* provided it is directed in the converse ordering $\supseteq$.) This is by the celebrated Hofmann-Mislove Theorem [7, Theorem 7.2.9].

Notice the parallel between well-filteredness and the property that every directed family of opens whose union contains a compact K already has an element containing K. Formally, we obtain the former from the latter by reversing the ordering of inclusion, replacing unions by intersection, and exchanging the roles of compacts and opens. Stably compact spaces are the class of spaces where this parallel can be made fully formal.

Say that a topological space X is *coherent* iff the intersection of two saturated compacts is compact. X is *locally compact* if and only if every element has a basis of saturated compact neighborhoods. That is, whenever $x \in U$ with U open, there is a saturated compact Q such that x is in the interior of Q, and $Q \subseteq U$. A *stably compact space* is a sober, coherent, locally compact and compact space. This is a much stronger property than just being sober.

Let X be a stably compact space. One may show that the complements of saturated compacts of X form a new topology, the so-called *cocompact topology*. Write X^d for X under its cocompact topology: this is the *de Groot dual* of X. Then X^d is again stably compact, and $X^{dd} = X$ [15, Corollary 2.13]. Moreover, the specialisation quasi-ordering of X^d is the converse $\geq$ of $\leq$.

6. Hard Constructions of Well Topologies

Let $\mathbf{CCCLat}$ be the full subcategory of $\mathbf{CLat}^{op}$ consisting of complete lattices (resp., $\mathbf{Loccc}$ the full subcategory of $\mathbf{Loc}$ consisting of frames) that satisfy the ascending chain condition. Proposition 3.2 states that Ω induces a functor from $\mathbf{Well}$ to $\mathbf{CCCLat}$, and to $\mathbf{Loccc}$.

Lemma 6.1 *The functor* pt *induces a functor from* $\mathbf{CCCLat}$ *(resp.,* $\mathbf{Loccc}$*) to* $\mathbf{Well}$*, right adjoint to* Ω.

Proof. Let us show that pt is a functor from $\mathbf{CCCLat}$, resp. $\mathbf{Loccc}$, to $\mathbf{Well}$. This boilds down to the fact that for every complete lattice L with the ascending chain condition, $\mathsf{pt}(L)$ is well. By Proposition 3.2, it suffices to show that every ascending chain $\mathcal{O}_{x_1} \subseteq \mathcal{O}_{x_2} \subseteq \ldots \subseteq \mathcal{O}_{x_k} \subseteq \ldots$ stabilizes. Notice that $\mathcal{O}_x \subseteq \mathcal{O}_y$ if and only if $x \leq y$: the if direction is clear; conversely, if $\mathcal{O}_x \subseteq \mathcal{O}_y$, then the filter $\uparrow x$ is completely prime, belongs to $\mathcal{O}_x$, so it belongs to $\mathcal{O}_y$, i.e., $y \in \uparrow x$, that is, $x \leq y$. It follows that $x_1 \leq x_2 \leq \ldots \leq x_k \leq \ldots$ is an ascending chain in L. So it stabilizes. □

Sobrification preserves the property of being well. In fact, we can say slightly more:

Proposition 6.2 *A space X is well if and only if its sobrification* $\mathsf{pt}(\Omega(X)) \cong \mathcal{S}(X)$ *is well.*

Proof. If X is well, then so is $\mathsf{pt}(\Omega(X))$, because Ω is a functor from $\mathbf{Well}$ to $\mathbf{Loccc}$, and pt is one from $\mathbf{Loccc}$ to $\mathbf{Well}$ (Lemma 6.1). It follows that $\mathcal{S}(X) \cong \mathsf{pt}(\Omega(X))$ is well, too. Conversely, assume $\mathcal{S}(X)$ is well, and let $U_1 \subseteq U_2 \subseteq \ldots \subseteq U_k \subseteq \ldots$ be an infinite ascending chain in X. Then $\Diamond U_1 \subseteq \Diamond U_2 \subseteq \ldots \subseteq \Diamond U_k \subseteq \ldots$ is an infinite ascending chain in $\mathcal{S}(X)$, so it stabilizes: for some $N \in \mathbb{N}$, for every $k \geq N$, $\Diamond U_k \subseteq \Diamond U_N$. For every $x \in U_k$, $\downarrow x$ is in $\Diamond U_k$, so it is in $\Diamond U_N$, therefore $x \in U_N$. So $U_k \subseteq U_N$, showing that the ascending chain $U_1 \subseteq U_2 \subseteq \ldots \subseteq U_k \subseteq \ldots$ also stabilizes. So X is well. □

6.1. Well Sober Spaces

Proposition 6.2 is crucial to our study. For the moment, it at least motivates a deeper study of those spaces that are both well and sober.

Proposition 6.3 *Every well sober space X is stably compact. Its upward-closed subsets coincide with its saturated compacts.*

Proof. First, X is trivially locally compact. Since X itself is open and well, X is compact. X is sober by assumption. It remains to show that X is coherent. This will be a trivial consequence of the second part of the proposition, since every intersection of upward-closed subsets is again upward-closed. Let therefore A be upward-closed in X. So A is saturated, i.e., A is the filtered intersection of the family $(U_i)_{i \in I}$ of all opens containing A. Since X is well, this is a family of saturated compacts. Since X is well-filtered, its intersection A is again saturated compact. ☐

Corollary 6.4 *Let X be sober and well. Then the cocompact topology on X is the Alexandroff topology of $\geq$.*

In particular, the topology of X is entirely determined by its specialisation quasi-ordering.

Corollary 6.5 *Let X be sober and well. The topology of X is exactly the upper topology of its specialisation quasi-ordering $\leq$. Every closed subset of X is finitary.*

Proof. The closed subsets of X, that is of X^{dd}, are the saturated compacts of X^d. By Corollary 6.4, the topology of X^d is the Alexandroff topology of $\geq$, so its saturated compacts are its finitary compacts. These are exactly the sets of the form $\downarrow E$, E finite. In other words, the closed subsets of X are exactly its finitary closed subsets.

Since all finitary closed subsets are closed in the upper topology, the topology of X is coarser than the upper topology. But the latter is the coarsest having $\leq$ as specialisation quasi-ordering. So the two topologies coincide. ☐

Recall that the sobrification of $\mathbb{N}$ is $\mathbb{N} \cup \{+\infty\}$, with opens $\uparrow n$, $n \in \mathbb{N}$. We have already noticed that this was the upper topology. Corollary 6.5 shows that this is no accident.

That X is *both* well and sober is essential. Note also that X is closed in its topology. Corollary 6.5 then implies that X can be written $\downarrow E$ for some finite E; that is, X has finitely many maximal elements and every element of X is below one of these. This is not the case of $\mathbb{N}$, which is well but not sober. But this is the case of its sobrification $\mathbb{N} \cup \{+\infty\}$. Again, Corollary 6.5 shows that this is no accident.

Definition 6.6 *The quasi-ordered set X has* property T *iff there is a finite subset E of X such that every element of X is less than or equal to some element of E.*

When X is well and sober, Corollary 6.5 also implies that for every $x, y \in X$, $\downarrow x \cap \downarrow y$, which is closed, is of the form $\downarrow E$, E finite. This is equivalent to the following property, a dual of Jung's property M [14, Definition, p.38]:

Definition 6.7 *The quasi-ordered set X has* property W *iff, for every $x, y \in X$, there is a finite subset E of maximal lower bounds of x and y, such that every lower bound of x and y is less than or equal to some element of E.*

Lemma 6.8 *Let X be a well topological space, $\leq$ its specialisation quasi-ordering, $\geq$ its converse, and $\equiv = (\leq \cap \geq)$. Then $\leq$ is well-founded: every infinite descending chain $\ldots \leq x_k \leq \ldots \leq x_2 \leq x_1$ stabilizes up to $\equiv$, meaning that there is an integer N such that $x_k \equiv x_N$ for every $k \geq N$.*

We establish the converse in Proposition 6.9 below. To this end, we need to define the *Hoare quasi-ordering* $\leq^\flat$ on the subsets of a set X quasi-ordered by $\leq$: $E \leq^\flat E'$ iff for every $x \in E$, there is an $x' \in E'$ such that $x \leq x'$. Equivalently, $E \leq^\flat E'$ iff $\downarrow E \subseteq \downarrow E'$. Define also the *multiset extension* of a quasi-ordering $\leq$. Let $\{\!|x_1, \ldots, x_n|\!\}$ be the multiset containing exactly the elements $x_1, \ldots, x_n$, and write $\uplus$ for multiset union. Let again $\equiv$ be $\leq \cap \geq$, and let $<= (\leq \setminus \equiv)$ be the strict part of $\leq$. For any two finite multisets M and M' of elements of X, define $M \leq^{mul} M'$ iff one may write $M = M_1 \uplus \{\!|x_1, \ldots, x_n|\!\}$, $M' = M'_1 \uplus \{\!|x'_1, \ldots, x'_n|\!\}$ so that $x_1 \equiv x'_1, \ldots, x_n \equiv x'_n$, and for every $x \in M_1$, there is an $x' \in M'_1$ such that $x \leq x'$. It is well-known that if (the strict part of) $\leq$ is well-founded then so is (the strict part of) $\leq^{mul}$. Equating every finite subset $E = \{x_1, \ldots, x_n\}$ with the finite multiset $\{\!|x_1, \ldots, x_n|\!\}$, where each x_i occurs exactly once, moreover, we have that $E \leq^\flat E'$ iff $E \leq^{mul} E'$.

Proposition 6.9 *Let $\leq$ be a quasi-ordering on X. If $\leq$ is well-founded and has property W, then X is well in its upper topology. Its closed subsets, except possibly X, are finitary.*

Proof. First, we show that: $(*)$ for every descending family $(\downarrow E_n)_{n \in \mathbb{N}}$, where each E_n is a finite subset of X, there is $k \in \mathbb{N}$ such that $\bigcap_{n \in N} \downarrow E_n = \downarrow E_k$. Note that, since $\downarrow E_{n+1} \subseteq \downarrow E_n$, for every $x \in E_{n+1}$, there is a $y \in E_n$ such that $x \leq y$, that is, $E_{n+1} \leq^\flat E_n$. Then $(*)$ follows since $\leq^\flat = \leq^{mul}$ is well-founded.

We obtain: $(**)$ for every filtered family $(\downarrow E_i)_{i \in I}$, where each E_i is finite, there is a finite subset $E \subseteq X$ such that $\bigcap_{i \in I} \downarrow E_i = \downarrow E$. Indeed, assume the contrary. We then build a descending sequence $\downarrow E'_n$, $n \in \mathbb{N}$, where each E'_n is some E_i, by induction on $n \in \mathbb{N}$. Let E'_0 be any E_i. Assuming E'_n has been built, for some $i \in I$ we must have $\downarrow E'_n \not\subseteq \downarrow E_i$, otherwise $\bigcap_{i \in I} \downarrow E_i = \downarrow E'_n$. Since $(\downarrow E_i)_{i \in I}$ is filtered, for some $j \in I$, $\downarrow E_j$ is contained in $\downarrow E'_n$ and in $\downarrow E_i$: let $E'_{n+1} = E_j$. By construction, the chain $(\downarrow E'_n)_{n \in \mathbb{N}}$ is strictly decreasing, contradicting $(*)$.

The closed subsets in the upper topology are the (arbitrary) intersections of subsets of the form $\downarrow E_i$, $i \in I$, E_i finite. The empty intersection is X. Each non-empty intersection $\bigcap_{i \in I} A_i$ can be written as a filtered intersection of non-empty finite intersections: $\bigcap_{i \in I} A_i = \bigcap_{J \subseteq I, J \neq \emptyset \text{ finite}} \bigcap_{i \in J} A_i$. By property W, every non-empty finite intersection $\bigcap_{i \in J} \downarrow E_i$ is of the form $\downarrow E_J$ for some finite subset E_J. By $(**)$, every filtered intersection of subsets of the form $\downarrow E_J$ is again of the form $\downarrow E$, E finite.

The closed subsets of the upper topology of X are therefore exactly those of the form $\downarrow E$, E finite, plus the whole of X. Taking complements in $(*)$, every infinite ascending chain of opens stabilizes: by Proposition 3.2, X is well. $\square$

Lemma 6.10 *Let $\leq$ be a quasi-ordering on X. If $\leq$ is well-founded and has property W, then the irreducible closed subsets F of X are of the form $\downarrow x$, $x \in X$, plus possibly X itself. If X additionally has property T, then the only irreducible closed sets are of the first kind.*

All this finally allows us to characterize exactly the well sober spaces in terms of their specialisation quasi-ordering:

Theorem 6.11 *The well sober spaces are exactly the spaces whose topology is the upper topology of a well-founded partial order that has properties W and T.*

6.2. Cartesian Products

We now have enough background to show that $\mathbf{Well}$ is finitely complete. We start with Cartesian products. Remember that $X_1 \times X_2$ is equipped with the *product topology*, which is generated by all *open rectangles* $U_1 \times U_2$, where U_1 is open in X_1 and U_2 is open in X_2. Alternatively, the product topology is the coarsest that makes the projections $\pi_i : X_1 \times X_2 \to X_i$ $(i = 1, 2)$ continuous. Theorem 6.11 makes the following almost immediate.

Lemma 6.12 *The product $X_1 \times X_2$ of two well sober spaces is well and sober.*

Theorem 6.13 *Let $X_1, \ldots, X_n$ be well topological spaces. Their product $\prod_{i=1}^n X_i$ is well. Its opens are all* finite *unions of open rectangles $\prod_{i=1}^n U_i$, $U_i \in \Omega(X_i)$.*

Proof. By induction on n. The essential case is $n = 2$. First, note that for every open U of a space X, $\eta_X(x) \in \mathcal{O}_U$ iff $x \in U$. In particular: $(*)$ $\eta_X^{-1}(\mathcal{O}_U) = U$.

Consider the continuous map $i = \eta_{X_1} \times \eta_{X_2} : X_1 \times X_2 \to \mathsf{pt}(\Omega(X_1)) \times \mathsf{pt}(\Omega(X_2))$. Let U any open subset of $X_1 \times X_2$. By definition, U can be written as $\bigcup_{i \in I} U_i^1 \times U_i^2$, where the U_i^1's are open in X_1 and the U_i^2's are open in X_2. By $(*)$, $U = \bigcup_{i \in I} \eta_{X_1}^{-1}(\mathcal{O}_{U_i^1}) \times \eta_{X_2}^{-1}(\mathcal{U}_{U_i^2}) = i^{-1}(\bigcup_{i \in I} \mathcal{O}_{U_i^1} \times \mathcal{U}_{U_i^2})$. Note that $\bigcup_{i \in I} \mathcal{O}_{U_i^1} \times \mathcal{O}_{U_i^2}$ is open in $\mathsf{pt}(\Omega(X_1)) \times \mathsf{pt}(\Omega(X_2))$. By Proposition 6.2, $\mathsf{pt}(\Omega(X_1))$ and $\mathsf{pt}(\Omega(X_2))$ are well. They are sober by construction. So by Lemma 6.12, $\mathsf{pt}(\Omega(X_1)) \times \mathsf{pt}(\Omega(X_2))$ is well (and sober), hence $\bigcup_{i \in I} \mathcal{O}_{U_i^1} \times \mathcal{O}_{U_i^2}$ is compact in $\mathsf{pt}(\Omega(X_1)) \times \mathsf{pt}(\Omega(X_2))$. The family $(\mathcal{O}_{U_i^1} \times \mathcal{O}_{U_i^2})_{i \in I}$ is an open cover of it. So there is a finite subset I_0 of I such that $\bigcup_{i \in I} \mathcal{O}_{U_i^1} \times \mathcal{O}_{U_i^2} = \bigcup_{i \in I_0} \mathcal{O}_{U_i^1} \times \mathcal{O}_{U_i^2}$. Then $U = i^{-1}(\bigcup_{i \in I_0} \mathcal{O}_{U_i^1} \times \mathcal{O}_{U_i^2}) = \bigcup_{i \in I_0} U_i^1 \times U_i^2$ is a finite union of open rectangles.

Since X_1 is well, U_i^1 is compact, and similarly for U_i^2, so $U_i^1 \times U_i^2$ is compact in $X_1 \times X_2$ by Tychonoff's Theorem. It follows that U, qua finite union of compacts, is also compact. So $X_1 \times X_2$ is well. $\square$

Corollary 6.14 $\mathbf{Well}$ *is finitely complete.*

Proof. By Theorem 6.13, it has all finite products. We need only verify that it has all equalizers of parallel pairs $f, f' : X \to Y$. Such an equalizer exists in $\mathbf{Top}$, and is given by $Z = \{x \in X | f(x) = f'(x)\}$, with the topology induced by X. Then Z is well by Lemma 4.2, and clearly forms an equalizer in $\mathbf{Well}$. $\square$

7. A Well Topology on $\mathbb{P}(X)$

Let us deal with the so-called Hoare powerdomain construction. For each topological space X, let its *Hoare powerdomain* $\mathcal{H}(X)$ (resp., $\mathcal{H}_\emptyset(X)$) be the space of all non-empty closed subsets (resp., all closed subsets) of X with the upper topology of the $\subseteq$ ordering. It has subbasic open sets $\Diamond U = \{F \in \mathcal{H}(X) | F \cap U \neq \emptyset\}$, U open in X.

$\mathcal{H}(X)$ is used in denotational semantics to model angelic non-determinism. Note that the closure of an element $F \in \mathcal{H}(X)$ is $\Box F = \mathcal{H}(X) \setminus \Diamond(\overline{F}) = \{F' \in \mathcal{H}(X) | F' \subseteq F\}$, and similarly in $\mathcal{H}_\emptyset(X)$. On finitary closed sets, $\downarrow E \subseteq \downarrow E'$ iff $E \leq^\flat E'$. The following is then immediate.

Proposition 7.1 *Let X be a well sober space. Then $\mathcal{H}(X)$ and $\mathcal{H}_\emptyset(X)$ are well and sober.*

Theorem 7.2 *Let X be a well topological space. Then $\mathcal{H}(X)$ and $\mathcal{H}_\emptyset(X)$ are well.*

Proof. Call *basic* open set any finite intersection of subbasic opens $\Diamond U$. Every open $\mathcal{U}$ of $\mathcal{H}(X)$ is the union of the basic opens $\mathcal{V}$ contained in $\mathcal{U}$. Fix a way of writing each basic open $\mathcal{V}_i$, say $\mathcal{V}_i = \bigcap_{j \in J_i} \Diamond V_{ij}$, where J_i is finite. Let $\widehat{\mathcal{V}_i} = \bigcap_{j \in J_i} \Diamond \mathcal{O}_{V_{ij}}$, and finally $\widehat{\mathcal{U}}$ be the union of all $\widehat{\mathcal{V}_i}$, $\mathcal{V}_i$ basic open contained in $\mathcal{U}$. By construction, if $\mathcal{U} \subseteq \mathcal{U}'$, then $\widehat{\mathcal{U}} \subseteq \widehat{\mathcal{U}'}$. For every ascending chain $\mathcal{U}_1 \subseteq \mathcal{U}_2 \subseteq \ldots \subseteq \mathcal{U}_k \subseteq \ldots$ in $\mathcal{H}(X)$, $\widehat{\mathcal{U}_1} \subseteq \widehat{\mathcal{U}_2} \subseteq \ldots \subseteq \widehat{\mathcal{U}_k} \subseteq \ldots$ is an ascending chain in $\mathcal{H}(\mathsf{pt}(\Omega(X)))$. So it stabilizes, using Proposition 6.2, Proposition 7.1, and Proposition 3.2.

Recall that $\eta_X^{-1}(\mathcal{O}_U) = U$ for every open U of X. So $\Diamond V_{ij} = \{F \in \mathcal{H}(X) | F \cap \eta_X^{-1}(\mathcal{O}_{V_{ij}}) \neq \emptyset\} =$

$\mathcal{H}(\eta_X)^{-1}(\Diamond\mathcal{O}_{V_{ij}})$, where $\mathcal{H}(\eta_X)$ maps $F \in \mathcal{H}(X)$ to $cl(\eta_X(F))$. Indeed, $\mathcal{H}(\eta_X)^{-1}(\Diamond\mathcal{O}_{V_{ij}}) = \{F \in \mathcal{H}(X) | cl(\eta_X(F)) \cap \mathcal{O}_{V_{ij}} \neq \emptyset\} = \{F \in \mathcal{H}(X) | \eta_X(F) \cap \mathcal{O}_{V_{ij}} \neq \emptyset\} = \{F \in \mathcal{H}(X) | F \cap \eta_X^{-1}(\mathcal{O}_{V_{ij}}) \neq \emptyset\}$. So $\mathcal{U} = \mathcal{H}(\eta_X)^{-1}(\widehat{\mathcal{U}})$ for every open subset $\mathcal{U}$ of $\mathcal{H}(X)$. In particular, the map $\mathcal{U} \mapsto \widehat{\mathcal{U}}$ is injective.

Since $\widehat{\mathcal{U}_1} \subseteq \widehat{\mathcal{U}_2} \subseteq \ldots \subseteq \widehat{\mathcal{U}_k} \subseteq \ldots$ stabilizes, $\mathcal{U}_1 \subseteq \mathcal{U}_2 \subseteq \ldots \subseteq \mathcal{U}_k \subseteq \ldots$ stabilizes, too. We conclude by Theorem 3.2. The argument is similar for $\mathcal{H}_\emptyset(X)$. ☐

This has the following surprising consequence:

Proposition 7.3 *Let X be a topological space, with specialisation quasi-ordering $\leq$. Let $\mathbb{P}(X)$ be the set of all subsets (resp. $\mathbb{P}^*(X)$ of all non-empty subsets) of X, quasi-ordered by the* topological Hoare quasi-ordering $\leq^{\flat *}$, *defined as: $A \leq^{\flat *} B$ iff $cl(A) \subseteq cl(B)$. Equip $\mathbb{P}(X)$, resp. $\mathbb{P}^*(X)$, with the corresponding upper topology.*

If X is well, then so are $\mathbb{P}(X)$ and $\mathbb{P}^*(X)$.

Proof. Let $\equiv^{\flat *}$ be the equivalence relation related to $\leq^{\flat *}$, and q the quotient map. Up to homeomorphism, $\mathbb{P}(X)/\equiv^{\flat *}$ is exactly $\mathcal{H}(X)$, and q maps each subset A to $cl(A)$. Note that q is continuous: the inverse image $q^{-1}(\Diamond U)$ is the set of all subsets A such that $cl(A) \cap U \neq \emptyset$, equivalently $cl(A) \subseteq \overline{U}$, equivalently $A \leq^{\flat *} \overline{U}$, since $\overline{U}$ is closed. So $q^{-1}(\Diamond O)$ is open in the upper topology.

We have actually just shown that $\Omega(q) : \Omega(\mathcal{H}(X)) \to \Omega(\mathbb{P}(X))$ maps $\Diamond O$ to $\downarrow^{\flat *} \overline{O}$. Recall that $\Omega(q)$ is a frame homomorphism, and is therefore entirely determined by this property. This is clearly a bijection, whose inverse is the unique frame homomorphism mapping $\downarrow^{\flat *} F$ to $\Diamond\overline{F}$, for each closed subset F of X.

Every ascending chain of opens $O_1 \subseteq O_2 \subseteq \ldots \subseteq O_k \subseteq \ldots$ of $\mathbb{P}(X)$ then induces an ascending chain of opens of $\mathcal{H}(X)$ through $\Omega(q)^{-1}$. By Theorem 7.2 and Proposition 3.2, the latter stabilizes. So the former, too. Hence $\mathbb{P}(X)$ is well. ☐

Corollary 7.4 *Let $\leq$ be a well quasi-ordering on X. Then $\mathbb{P}(X)$, resp. $\mathbb{P}^*(X)$, with the upper topology of $\leq^\flat$, is well.*

This is remarkable, as in general $\leq^\flat$ is *not* a well quasi-ordering on $\mathbb{P}(X)$. The standard counterexample is Rado's example [21]. Let X_{Rado} be the set $\{(m,n) \in \mathbb{N}^2 | m < n\}$, ordered by $\leq_{Rado}$: $(m,n) \leq_{Rado} (m',n')$ iff $m = m'$ and $n \leq n'$, or $n < m'$. It is well-known that $\leq_{Rado}$ is a well quasi-ordering. However, $\mathcal{H}(X_{Rado}) \cong \mathbb{P}(X_{Rado})$ is not well quasi-ordered by $\leq^\flat_{Rado}$. (We follow here [5, Example 3.2].) Indeed, let $\mathbb{C}_1$ be the set of sets of the form $\overline{\phi}_{i,j} = \{(m,n) \in X_{Rado} | m \leq j \wedge (i = m \Rightarrow n < j)\}$. It is easy to see that they are downward-closed, hence closed in the Alexandroff topology. Let $\overline{\psi}_j = \bigcap_{i=0}^{j-1} \overline{\phi}_{i,j}$; this is again closed, and non-empty since every (j,k) with $j < k$ is in it. But there is no ascending subsequence of $\overline{\psi}_0, \overline{\psi}_1, \ldots, \overline{\psi}_k, \ldots$. Indeed, if $j < k$ then $(j,k) \in \overline{\psi}_j$, but $(j,k) \notin \overline{\psi}_k$, else (j,k) would be in $\overline{\phi}_{j,k}$, which would imply $k < j$.

A trivial consequence of this is that $\mathcal{H}(X)$ and $\mathcal{H}_\emptyset(X)$ are in general not Alexandroff-discrete, even when X is. A more important observation is that choosing the right topology (here, the upper topology) matters.

The *Smyth space* $\mathcal{Q}(X)$ usually models demonic non-determinism, and is defined as the set of non-empty compact saturated subsets of X, ordered by reverse inclusion $\supseteq$, and equipped with the corresponding Scott topology. The latter is generated by basic opens $\Box U = \{Q \in \mathcal{Q}(X) | Q \subseteq U\}$, U open in X, as soon as X is well-filtered and locally compact. Contrarily to $\mathcal{H}(X)$, $\mathcal{Q}(X)$ is in general not well.

However, consider the smaller set $\mathcal{O}(X)$ of opens (remember that every open is compact), equipped with the upper topology of $\supseteq$. Clearly, $\mathcal{O}(X) \cong \mathcal{H}_\emptyset(X)$, by the homeomorphism sending an open set to its complement. By Corollary 7.4, $\mathcal{O}(X)$ is therefore well. Now note that if X is Alexandroff-discrete, then opens coincide with finitary compacts $\uparrow E$. Let $\mathbb{P}_{fin}(X)$ the set of finite subsets of X, which we equip with the upper topology of the Smyth quasi-ordering $\leq^\sharp$.

Lemma 7.5 *Let X be Alexandroff-discrete, with a well specialisation quasi-ordering. The map $\uparrow$ that sends $E \in \mathbb{P}_{fin}(X)$ to $\uparrow E \in \mathcal{O}(X)$ is a homeomorphism.*

Using this, the fact that $\mathcal{O}(X) \cong \mathcal{H}_\emptyset(X)$, and Theorem 7.2, we get:

Corollary 7.6 *Let $\leq$ be a well quasi-ordering on X, and equip X with its Alexandroff topology. Then $\mathbb{P}_{fin}(X)$, with the upper topology of $\leq^\sharp$, is well.*

Again, this contrasts with the theory of well-quasi orderings. Rado's example (see above) shows that $\leq^\sharp$ is in general not a well quasi-ordering on $\mathbb{P}_{fin}(X)$. It would be if $\leq$ were ω^2-wqo [13].

8. A New Data Structure for Coverability?

Consider the following argument. Start from a well topological space X, e.g., $\mathbb{N}^k$ with its Alexandroff topology (this is the space of markings of a Petri net). By Proposition 6.2, $\mathcal{S}(X)$ is well. By Corollary 6.5, its opens are exactly the finitary closed subsets $\downarrow E$, $E \subseteq \mathcal{S}(X)$. We equate X with a subspace of $\mathcal{S}(X)$, i.e., we equate $x \in X$ with $\eta_X(x) \in \mathcal{S}(X)$. For instance, $\mathcal{S}(\mathbb{N}^k) = (\mathbb{N} \cup \{+\infty\})^k$, as we have seen. Now the topology of X is exactly the topology induced on X by that of $\mathcal{S}(X)$, so we have a way of representing all opens of X using a finite set E, of elements of $\mathcal{S}(X)$, as the complement in X of $X \cap \downarrow E$. This is clear on

$\mathbb{N}^k$: for example, with $k = 3$, we may represent the upward-closed set (open in $\mathbb{N}^3$) $\uparrow \{(2, 3, 5)\}$ as the complement of $X \cap \downarrow \{(1, +\infty, +\infty), (+\infty, 2, +\infty), (+\infty, +\infty, 4)\}$.

This is completely general: we can *always* represent opens of well topological spaces X as complements of sets of the form $X \cap \downarrow E$, E a finite subset of $\mathcal{S}(X)$. This is important for those well topological spaces, e.g., $\mathbb{P}(X)$, that do not arise from well quasi-orderings, and where opens cannot be represented as $\uparrow E$ (E finite $\subseteq X$). Even on well quasi-ordered spaces such as $\mathbb{N}^k$, this may provide an alternate representation of sets of the form $\mathrm{Pre}^{\exists *}\delta(V)$ or $I\,[\![F]\!]_\delta\,\rho$. E.g., on $\mathbb{N}^k$, when δ is the transition relation of a Petri net (taken as a finite set of rewrite rules $\vec{m}_i \to \vec{n}_i$, $1 \leq i \leq \ell$, of vectors in $\mathbb{N}^k$, where $\vec{m}_i, \vec{n}_i \in \mathbb{N}^k$), we may compute $\mathrm{Pre}^{\exists}\delta(\overline{X \cap \downarrow E}) = \overline{\mathrm{Pre}^{\forall}\delta(X \cap \downarrow E)}$ and $\mathrm{Pre}^{\forall}\delta(\overline{X \cap \downarrow E}) = \overline{\mathrm{Pre}^{\exists}\delta(X \cap \downarrow E)}$ by the formulae:

$$\begin{aligned}
\mathrm{Pre}^{\exists}\delta(X \cap \downarrow E) &= X \cap \downarrow \{\vec{p} + \vec{m}_i - \vec{n}_i \\
&\qquad\qquad \mid \vec{p} \in E, 1 \leq i \leq \ell \cdot \vec{p} \geq \vec{n}_i\} \\
\mathrm{Pre}^{\forall}\delta(X \cap \downarrow E) &= X \cap \bigcap_{i=1}^{\ell} \Big(\complement\!\uparrow \vec{m}_i \cup \downarrow \{\vec{p} + \vec{m}_i - \vec{n}_i \\
&\qquad\qquad \mid \vec{p} \in E \cdot \vec{p} \geq \vec{n}_i\}\Big)
\end{aligned}$$

(We leave the computation of finite unions and intersections of sets of the form $\downarrow E$, and of the complement $\complement\!\uparrow \vec{m}_i$ of $\uparrow \vec{m}_i$ in $\mathcal{S}(X)$, as an exercise to the reader.)

9. Conclusion

We have laid down the first steps towards a theory of well topological spaces, generalizing well quasi-orderings. Well topological spaces enjoy many nice properties. Every finite product, equalizer, subspace, finite coproduct, coequalizer, quotient, retract of well topological spaces is again well. We have also characterized those well topological spaces that are sober, as being exactly those upper topologies of well-founded quasi-orderings satisfying properties W and T. And we have shown that a space is well iff its sobrification is well. Finally, the Hoare powerdomain of a well space is well, which implies the surprising property that the set $\mathbb{P}(X)$ of all subsets of a well space X, even infinite ones, under the upper topology of the Hoare quasi-ordering $\leq^\flat$, is well, although $\leq^\flat$ is not in general a well quasi-ordering. Similarly, the set $\mathbb{P}_{fin}(X)$ of finite subsets of X is well under the Smyth quasi-ordering $\leq^\sharp$. An intriguing question stems from the fact that the sobrification of $\mathbb{N}^k$ is $(\mathbb{N} \cup \{+\infty\})^k$, and that this is exactly the space of objects labelling Karp-Miller trees [16]. Is there a connection between sobrification and the Karp-Miller construction for Petri nets and VASS?

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A. Proofs of Theorems

Proposition 3.5 *Every well-structured transition system (X, δ) is a topological well-structured transition system and conversely, where X is seen with its Alexandroff topology.*

Proof. By Proposition 3.1, X is well. Whenever V is open, i.e., upward-closed, we claim that $\mathrm{Pre}^{\exists}\delta(V)$ is upward-closed, therefore δ is lower semi-continuous. Indeed, for every $x \in \mathrm{Pre}^{\exists}\delta(V)$, by definition there is an $y \in V$ such that $x \ \delta \ y$. Whenever $x \leq x'$, there is a y' such that $x' \ \delta \ y'$ and $y \leq y'$. Since V is upward-closed, $y' \in V$, so $x' \in \mathrm{Pre}^{\exists}\delta(V)$.

Conversely, assume δ to be lower semi-continuous, and let V be the Alexandroff open $\uparrow y$: since $\mathrm{Pre}^{\exists}\delta(V)$ is Alexandroff-open, i.e., upward-closed, whenever $x \ \delta \ y$ and $x \leq x'$, it must be that $x' \in \mathrm{Pre}^{\exists}\delta(V)$, whence there is an element y' such that $x' \ \delta \ y'$ such that $y' \in V$, that is, $y \leq y'$. □

Lemma 4.1 *Every topology coarser than a well topology is well.*

Proof. Every open U in the coarser topology is open in the well topology, hence compact in the well topology. Every cover of U by opens in the coarser topology is a cover by opens in the well topology, hence we can extract a finite subcover ("a finer topology has more opens and less compacts"). So U is compact in the coarser topology. □

Corollary 4.3 *Any retract of a well topological space is well.*

Proof. Z is homeomorphic to the subspace $s(Z)$. Apply Lemma 4.2, observing that being well is invariant under homeomorphisms. □

Lemma 4.4 *Let $X_1, \ldots, X_k$ be k well topological spaces. Their coproduct $X_1 + \ldots + X_k$ is well.*

Proof. Because $\Omega(X) \cong \Omega(X_1) \times \ldots \times \Omega(X_k)$ has the ascending chain condition. Then use Proposition 3.2. □

Lemma 4.5 *Let $q : X \to Z$ be a surjective continuous map. If X is well then so is Z.*

Proof. For every open U of Z, $q^{-1}(U)$ is open in X, hence compact. Since q is surjective, $U = q(q^{-1}(U))$, so U is compact. □

Lemma 4.6 *Let X be a well topological space, $\equiv$ an equivalence relation on X. Then $X/{\equiv}$ is well.*

Proof. By Lemma 4.5, since $q_{\equiv}$ is continuous and surjective. □

Lemma 6.8 *Let X be a well topological space, $\leq$ its specialisation quasi-ordering, $\geq$ its converse, and $\equiv = (\leq \cap \geq)$. Then $\leq$ is well-founded: every infinite descending chain $\ldots \leq x_k \leq \ldots \leq x_2 \leq x_1$ stabilizes up to $\equiv$, meaning that there is an integer N such that $x_k \equiv x_N$ for every $k \geq N$.*

Proof. For every i, the complement $U_i = \overline{\downarrow x_i}$ is open, since it is open in the upper topology of $\leq$, which is coarser than the topology of X. Moreover, $U_1 \subseteq U_2 \subseteq \ldots \subseteq U_k \subseteq \ldots$. By Proposition 3.2, the latter stabilizes. So the chain $(x_k)_{k \in \mathbb{N}}$ stabilizes up to $\equiv$. □

Lemma 6.10 *Let $\leq$ be a quasi-ordering on X. If $\leq$ is well-founded and has property W, then the irreducible closed subsets F of X are of the form $\downarrow x$, $x \in X$, plus possibly X itself. If X additionally has property T, then the only irreducible closed sets are of the first kind.*

Proof. By Proposition 6.9, $F = \downarrow E$ for some finite E or $F = X$. In the first case, since F is irreducible, F, hence E is not empty. Without loss of generality, we may assume that E is an antichain. Write $E = \{x_1, \ldots, x_n\}$. If $n \geq 2$, then $F = \downarrow E$ is the union of the two closed subsets $\downarrow x_1$ and $\downarrow \{x_2, \ldots, x_n\}$, contradicting the fact that F is irreducible. So $n = 1$.

If X has property T, then X itself is of the form $\downarrow E$, so cannot be irreducible unless it is of the form $\downarrow x$ again. □

Lemma 6.12 *The product $X_1 \times X_2$ of two well sober spaces is well and sober.*

Proof. Let $\leq_1$ and $\leq_2$ the specialisation orderings of X_1 and X_2 respectively. By Lemma 6.8, $\leq_1$ and $\leq_2$ are well-founded and have properties W and T. It follows that the same holds for $\leq_1 \times \leq_2$, where $(x_1, x_2) \ (\leq_1 \times \leq_2) \ (x_1', x_2')$ iff $x_1 \leq_1 x_1'$ and $x_2 \leq_2 x_2'$.

Remember that a function is continuous iff the inverse image of every closed set is closed. It follows that the closed sets of $X_1 \times X_2$ are generated by the subsets of the form $\downarrow x_1 \times X_2$ and $X_1 \times \downarrow x_2$, $x_1 \in X_1$, $x_2 \in X_2$. By Corollary 6.5, the closed set X_1 can be written $\downarrow E_1$, E_1 finite. Similarly $X_2 = \downarrow E_2$, E_2 finite. So the closed sets of $X_1 \times X_2$ are generated by those of the form $\downarrow x_1 \times \downarrow E_2 = \downarrow (\{x_1\} \times E_2)$ and $\downarrow E_1 \times \downarrow x_2 = \downarrow (E_1 \times \{x_2\})$, which are closed in the upper topology of $\leq_1 \times \leq_2$. So the product topology is coarser than the upper topology. Since they have the same specialisation quasi-ordering, they are equal. We conclude by Theorem 6.11. □

Proposition 7.1 *Let X be a well sober space. Then $\mathcal{H}(X)$ and $\mathcal{H}_{\emptyset}(X)$ are well and sober.*

Proof. By Lemma 6.8, the specialisation ordering $\leq$ is well-founded and has properties W and T. The quasi-ordering $\leq^{\flat}$ coincides with $\leq^{mul}$, and is therefore also well-founded. Since all closed subsets of X are finitary by Corollary 6.5, $\subseteq$ is well-founded on $\mathcal{H}(X)$. Property W is trivial in $\mathcal{H}(X)$, since every two closed subsets F_1, F_2 have a greatest lower bound $F_1 \cap F_2$ if non-empty, none otherwise. Property T is also trivial, since X is the top element of $\mathcal{H}(X)$. We reason similarly on $\mathcal{H}_\emptyset(X)$. We conclude by Theorem 6.11. $\square$

Corollary 7.4 *Let $\leq$ be a well quasi-ordering on X. Then $\mathbb{P}(X)$, resp. $\mathbb{P}^*(X)$, with the upper topology of $\leq^\flat$, is well.*
Proof. Let X be equipped with the Alexandroff topology of $\leq$. By Proposition 3.1, X is well. For every subset A, $cl(A)$ equals $\downarrow A$, since every downward-closed set is closed. So $\leq^\flat$ coincides with $\leq^{\flat *}$, and we conclude by Proposition 7.3. $\square$

Lemma 7.5 *Let X be Alexandroff-discrete, with a well specialisation quasi-ordering. The map $\uparrow$ that sends $E \in \mathbb{P}_{fin}(X)$ to $\uparrow E \in \mathcal{O}(X)$ is a homeomorphism.*
Proof. The topology of $\mathcal{O}(X)$ is generated by sets $\{V | V \not\supseteq U\}$, when $U = \uparrow E'$ ranges over opens of X. The inverse image of the latter by $\uparrow$ is $\{E | \exists x \in E' \cdot \forall y \in E \cdot y \not\leq x\}$, the set of finite subsets E such that $E \not\leq^\sharp E'$, so $\uparrow$ is continuous. Conversely, the direct image of $\{E | \exists x \in E' \cdot \forall y \in E \cdot y \not\leq x\}$ by $\uparrow$ is $\{V | V \not\supseteq \uparrow E'\}$. $\square$