Equivalence of Deterministic One-Counter Automata is NL-complete

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Abstract

We prove that language equivalence of deterministic one-counter automata is NL-complete. This improves the superpolynomial time complexity upper bound shown by Valiant and Paterson in 1975. Our main contribution is to prove that two deterministic one-counter automata are inequivalent if and only if they can be distinguished by a word of length polynomial in the size of the two input automata.

1 Introduction

In theoretical computer science, one of the most fundamental decision problems is the *equivalence problem* which asks whether two given machines behave equivalently. Among the various models of computation – such as Turing machines, random access machines and loop programs, just to mention a few of them – the equivalence problem already becomes undecidable when one imposes strong restrictions on their time and space consumption.

Emerging from formal language theory, a classical model of computation is that of pushdown automata. A folklore result is that already universality (and hence equivalence) of pushdown automata is undecidable. Concerning *deterministic pushdown automata (dpda)*, it is fair to say that the computer science community knows very little about the equivalence problem and its complexity.

Oyamaguchi proved that the equivalence problem for real-time dpda (dpda without ε-transitions) is decidable [17]. It took significant further innovation to show the decidability for general dpda, which is the celebrated result by Sénizergues [19], see also [20]. A couple of years later, Stirling showed that dpda equivalence is in fact primitive recursive [22], and his bound is still the best known upper bound for this problem. Probably due to its intricacy, this fundamental problem has not attracted too much research in the past ten years; only recently a simplified proof has been announced [13], with no substantial improvement of the complexity bound.

It is burdensome to realize that for equivalence of dpda there is still an enormous complexity gap, where the mentioned upper bound is far from the best known lower bound, i.e. from P-hardness (which straightforwardly follows from P-hardness of the emptiness problem).

The same complexity gap persists even for real-time dpda. Thus, further subclasses of dpda have been studied. A coNP upper bound is known [21] for *finite-turn dpda* which are dpda where the number of switches between pushing and popping phases is bounded. For *simple dpda* (which are single state and real-time dpda), equivalence is even decidable in polynomial time [12] (see [4] for the currently best known upper bound).

Deterministic one-counter automata (doca) are one of the simplest infinite-state computational models, extending deterministic finite automata just with one nonnegative integer counter; doca are thus dpda over a singleton stack alphabet plus a bottom stack symbol. Doca were first studied by Valiant and Paterson in 1975 [23]; they showed that equivalence is decidable in time $2^{O(\sqrt{n \log n})}$, and a simple analysis of their proof reveals that the equivalence problem is in PSPACE. The problem is easily shown to be NL-hard, there is however an exponential gap between NL and PSPACE. There were attempts to settle the complexity of the doca equivalence problem (later we mention some) but the problem proved intricate; only recently NL-completeness was established for real-time one-counter automata [2] but it was far from clear if and how the proof can be extended to the general case.

Let us mention that a convenient and equi-succinct way to present a doca is to partition the control states (and thus the configurations) into *stable states*, in which the automaton waits for a letter to be read, and into *reset states*, in which the counter is reset to zero and the residue class of the current counter value modulo some specified number determines the successor (stable) state. Technically speaking, the difference between deterministic one-counter automata and their real-time variant is the lack of reset states in the real-time case. The presence of reset states substantially increases the difficulty of the equivalence problem.

One reason seems to be that a doca can exhibit a behaviour with exponential periodicity, demonstrated by the following example (which slightly adapts the version from [23]). We take a family $(\mathcal{A}_n)_{n\geq 1}$ where $\mathcal{A}_n$ is a doca accepting the regular language $L_n = \{a^m b_i \mid 1 \leq i \leq n, m \equiv 0 \pmod{p_i}\}$, where p_i denotes the i^{th} prime number. The index of the Myhill-Nerode congruence of L_n is obviously $2^{\Omega(n)}$ but we can easily construct $\mathcal{A}_n$ with $O(n^2 \log n)$ states. The example also demonstrates that doca are exponentially more succint than their real-time variant, since one can prove that real-time deterministic one-counter automata accepting L_n have $2^{\Omega(n)}$ states. It is also easy to show that doca are strictly more expressive than their real-time variant. Analogous expressiveness and succinctness results hold for dpda and real-time dpda, respectively.

As mentioned above, this increase in difficulty in the presence of ε-transitions is confirmed by the fact that it took more than a decade to lift the decidability of real-time dpda [17] to the general case [19, 20].

Our contribution and overview. The main result of this paper is that equivalence of doca is NL-complete, thus closing the exponential complexity gap that has been existing for over thirty-five years ever since doca were introduced.

The above-mentioned exponential behavior of doca is reflected in our central notion of *extended deterministic transition system* $\mathcal{T}_{\mathsf{ext}}(\mathcal{A})$ that is attached to each doca $\mathcal{A}$. This system includes a special finite deterministic transition system which might be exponentially large in the size of $\mathcal{A}$ and which corresponds to the *special-mode* variant of stable configurations. Roughly speaking, in the special mode we do not count with reaching the zero value in the counter unless a reset state is visited, and each reset-state visit finishes the special mode. Hence the special mode assumes that the counter is positive and it only requires to remember finite information which is sufficient to perform the resets correctly; in more detail, only the current control state and the current residue classes of the counter value w.r.t. the numbers associated with reset states are needed.

For understanding the shortest words distinguishing two stable inequivalent configurations of $\mathcal{A}$, it turns out useful to include also the special-mode variants of the configurations in the study. This allows us to show that *shortest distinguishing words* for two zero configurations have *polynomial length*.

In Section 2 we introduce basic definitions and state our main result that equivalence of doca is NL-complete. A proof of the central claim on polynomial length is given in Section 3 which is in turn divided into the following parts. We give a brief overview of shortest positive paths in the transition system of a doca in Section 3.1; this is the only part which is derived directly from [23]. In Section 3.2 we introduce the above mentioned central notion $\mathcal{T}_{\mathsf{ext}}(\mathcal{A})$, and we make a straightforward analysis of some useful related notions in Sections 3.3–3.7. In particular, in Section 3.4 we study the *independence level* of a configuration, as the length of a shortest distinguishing word for the configuration and its special-mode variant. This allows us to make various useful observations, e.g. about *linear relations* between counter values of configurations with the same independence level in Section 3.7.

Sections 3.8 and 3.9 contain the main argument. Sections 3.8 shows that when following a shortest distinguishing word for two zero configurations, we cannot get a long *line-climbing* segment in which the counter values grow at both sides, keeping a linear relation entailed by keeping the same independence levels. Section 3.9 then shows that a shortest distinguishing word for two zero configurations cannot be long without having a long line-climbing segment.

In Section 4 we add a remark on the *regularity problem*. In Appendix we sketch the standard ideas of showing that the deterministic one-counter automata as introduced in [23] and the above-mentioned reset model that we work with are equi-succinct. We also make clear that our simple form of language equivalence, called *trace equivalence*, does not bring any loss of generality.

Related work. As mentioned above, doca were introduced by Valiant and Paterson in [23], where the above-mentioned $2^{O(\sqrt{n \log n})}$ time upper bound for language equivalence was proven. Polynomial time algorithms for language equivalence and inclusion for strict subclasses of doca were provided in [10, 11]. In [1, 5] polynomial time learning algorithms were presented for doca. Simulation and bisimulation problems on one-counter automata were studied in [3, 14, 15, 16]. In recent years one-counter automata have attracted a lot of attention in the context of formal verification [9, 7, 6, 8].

Remark: In [1, 18] it is stated that equivalence of doca can be decided in polynomial time. Unfortunately, the proofs provided in [1, 18] were not exact enough to be verified, and they raise several questions which are unanswered to date.

2 Definitions and results

By $\mathbb{N}$ we denote the set $\{0, 1, 2, \ldots\}$ of non-negative integers, and by $\mathbb{Z}$ the set of all integers. For a finite set X, by $|X|$ we denote its cardinality.

By Σ^* we denote the set of finite sequences of elements of Σ, i.e. of *words* over Σ. For $w \in \Sigma^*$, $|w|$ denotes the length of w. By ε we denote the empty word; hence $|\varepsilon| = 0$. If $w = uv$ then u is a *prefix* of w and v is a *suffix* of w.

By $\div$ we denote integer division; for $m, n \in \mathbb{N}$ where $n > 0$ we have $m = (m \div n) \cdot n + (m \bmod n)$. We use "mod" in two ways, clarified by the following example: $3 = 18 \bmod 5$, $8 \neq 18 \bmod 5$, $3 \equiv 18 \pmod 5$, $8 \equiv 18 \pmod 5$. For $m \in \mathbb{Z}$, $|m|$ denotes the absolute value of m.

We use ω to stand for infinity; we stipulate $z < \omega$ and $\omega + z = z + \omega = \omega$ for all $z \in \mathbb{Z}$.

A *deterministic labelled transition system*, a *det-LTS* for short, is a tuple

$$\mathcal{T} = (S_{\mathsf{St}}, S_\varepsilon, \Sigma, (\overset{a}{\mapsto})_{a \in \Sigma}, \overset{\varepsilon}{\mapsto})$$

where S_{St} and S_ε are (maybe infinite) disjoint sets of *stable states* and *unstable states*, respectively, Σ is a finite *alphabet*, $\overset{a}{\mapsto} \subseteq S_{\mathsf{St}} \times (S_{\mathsf{St}} \cup S_\varepsilon)$, for $a \in \Sigma$, and $\overset{\varepsilon}{\mapsto} \subseteq S_\varepsilon \times S_{\mathsf{St}}$ are sets of *labelled transitions*; for each $s \in S_\varepsilon$ there is precisely one $t \in S_{\mathsf{St}}$ such that $s \overset{\varepsilon}{\mapsto} t$, whereas for any $s \in S_{\mathsf{St}}$ and $a \in \Sigma$ there is at most one $t \in S_{\mathsf{St}} \cup S_\varepsilon$ such that $s \overset{a}{\mapsto} t$. For all $w \in \Sigma^*$, we define relations $\xrightarrow{w} \subseteq S \times S$, where $S = S_{\mathsf{St}} \cup S_\varepsilon$, inductively: $s \xrightarrow{\varepsilon} s$ for each $s \in S$; if $s \overset{\varepsilon}{\mapsto} t$ then $s \xrightarrow{\varepsilon} t$; if $s \overset{a}{\mapsto} t$ ($a \in \Sigma$) then $s \xrightarrow{a} t$; if $s \xrightarrow{u} s'$ and $s' \xrightarrow{v} t$ ($u, v \in \Sigma^*$) then $s \xrightarrow{uv} t$.

By $s \xrightarrow{w}$ we denote that w is *enabled in* s, i.e. $s \xrightarrow{w} t$ for some t.

Given $\mathcal{T} = (S_{\mathsf{St}}, S_\varepsilon, \Sigma, (\overset{a}{\mapsto})_{a \in \Sigma}, \overset{\varepsilon}{\mapsto})$, *trace equivalence* $\sim$ on $S = S_{\mathsf{St}} \cup S_\varepsilon$ is defined as follows:

$$s \sim t \quad \text{if} \quad \forall w \in \Sigma^* : s \xrightarrow{w} \Leftrightarrow t \xrightarrow{w}.$$

Hence two states are equivalent iff they enable the same set of words (also called traces). A word $w \in \Sigma^*$ is a *non-equivalence witness for* (s, t), a *witness for* (s, t) for short, if w is enabled in precisely one of s, t.

Remark. By the above definitions, $s \overset{\varepsilon}{\mapsto} t$ implies $s \sim t$. This could suggest merging the states s and t but we keep them separate since this is convenient in the definitions of det-LTSs generated by deterministic one-counter automata, as given below.

We put $\Sigma^{\leq i} = \{w \in \Sigma^*;\ |w| \leq i\}$, and we note that $\sim = \bigcap \{\sim_i \mid i \in \mathbb{N}\}$ where the equivalences $\sim_0 \supseteq \sim_1 \supseteq \sim_2 \supseteq \ldots$ are defined as follows:

$$s \sim_i t \quad \text{if} \quad \forall w \in \Sigma^{\leq i} : s \xrightarrow{w} \Leftrightarrow t \xrightarrow{w}.$$

Each pair of states (s, t) has the *equivalence level*, the *eqlevel* for short, $\mathsf{EqL}(s, t) \in \mathbb{N} \cup \{\omega\}$:

$$\mathsf{EqL}(s,t) = \begin{cases} \omega & \text{if } s \sim t, \\ \max\{j \in \mathbb{N} \mid s \sim_j t\} & \text{otherwise.} \end{cases}$$

We also write $s \overset{e}{\longleftrightarrow} t$ instead of $\mathsf{EqL}(s,t) = e$ (where $e \in \mathbb{N} \cup \{\omega\}$). We note that the length of any *shortest* witness for (s,t), where $s \not\sim t$, is $\mathsf{EqL}(s,t)+1$. We also highlight the next simple fact (valid since our LTSs are *deterministic*).

Observation 1. *Suppose $s \xrightarrow{w} s'$ and $t \xrightarrow{w} t'$ in a given det-LTS. Then we have:*

1. $\mathsf{EqL}(s',t') \geq \mathsf{EqL}(s,t) - |w|$. *(Hence $s' \sim t'$ if $s \sim t$.)*
2. *If w is a (proper) prefix of a witness for (s,t) then* $\mathsf{EqL}(s',t') = \mathsf{EqL}(s,t) - |w|$.

A *deterministic one-counter automaton*, a *doca* for short, is a tuple

$$\mathcal{A} = (Q_{\mathsf{St}}, Q_{\mathsf{Res}}, \Sigma, \delta, (\mathsf{per}_s)_{s \in Q_{\mathsf{Res}}}, (\mathsf{goto}_s)_{s \in Q_{\mathsf{Res}}})$$

where Q_{St} and Q_{Res} are disjoint finite sets of *stable control states* and *reset control states*, respectively, Σ is a finite *alphabet*, $\delta \subseteq Q_{\mathsf{St}} \times \Sigma \times \{0,1\} \times (Q_{\mathsf{St}} \cup Q_{\mathsf{Res}}) \times \{-1,0,1\}$ is a set of *(transition) rules*, $\mathsf{per}_s \in \mathbb{N}$ are *periods* satisfying $1 \leq \mathsf{per}_s \leq |Q_{\mathsf{St}}|$, and $\mathsf{goto}_s : \{0,1,2,\ldots,\mathsf{per}_s{-}1\} \to Q_{\mathsf{St}}$ are *reset mappings*. For each $p \in Q_{\mathsf{St}}$, $a \in \Sigma$, $c \in \{0,1\}$ there is at most one pair (q,j) (where $q \in Q_{\mathsf{St}} \cup Q_{\mathsf{Res}}$, $j \in \{-1,0,1\}$) such that $(p,a,c,q,j) \in \delta$; moreover, if $c = 0$ then $j \neq -1$. The tuples $(p,a,0,q,j) \in \delta$ are called the *zero rules*, the tuples $(p,a,1,q,j) \in \delta$ are the *positive rules*.

A doca $\mathcal{A} = (Q_{\mathsf{St}}, Q_{\mathsf{Res}}, \Sigma, \delta, (\mathsf{per}_s)_{s \in Q_{\mathsf{Res}}}, (\mathsf{goto}_s)_{s \in Q_{\mathsf{Res}}})$ defines the det-LTS

$$\mathcal{T}(\mathcal{A}) = (Q_{\mathsf{St}} \times \mathbb{N}, Q_{\mathsf{Res}} \times \mathbb{N}, \Sigma, (\overset{a}{\mapsto})_{a \in \Sigma}, \overset{\varepsilon}{\mapsto}) \tag{1}$$

where $\overset{a}{\mapsto}$ and $\overset{\varepsilon}{\mapsto}$ are defined by the following (deduction) rules.

1. If $(p,a,1,q,j) \in \delta$ and $n > 0$ then $(p,n) \overset{a}{\mapsto} (q,n{+}j)$.
2. If $(p,a,0,q,j) \in \delta$ then $(p,0) \overset{a}{\mapsto} (q,j)$. (Recall that $j \in \{0,1\}$ in this case.)
3. If $s \in Q_{\mathsf{Res}}$ and $n \geq 0$ then $(s,n) \overset{\varepsilon}{\mapsto} (q,0)$ where $q = \mathsf{goto}_s(n \bmod \mathsf{per}_s)$.

An example of a doca with the respective det-LTS is sketched in Fig. 1.

By a *configuration* C of the doca $\mathcal{A}$ we mean (p,m), usually written as $p(m)$, where p is its control state and $m \in \mathbb{N}$ is its *counter value*. If $C = p(0)$ then it is a *zero configuration*. If $p \in Q_{\mathsf{St}}$ then $C = p(m)$ is a *stable configuration*; if $p \in Q_{\mathsf{Res}}$ then $p(m)$ is a *reset configuration*.

The definition of (general) det-LTSs induces the relations $\xrightarrow{w}$ ($w \in \Sigma^*$) on $Q \times \mathbb{N}$ where $Q = Q_{\mathsf{St}} \cup Q_{\mathsf{Res}}$. We are interested in the *doca equivalence problem*, denoted

$\mathsf{Doca\text{-}Eq}$:

Instance: A doca $\mathcal{A}$ and two stable zero configurations $p(0), q(0)$.

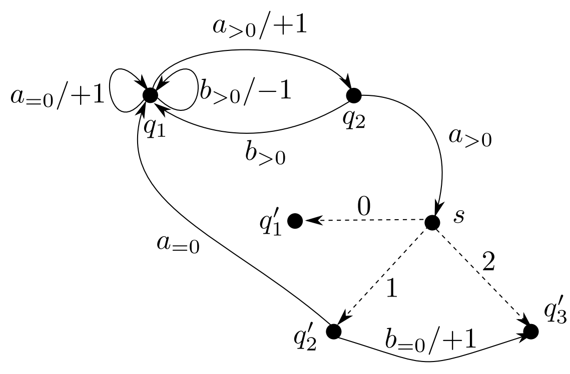
Question: Is $p(0) \sim q(0)$ in $\mathcal{T}(\mathcal{A})$?

Our main aim is to show the following theorem.

Theorem 2. *There is a polynomial* $\text{POLY} : \mathbb{N} \to \mathbb{N}$ *such that for any $\mathsf{Doca\text{-}Eq}$ instance $\mathcal{A}, p(0), q(0)$ where $\mathcal{A}$ has k control states we have that $p(0) \not\sim q(0)$ implies* $\mathsf{EqL}(p(0),q(0)) \leq \text{POLY}(k)$.

Using Theorem 2, we easily get the next theorem.

Theorem 3. *$\mathsf{Doca\text{-}Eq}$ is NL-complete.*



$Q_{\mathsf{St}} = \{q_1, q_2, q'_1, q'_2, q'_3\}$
$Q_{\mathsf{Res}} = \{s\}$
$\Sigma = \{a, b\}$
$\delta = \{\ (q_1, a, 1, q_2, 1),$
$(q_1, a, 0, q_1, 1),$
$(q_1, b, 1, q_1, -1),$
$(q_2, a, 1, s, 0),$
$(q_2, b, 1, q_1, 0),$
$(q'_2, a, 0, q_1, 0),$
$(q'_2, b, 0, q'_3, 1)\ \}$
$\mathsf{per}_s = 3$
$\mathsf{goto}_s(0) = q'_1$
$\mathsf{goto}_s(1) = q'_2$
$\mathsf{goto}_s(2) = q'_3$

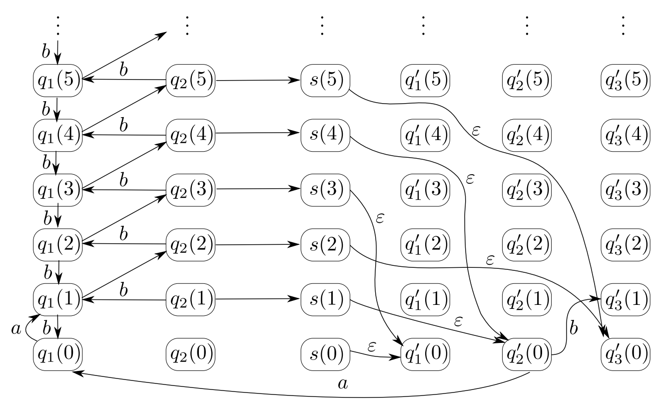


Figure 1: A doca $\mathcal{A}$, presented by a graph, and a fragment of $\mathcal{T}(\mathcal{A})$

Proof. The lower bound follows easily from NL-hardness of digraph reachability.

On the other hand, given a Doca-Eq instance $\mathcal{A}, p(0), q(0)$, a nondeterministic algorithm can perform the phases $j = 0, 1, 2, \dots$ described as follows. In phase j, there is a pair $(p_j(m_j), q_j(n_j))$ in memory, the counter values m_j, n_j written in binary; for $j = 0$ we have $(p_j(m_j), q_j(n_j)) = (p(0), q(0))$. If $\mathsf{EqL}(p_j(m_j), q_j(n_j)) > 0$ then a letter a is nondeterministically chosen, and $(p_j(m_j), q_j(n_j))$ is replaced with $(p_{j+1}(m_{j+1}), q_{j+1}(n_{j+1}))$ where $p_j(m_j) \xrightarrow{a} p_{j+1}(m_{j+1})$ and $q_j(m_j) \xrightarrow{a} q_{j+1}(m_{j+1})$.

If $p(0) \not\sim q(0)$ then Theorem 2 guarantees that a pair $(p_j(m_j), q_j(n_j))$ with $\mathsf{EqL}(p_j(m_j), q_j(n_j)) = 0$ can be thus reached by using only logarithmic space.

Hence Doca-Eq is in co-NL. Since NL=co-NL, we are done. □

3 Proof of Theorem 1

Convention. When considering a doca $\mathcal{A}$, we will always tacitly assume the notation

$$\mathcal{A} = (Q_{\mathsf{St}}, Q_{\mathsf{Res}}, \Sigma, \delta, (\mathsf{per}_s)_{s \in Q_{\mathsf{Res}}}, (\mathsf{goto}_s)_{s \in Q_{\mathsf{Res}}}) \tag{2}$$

if not said otherwise. We also reserve k for denoting the number of control states, i.e.

$$k = |Q_{\mathsf{St}}| + |Q_{\mathsf{Res}}|.$$

To be more concise in the later reasoning concerning a given doca $\mathcal{A}$, we use the words "few", "small", or "short" when we mean that the relevant quantity is bounded by a polynomial in k; the polynomial is always independent of $\mathcal{A}$. By a small rational number we mean $\rho = \frac{a}{b}$ or $\rho = -\frac{a}{b}$ where $a, b \in \mathbb{N}$ are small. We also say that

a set is small if its cardinality is a small number.

We note that if all elements of a set X of (integer or rational) numbers are small then X is a small set; the opposite is not true in general. We often tacitly use the fact that

a quantity arising as the sum or the product of two small quantities is also small.

Though these expressions might look informal, they can be always easily replaced by the formal statements which they abridge. By this convention, Theorem 2 says that the eqlevel of any pair of zero configurations is small when finite.

Remark. It will be always obvious that we could calculate a concrete respective polynomial whenever we use "few", "small", "short" in our claims. But such calculations would add tedious technicalities, and they would be not particularly rewarding w.r.t. the degree of the polynomials. We thus prefer a transparent concise proof which avoids technicalities whenever possible.

3.1 Shortest positive paths in $\mathcal{T}(\mathcal{A})$

We first define the notion of paths in general det-LTSs, and then we look at special paths in $\mathcal{T}(\mathcal{A})$, for a doca $\mathcal{A}$.

Definition 4. *Given a det-LTS* $\mathcal{T} = (S_{\mathsf{St}}, S_\varepsilon, \Sigma, (\overset{a}{\mapsto})_{a\in\Sigma}, \overset{\varepsilon}{\mapsto})$*, a* path *in* $\mathcal{T}$ *is a sequence*

$$s_0 \xrightarrow{a_1} s_1 \xrightarrow{a_2} \ldots \xrightarrow{a_z} s_z \ (z \in \mathbb{N})$$

where $s_i \in S_{\mathsf{St}}$ *and* $a_i \in \Sigma$ *(for all* $i, 0 \leq i \leq z$*); it is a path* from its start s_0 to its end s_z*. For any* i_1, i_2*, where* $0 \leq i_1 \leq i_2 \leq z$*, the sequence* $s_{i_1} \xrightarrow{a_{i_1+1}} s_{i_1+1} \xrightarrow{a_{i_1+2}} \cdots \xrightarrow{a_{i_2}} s_{i_2}$ *is a* subpath *of the above path. Slightly abusing notation, we will also use* $s \xrightarrow{w}$ *and* $s \xrightarrow{w} t$ $(s, t \in S_{\mathsf{St}})$ *to denote paths.*

We also refer to $s \xrightarrow{a} t$ *where* $s, t \in S_{\mathsf{St}}$ *and* $a \in \Sigma$ *as to a* step*. If* $s \overset{a}{\mapsto} t$ *then it is a* simple step*; if* $s \overset{a}{\mapsto} s' \overset{\varepsilon}{\mapsto} t$ *then it is a* combined step*. The* length of a path $s_0 \xrightarrow{a_1} s_1 \xrightarrow{a_2} \ldots \xrightarrow{a_z} s_z$ *is* z*, i.e. the number of its steps.*

When discussing the det-LTS $\mathcal{T}(\mathcal{A})$ for a doca $\mathcal{A}$, we use the term *reset steps* instead of combined steps. We now concentrate on positive paths in $\mathcal{T}(\mathcal{A})$, defined as follows.

Definition 5. *Given a doca* $\mathcal{A}$ *(in notation (2)), a* path

$$p_0(m_0) \xrightarrow{a_1} p_1(m_1) \xrightarrow{a_2} \cdots \xrightarrow{a_z} p_z(m_z) \tag{3}$$

in $\mathcal{T}(\mathcal{A})$ *is* positive *if each step* $p_i(m_i) \xrightarrow{a_{i+1}} p_{i+1}(m_{i+1})$ $(0 \leq i < z)$ *is simple and is induced by a positive rule* $(p_i, a_{i+1}, 1, p_{i+1}, j) \in \delta$ *(where* $j = m_{i+1} - m_i$*).*

The effect *(or the* counter change*)* of the path *(3) is* $m_z - m_0$*; if the path is positive, its effect is an integer in the interval* $[-z, z]$*. The path (3) is a* control state cycle *if it is positive and we have* $z > 0$ *and* $p_z = p_0$*.*

We note that if (3) is positive then there is no reset step in the path and $m_i > 0$ for all $i, 0 \leq i < z$; but we can have $m_z = 0$.

The next lemma can be easily derived from Lemma 2 in [23]; we thus only sketch the idea. The claim of the lemma is illustrated in Fig. 2.

Lemma 6. *If there is a positive path from* $p(m)$ *to* $q(n)$ *in* $\mathcal{T}(\mathcal{A})$ *then some of the shortest positive paths from* $p(m)$ *to* $q(n)$ *is of the form*

$$p(m) \xrightarrow{u_1} p'(m') \xrightarrow{v^i} p'(m'+id) \xrightarrow{u_2} q(n)$$

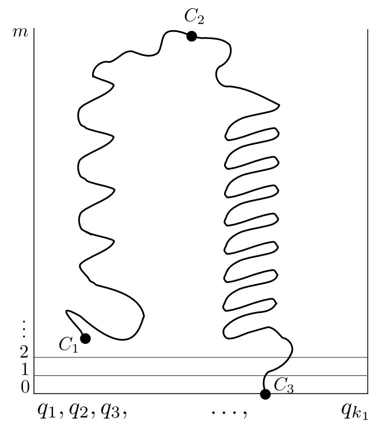


Figure 2: Shortest positive paths in $\mathcal{T}(\mathcal{A})$, one from a configuration C_1 to C_2 and one from C_2 to a zero configuration C_3. (Only the stable control states $q_1, q_2, \dots, q_{k_1}$ are depicted.)

where u_1 is a short word, called the pre-phase*, $p'(m') \xrightarrow{v} p'(m'+d)$ is a short control state cycle with the effect $d \in \mathbb{Z}$, and u_2 is a short word, called the* post-phase*. (The cycle v is repeated i times, where $i \geq 0$.)*

Proof. (Sketch.) Lemma 2 in [23] considers the case when $m \geq n + k^2$. The cycle v shown by that lemma has the length in $\{1, 2, \dots, k\}$ and the effect in $\{-1, -2, \dots, -k\}$. The length of the pre-phase plus the post-phase is bounded by k^2. The idea is to use a most effective control state cycle for repeating (with the largest ratio $\frac{|\text{effect}|}{\text{length}}$), and to add the "cost" of reaching that cycle from $p(m)$ and of reaching $q(n)$ from the end of the repeated cycle. The technical details can be found in [23].

The situation with $n \geq m + k^2$ is handled symmetrically. Having solved the case $|n - m| \geq k^2$, the case $|n-m| < k^2$ is obvious, as can be seen in Fig. 2: if a long path is going up via a short cycle with a positive effect d_1 and then down via another short cycle with a negative effect $-d_2$, then it can be shortened by removing d_2 copies of the first cycle and d_1 copies of the second cycle. Hence $|n - m| < k^2$ implies that there is a short positive path $p(m) \xrightarrow{u_1} q(n)$ (with $v = u_2 = \varepsilon$). $\square$

It is useful to highlight the following corollary of the previous lemma.

Corollary 7. *If $|m - n|$ is small and there is a positive path from $p(m)$ to $q(n)$ then there is a short positive path from $p(m)$ to $q(n)$.*

3.2 The extended det-LTS $\mathcal{T}_{\mathsf{ext}}(\mathcal{A})$

We now introduce a central notion, the det-LTS $\mathcal{T}_{\mathsf{ext}}(\mathcal{A})$, which extends the det-LTS $\mathcal{T}(\mathcal{A})$ defined in (1), for a given doca $\mathcal{A} = (Q_{\mathsf{St}}, Q_{\mathsf{Res}}, \Sigma, \delta, (\mathsf{per}_s)_{s \in Q_{\mathsf{Res}}}, (\mathsf{goto}_s)_{s \in Q_{\mathsf{Res}}})$.

Before giving a formal definition, we give an intuitive explanation. Let us (temporarily) imagine that $\mathcal{A}$ has also a *special mode of behaviour*, besides the *normal mode* defined previously; let any configuration $p(m)$ have its special-mode analogue $\overline{p}(m)$. For any *positive* counter value $m > 0$, each transition $p(m) \overset{a}{\mapsto} q(m+j)$ ($a \in \Sigma$) induces the transition $\overline{p}(m) \overset{a}{\mapsto} \overline{q}(m+j)$. Further, any transition $s(m) \overset{\varepsilon}{\mapsto} q(0)$ ($s \in Q_{\mathsf{Res}}$, $m \geq 0$) induces $\overline{s}(m) \overset{\varepsilon}{\mapsto} q(0)$; hence the special mode is finished by any reset step, after which the normal mode applies. *A crucial property of the special mode* is that whenever a configuration $\overline{p}(0)$, where $p \in Q_{\mathsf{St}}$, is entered (by a non-reset step), a multiple (the least common multiple, say) $\Delta \in \mathbb{N}$ of all periods per_s, $s \in Q_{\mathsf{Res}}$, is silently added to the counter (we put $\Delta = 1$ when $Q_{\mathsf{Res}} = \emptyset$). Hence the zero rules are never used in the special mode since the counter is always positive (until a possible reset step is performed). If we added the special-mode configurations and the respective transitions to $\mathcal{T}(\mathcal{A})$, we would easily observe that

- $p(m) \sim_m \overline{p}(m)$ (thus $\mathsf{EqL}(p(m), \overline{p}(m)) \geq m$);

- $p(m) \not\sim \overline{p}(m)$ iff there is a positive path $p(m) \xrightarrow{u} q(0)$ (and thus $\overline{p}(m) \xrightarrow{u} \overline{q}(0) = \overline{q}(\Delta)$) for some $q \in Q_{\mathsf{St}}$ such that $q(0) \not\sim \overline{q}(0)$ (i.e. $q(0) \not\sim \overline{q}(\Delta)$);
- if $m \equiv m' \pmod{\mathsf{per}_s}$ for all $s \in Q_{\mathsf{Res}}$ then $\overline{p}(m) \sim \overline{p}(m')$;
- if $s \in Q_{\mathsf{Res}}$ and $m \equiv m' \pmod{\mathsf{per}_s}$ then $\overline{s}(m) \sim \overline{s}(m')$.

In the special mode of $\mathcal{A}$, the concrete value m of the counter is not important once we know the tuple $(c_s)_{s\in Q_{\mathsf{Res}}}$ where $c_s = m \bmod \mathsf{per}_s$; in a reset configuration $\overline{s}(m)$, knowing just $c = m \bmod \mathsf{per}_s$ is sufficient.

We do not formalize the above notions and claims, since they only serve us for a better understanding of the definition of $\mathcal{T}_{\mathsf{ext}}(\mathcal{A})$ given below. The det-LTS $\mathcal{T}_{\mathsf{ext}}(\mathcal{A})$ arises from $\mathcal{T}(\mathcal{A})$ by adding a finite set Q_{Mod} of stable states and a finite set Q_{FixRes} of unstable states and the transitions defined below. The transitions from Q_{Mod} will only lead to $Q_{\mathsf{Mod}} \cup Q_{\mathsf{FixRes}}$, whereas the ε-transitions from Q_{FixRes} lead to zero configurations in $\mathcal{T}(\mathcal{A})$. There are no transitions leading from the configurations in $\mathcal{T}(\mathcal{A})$ to $Q_{\mathsf{Mod}} \cup Q_{\mathsf{FixRes}}$, and the subgraph of $\mathcal{T}_{\mathsf{ext}}(\mathcal{A})$ arising by the restriction to the configurations of $\mathcal{T}(\mathcal{A})$ is $\mathcal{T}(\mathcal{A})$ itself. We thus also safely use the same symbols $\overset{a}{\mapsto}$, $\overset{\varepsilon}{\mapsto}$ in both $\mathcal{T}(\mathcal{A})$ and $\mathcal{T}_{\mathsf{ext}}(\mathcal{A})$. An example is sketched in Fig. 3.

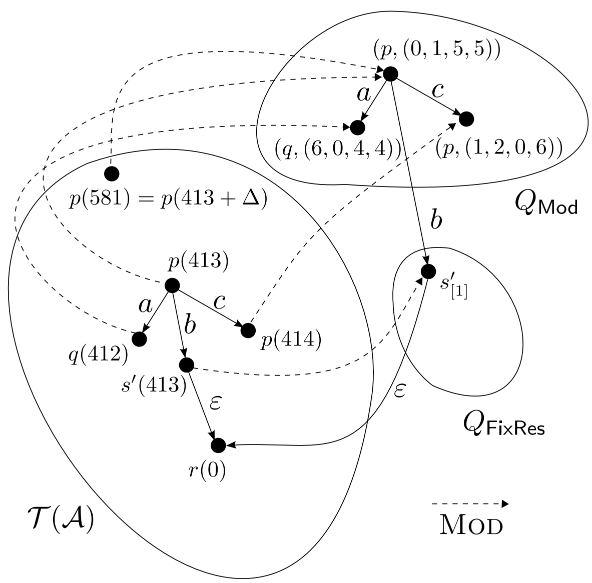


Figure 3: A fragment of $\mathcal{T}_{\mathsf{ext}}(\mathcal{A})$ where: $\{p,q,r\} \subseteq Q_{\mathsf{St}}$, $Q_{\mathsf{Res}} = \{s,s',s'',s'''\}$, $\{a,b,c\} \subseteq \Sigma$, $\{(p,a,1,q,-1),(p,b,1,s',0),(p,c,1,p,1)\} \subseteq \delta$, $(\mathsf{per}_s, \mathsf{per}_{s'}, \mathsf{per}_{s''}, \mathsf{per}_{s'''}) = (7,4,6,8)$, $\mathsf{goto}_{s'}(1) = r$, $\Delta = lcm\{7,4,6,8\} = 168$.

Definition 8. *Given a doca $\mathcal{A} = (Q_{\mathsf{St}}, Q_{\mathsf{Res}}, \Sigma, \delta, (\mathsf{per}_s)_{s\in Q_{\mathsf{Res}}}, (\mathsf{goto}_s)_{s\in Q_{\mathsf{Res}}})$, with the associated det-LTS $\mathcal{T}(\mathcal{A}) = (Q_{\mathsf{St}} \times \mathbb{N}, Q_{\mathsf{Res}} \times \mathbb{N}, \Sigma, (\overset{a}{\mapsto})_{a\in\Sigma}, \overset{\varepsilon}{\mapsto})$, we define the det-LTS*

$$\mathcal{T}_{\mathsf{ext}}(\mathcal{A}) = ((Q_{\mathsf{St}} \times \mathbb{N}) \cup Q_{\mathsf{Mod}}, (Q_{\mathsf{Res}} \times \mathbb{N}) \cup Q_{\mathsf{FixRes}}, \Sigma, (\overset{a}{\mapsto})_{a\in\Sigma}, \overset{\varepsilon}{\mapsto})$$

as the extension of $\mathcal{T}(\mathcal{A})$ where

- $Q_{\mathsf{Mod}} = \{(p,(c_s)_{s\in Q_{\mathsf{Res}}}) \mid p \in Q_{\mathsf{St}}, 0 \le c_s \le \mathsf{per}_s - 1\}$,
- $Q_{\mathsf{FixRes}} = \{s_{[c]} \mid s \in Q_{\mathsf{Res}}, 0 \le c \le \mathsf{per}_s - 1\}$, *and*
- *the additional transitions are defined by the following (deduction) rules:*
  1. *If $(p,a,1,q,j) \in \delta$ and $q \in Q_{\mathsf{St}}$ then for each $(p,(c_s)_{s\in Q_{\mathsf{Res}}}) \in Q_{\mathsf{Mod}}$ we have*

$$(p,(c_s)_{s\in Q_{\mathsf{Res}}}) \overset{a}{\mapsto} (q,(c'_s)_{s\in Q_{\mathsf{Res}}})$$

where $c'_s = (c_s{+}j) \bmod \mathsf{per}_s$ *for each* $s \in Q_{\mathsf{Res}}$.

2. *If* $(p,a,1,s',j) \in \delta$ *and* $s' \in Q_{\mathsf{Res}}$ *then for each* $(p,(c_s)_{s\in Q_{\mathsf{Res}}}) \in Q_{\mathsf{Mod}}$ *we have*

$$(p,(c_s)_{s\in Q_{\mathsf{Res}}}) \overset{a}{\mapsto} s'_{[c]}$$

where $c = (c_{s'}{+}j) \bmod \mathsf{per}_{s'}$.

3. *For each* $s_{[c]} \in Q_{\mathsf{FixRes}}$ *we have* $s_{[c]} \overset{\varepsilon}{\mapsto} q(0)$ *where* $q = \mathsf{goto}_s(c)$.

A configuration C *is a state in* $\mathcal{T}_{\mathsf{ext}}(\mathcal{A})$. *If* $C \in Q_{\mathsf{Mod}}$ *or* $C = p(m)$ *where* $p \in Q_{\mathsf{St}}$ *then* C *is* stable, *otherwise* C *is* unstable.

Moreover, we define the mapping

$$\mathsf{Mod} : ((Q_{\mathsf{St}} \cup Q_{\mathsf{Res}}) \times \mathbb{N}) \to (Q_{\mathsf{Mod}} \cup Q_{\mathsf{FixRes}}):$$

- *if* $p \in Q_{\mathsf{St}}$ *then* $\mathsf{Mod}(p(m)) = (p,(c_s)_{s\in Q_{\mathsf{Res}}}) \in Q_{\mathsf{Mod}}$ *where* $c_s = m \bmod \mathsf{per}_s$ *for all* $s \in Q_{\mathsf{Res}}$;
- *if* $s \in Q_{\mathsf{Res}}$ *then* $\mathsf{Mod}(s(m)) = s_{[c]} \in Q_{\mathsf{FixRes}}$ *where* $c = m \bmod \mathsf{per}_s$.

We note that the cardinality of Q_{Mod} might be exponential in k (i.e. in the number of control states of $\mathcal{A}$). On the other hand, Q_{FixRes} is small; this is a crucial fact for some claims in the next auxiliary propositions. We stipulate $\min\emptyset = \omega$, and recall that $z + \omega = \omega$ for any $z \in \mathbb{N}$.

Proposition 9.

1. *If* $(p,(c_s)_{s\in Q_{\mathsf{Res}}}) \xrightarrow{w} (q,(c'_s)_{s\in Q_{\mathsf{Res}}})$ *then for each* $(p,(d_s)_{s\in Q_{\mathsf{Res}}}) \in Q_{\mathsf{Mod}}$ *we have* $(p,(d_s)_{s\in Q_{\mathsf{Res}}}) \xrightarrow{w} (q,(d'_s)_{s\in Q_{\mathsf{Res}}})$ *where* $d'_s{-}c'_s \equiv d_s{-}c_s \pmod{\mathsf{per}_s}$ *for all* $s \in Q_{\mathsf{Res}}$.
2. *If* $(p,(c_s)_{s\in Q_{\mathsf{Res}}}) \xrightarrow{w} s'_{[c]}$ *then for each* $(p,(d_s)_{s\in Q_{\mathsf{Res}}}) \in Q_{\mathsf{Mod}}$ *we have* $(p,(d_s)_{s\in Q_{\mathsf{Res}}}) \xrightarrow{w} s'_{[d]}$ *where* $d{-}c \equiv d_{s'}{-}c_{s'} \pmod{\mathsf{per}_{s'}}$.
3. *For any* $s \in Q_{\mathsf{Res}}$ *we have* $s(m) \sim \mathsf{Mod}(s(m))$.
4. *If* $p \in Q_{\mathsf{St}}$ *then* $\mathsf{EqL}(p(m), \mathsf{Mod}(p(m))) =$ $= \min\{z + \mathsf{EqL}(q(0), \mathsf{Mod}(q(0))) \mid q \in Q_{\mathsf{St}}$ *and* z *is the length of a positive path from* $p(m)$ *to* $q(0)\}$.
5. *For any* $p \in Q_{\mathsf{St}}$, $m \in \mathbb{N}$, *and* $w \in \Sigma^*$ *there is some small positive* $d \in \mathbb{N}$ *such that*
   - *either for each* m' *such that* $m' \equiv m \pmod{d}$ *we have that* $\mathsf{Mod}(p(m'))$ *enables* w,
   - *or for each* m' *such that* $m' \equiv m \pmod{d}$ *we have that* $\mathsf{Mod}(p(m'))$ *does not enable* w.

Proof. Points 1 and 2 can be easily shown by induction on $|w|$, using Def. 8.

Point 3 is obvious since $s(m) \overset{\varepsilon}{\mapsto} q(0)$ and $\mathsf{Mod}(s(m)) \overset{\varepsilon}{\mapsto} q(0)$ for the appropriate $q \in Q_{\mathsf{St}}$.

Point 4:

We first note that if $p(m) \xrightarrow{w} q(n)$ is a positive path (in $\mathcal{T}(\mathcal{A})$) then we have $\mathsf{Mod}(p(m)) \xrightarrow{w} \mathsf{Mod}(q(n))$ (as can be easily shown by induction on $|w|$).

One part of the equality, namely $\mathsf{EqL}(p(m), \mathsf{Mod}(p(m))) \leq \min\{\dots\}$, is thus clear; it remains to show

$$\mathsf{EqL}(p(m), \mathsf{Mod}(p(m))) \geq \min\{z + \mathsf{EqL}(q(0), \mathsf{Mod}(q(0))) \mid \dots\}. \tag{4}$$

The case where $p(m) \sim \mathsf{Mod}(p(m))$ is trivial. We thus further consider only the cases $p(m) \not\sim \mathsf{Mod}(p(m))$, and we proceed by induction on $\mathsf{EqL}(p(m), \mathsf{Mod}(p(m)))$. If $\mathsf{EqL}(p(m), \mathsf{Mod}(p(m))) = 0$ then we obviously must have $m = 0$, and (4) is trivial in any case with $m = 0$.

Let us now assume $m > 0$, and let av ($a \in \Sigma$) be a shortest witness for $(p(m), \mathsf{Mod}(p(m)))$. We must have some $(p,a,1,q,j) \in \delta$, and thus $p(m) \overset{a}{\mapsto} q(m{+}j)$ and $\mathsf{Mod}(p(m)) \overset{a}{\mapsto} \mathsf{Mod}(q(m{+}j))$

(as can be easily checked). Point 3 excludes the case $q \in Q_{\mathsf{Res}}$, hence $q \in Q_{\mathsf{St}}$. By recalling Observation 1(2), and using the induction hypothesis for $q(m+j), \mathsf{Mod}(q(m+j))$, we finish the proof easily: $\mathsf{EqL}(p(m), \mathsf{Mod}(p(m))) = 1 + \mathsf{EqL}(q(m+j), \mathsf{Mod}(q(m+j))) \geq 1 + z + \mathsf{EqL}(q'(0), \mathsf{Mod}(q'(0)))$ where z is the length of some positive path from $q(m+j)$ to $q'(0)$, and $1+z$ is thus the length of some positive path from $p(m)$ to $q'(0)$.

Point 5:

By recalling Points 1 and 2, we easily note the following fact:
If $\mathsf{Mod}(p(m_1)) \xrightarrow{u} C \in Q_{\mathsf{Mod}} \cup Q_{\mathsf{FixRes}}$ then for any m_2 there is $C' \in Q_{\mathsf{Mod}} \cup Q_{\mathsf{FixRes}}$ such that $\mathsf{Mod}(p(m_2)) \xrightarrow{u} C'$; moreover, if $\mathsf{Mod}(p(m_1)) \xrightarrow{u} s_{[c]}$ then $\mathsf{Mod}(p(m_2)) \xrightarrow{u} s_{[c]}$ for any m_2 such that $m_2 \equiv m_1 \pmod{\mathsf{per}_s}$.

Hence if $w = uv$ where $\mathsf{Mod}(p(m)) \xrightarrow{u} s_{[c]}$ then the claim is satisfied by $d = \mathsf{per}_s$, and otherwise it is satisfied even by $d = 1$. □

We recall that $C \xleftrightarrow{e} C'$ means $\mathsf{EqL}(C, C') = e$.

Proposition 10.

1. *For any $p, q \in Q_{\mathsf{St}}$ and $m, n \in \mathbb{N}$ there are small positive $d_1, d_2 \in \mathbb{N}$ such that for any $m', n' \in \mathbb{N}$ we have: if $m' \equiv m \pmod{d_1}$ and $n' \equiv n \pmod{d_2}$ then*

$$\mathsf{EqL}(\mathsf{Mod}(p(m')), \mathsf{Mod}(q(n'))) \leq \mathsf{EqL}(\mathsf{Mod}(p(m)), \mathsf{Mod}(q(n))).$$

2. *The set $\{\, e \mid$ there are $C, C' \in Q_{\mathsf{Mod}}$ s.t. $C \xleftrightarrow{e} C' \}$ is small.*

Proof. Point 1:
If $\mathsf{Mod}(p(m)) \sim \mathsf{Mod}(q(n))$ then the claim is trivial. We thus assume $\mathsf{Mod}(p(m)) \not\sim \mathsf{Mod}(q(n))$ and let w be a shortest witness for $(\mathsf{Mod}(p(m)), \mathsf{Mod}(q(n)))$.

By Prop. 9(5), p, m, w give rise to d_1 and q, n, w give rise to d_2 such that precisely one of $\mathsf{Mod}(p(m')), \mathsf{Mod}(q(n'))$ enables w when $m' \equiv m \pmod{d_1}$ and $n' \equiv n \pmod{d_2}$. In this case w is a witness (not necessarily a shortest) for $(\mathsf{Mod}(p(m')), \mathsf{Mod}(q(n')))$, and the claim thus follows.

Point 2:
It is obvious that the set in Point 2 is equal to

$$\{\, e \mid \mathsf{Mod}(p(m)) \xleftrightarrow{e} \mathsf{Mod}(q(n)) \text{ for some } p, q \in Q_{\mathsf{St}}, m, n \in \mathbb{N}\}.$$

With every tuple (p, m, q, n) we associate a fixed tuple (d_1, d_2, c_1, c_2) where d_1, d_2 are those guaranteed by Point 1, and $c_1 = m \bmod d_1$, $c_2 = n \bmod d_2$. If two tuples (p, m_1, q, n_1), (p, m_2, q, n_2) have the same associated tuple (d_1, d_2, c_1, c_2) then $\mathsf{EqL}(\mathsf{Mod}(p(m_1)), \mathsf{Mod}(q(n_1))) = \mathsf{EqL}(\mathsf{Mod}(p(m_2)), \mathsf{Mod}(q(n_2)))$, as follows by applying Point 1 in both directions. Since the number of possible tuples $(p, q, d_1, d_2, c_1, c_2)$ is small, we are done. □

The next proposition can be proved analogously as the previous one.

Proposition 11.

1. *For any $p, q \in Q_{\mathsf{St}}$ and $m, n \in \mathbb{N}$ there is some small positive $d \in \mathbb{N}$ such that for any $m' \in \mathbb{N}$ we have: if $m' \equiv m \pmod{d}$ then*

$$\mathsf{EqL}(\mathsf{Mod}(p(m')), q(n)) \leq \mathsf{EqL}(\mathsf{Mod}(p(m)), q(n)).$$

2. *For any (fixed) $q(n)$, the set $\{\, e \mid$ there is $C \in Q_{\mathsf{Mod}}$ s.t. $C \xleftrightarrow{e} q(n)\}$ is small.*

3.3 Eqlevels of pairs of zero configurations

Let us recall $\mathcal{T}_{\mathsf{ext}}(\mathcal{A})$ defined in Def. 8. We could view the elements of $Q_{\mathsf{Mod}} \cup Q_{\mathsf{FixRes}}$ as additional control states of $\mathcal{A}$; in these states the counter value would play no role and could be formally viewed as zero. This observation justifies the name "zero configurations" in the following definition.

Definition 12. *Given a doca $\mathcal{A}$ as in (2), with the associated $\mathcal{T}_{\mathsf{ext}}(\mathcal{A})$ by Def. 8, a state C in $\mathcal{T}_{\mathsf{ext}}(\mathcal{A})$ is a* zero configuration *if either $C \in Q_{\mathsf{Mod}} \cup Q_{\mathsf{FixRes}}$ or $C = p(0)$ where $p \in Q_{\mathsf{St}} \cup Q_{\mathsf{Res}}$.*

We define the set $\mathrm{ZE} \subseteq \mathbb{N}$ (Zero configurations Eqlevels) as follows:

$$\mathrm{ZE} = \{\, e \in \mathbb{N} \mid \textit{there are two stable zero configurations } C, C' \textit{ s.t. } C \overset{e}{\longleftrightarrow} C' \}.$$

We thus have $\mathrm{ZE} = \mathrm{E}_1 \cup \mathrm{E}_2 \cup \mathrm{E}_3$ where

$\mathrm{E}_1 = \{\, e \in \mathbb{N} \mid p(0) \overset{e}{\longleftrightarrow} q(0)$ for some $p, q \in Q_{\mathsf{St}} \}$,
$\mathrm{E}_2 = \{\, e \in \mathbb{N} \mid p(0) \overset{e}{\longleftrightarrow} C$ for some $p \in Q_{\mathsf{St}}, C \in Q_{\mathsf{Mod}} \}$,
$\mathrm{E}_3 = \{\, e \in \mathbb{N} \mid C \overset{e}{\longleftrightarrow} C'$ for some $C, C' \in Q_{\mathsf{Mod}} \}$.

Since the set $\{\, p(0) \mid p \in Q_{\mathsf{St}} \}$ is obviously small, by Prop. 10(2) and 11(2) we easily derive the following claim.

Lemma 13. *The set* ZE *is small.*

The lemma does not claim that the elements of ZE are small numbers. This will be shown in the following subsections; i.e., we will prove the next theorem which strengthens Theorem 2.

Theorem 14. *There is a polynomial* $\textsc{poly} : \mathbb{N} \to \mathbb{N}$ *such that* $\max\{\, e \mid e \in \mathrm{ZE}\} \leq \textsc{poly}(k)$ *(for any doca $\mathcal{A}$ with k control states).*

Let $e_0 < e_1 < e_2 < \cdots < e_f$ be the ordered elements of ZE. We have shown that f is small but we have not yet shown that all e_i are small numbers. W.l.o.g. we can assume $e_0 = 0$ (by adding two special control states, say). For proving Theorem 14 it thus suffices to show that the "gaps" between e_i and e_{i+1}, i.e. the differences $e_{i+1} - e_i$, are small. We will later contradict the existence of a large gap between $e_i = e_D$ (Down) and $e_{i+1} = e_U$ (Up) depicted in Figure 4.

$$e_0 - - e_1 - \cdots - e_D - - - - - - - - - - - - - - - - - - - e_U - \cdots - e_f$$

Figure 4: Assumption of a large gap in ZE (to be contradicted later)

But we first explore some further notions related to a given doca $\mathcal{A}$ and the det-LTS $\mathcal{T}_{\mathsf{ext}}(\mathcal{A})$.

3.4 Independence level

We assume a doca $\mathcal{A}$ as in (2), and explore a notion which we have already touched on implicitly.

Definition 15. *For $p \in Q_{\mathsf{St}}$, $m \in \mathbb{N}$ we put*

$$\mathsf{IL}(p(m)) = \mathsf{EqL}(p(m), \mathsf{Mod}(p(m))).$$

$\mathsf{IL}(p(m))$ can be understood as an "Independence Level" of $p(m)$ w.r.t. the concrete value m.

Proposition 16. *For each $p(m)$ with $\mathsf{IL}(p(m)) < \omega$ there are small rational numbers ρ, σ (of the type $\frac{a}{b}$, $-\frac{a}{b}$ where $a, b \in \mathbb{N}$ are small) and some $q \in Q_{\mathsf{St}}$ such that*

$$\mathsf{IL}(p(m)) = \rho \cdot m + \sigma + \mathsf{IL}(q(0)).$$

Moreover, we can require $\rho \geq 0$, $\rho \cdot m + \sigma \geq 0$, and if m is larger than a small bound then $\rho > 0$.

Convention. We will further assume that each $p(m)$ with $\mathsf{IL}(p(m)) < \omega$ has a fixed associated equality $\mathsf{IL}(p(m)) = \rho \cdot m + \sigma + e$ where $e = \mathsf{IL}(q(0)) \in \mathrm{ZE}$ and ρ, σ, q have the claimed properties.

Proof. Suppose $\mathsf{IL}(p(m)) < \omega$. If $m = 0$ then we can take $\rho = \sigma = 0$ and $q = p$. If $m > 0$ then Prop. 9(4) implies that there is some $q \in Q_{\mathsf{St}}$ such that $\mathsf{IL}(p(m)) = |w| + \mathsf{IL}(q(0))$ where $p(m) \xrightarrow{w} q(0)$ is a shortest positive path from $p(m)$ to $q(0)$. (Recall the path from C_2 to C_3 in Fig. 2 as an example.) By Lemma 6 we can assume that w is in the form $u_1 v^i u_2$, for a short prefix u_1, a short repeated cycle v, and a short suffix u_2. Hence $|w| = i \cdot |v| + |u_1| + |u_2|$, and $m = i \cdot (-d) - d_1 - d_2$ where d, d_1, d_2 are the effects of (i.e. the counter changes caused by) v, u_1, u_2, respectively. We note that $d_2 \leq 0$ and that we can assume $d < 0$. Since $i = \frac{m+d_1+d_2}{-d}$, we get $|w| = \frac{|v|}{-d} \cdot m + \frac{|v| \cdot (d_1+d_2)}{-d} + |u_1| + |u_2|$. As $\mathsf{IL}(p(m)) = |w| + \mathsf{IL}(q(0))$, all the claims follow easily. $\square$

Figure 5 depicts $\mathsf{IL}(p(m))$ for a fixed $p \in Q_{\mathsf{St}}$ and for a few values m, by using black circles $\bullet$; e_1, e_2, e_3 are elements of ZE corresponding to $\mathsf{IL}(q(0))$ for several q. There might be some "irregular" values $\mathsf{IL}(p(m)) = z + \mathsf{IL}(q(0))$ for small m and small z but for m larger than a small bound the values $\mathsf{IL}(p(m))$ lie on few lines, starting near some e_j and having small slopes. (In fact, we have $1 \leq \frac{|v|}{|\mathsf{effect}(v)|} \leq k$ for the respective cycles v in $w = u_1 v^i u_2$; the unit-length for the vertical axis is thus smaller than for the horizontal axis in Fig. 5.) The circles $\bullet$ and $\circ$ on one depicted line can correspond to the pairs $(m_0, z_0 + \mathsf{IL}(q(0)))$, $(m_0 + d, z_0 + d' + \mathsf{IL}(q(0)))$, $(m_0 + 2d, z_0 + 2d' + \mathsf{IL}(q(0)))$, $\ldots$ where $d = |\mathsf{effect}(v)|$ and $d' = |v|$ (and $z_0 = |u_1 u_2|$, $z_0 + d' = |u_1 v u_2|$, $z_0 + 2d' = |u_1 v^2 u_2|, \ldots$). A white circle $\circ$ depicts that the respective value, corresponding to a positive path $p(m_0+id) \xrightarrow{u_1 v^i u_2} q(0)$, is not $\mathsf{IL}(p(m_0+id))$ since there is another, and shorter, witness in this case.

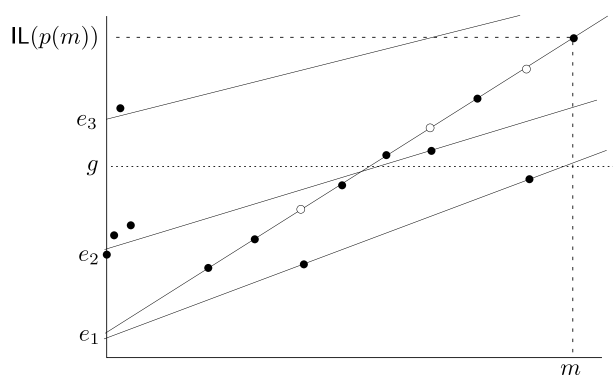


Figure 5: Illustrating $\mathsf{IL}(p(m))$ as a function of m

We now observe some further facts for later use.

Proposition 17.

1. *For each $g \in \mathbb{N}$ there are only few $p(m)$ such that $\mathsf{IL}(p(m)) = g$.*

2. *For any $p(m)$ where $\mathsf{IL}(p(m)) < \omega$ there are some small numbers $\mathsf{base} \geq 0$ and $\mathsf{per} > 0$ such that the following condition holds:*
   for any m' such that $\mathsf{base} \leq m' < m$ and $m' \equiv m \pmod{\mathsf{per}}$ we have $\mathsf{IL}(p(m')) < \mathsf{IL}(p(m))$.

Proof. Point 1 is intuitively clear from the horizontal line at level g in Fig. 5. Formally, we look when we can have $g = \rho \cdot m + \sigma + e$ where $\mathsf{IL}(p(m)) = \rho \cdot m + \sigma + e$ is the equality associated with some $p(m)$ (by Convention after Prop. 16). Since there are only few possibilities for ρ, σ, e, and we can have $\rho = 0$ only for few (small) values m, there are only few possible m which might fit.

Point 2 has been intuitively shown by the line with black and white circles in Fig. 5 and by the respective discussion. To be more formal, we recall that $\mathsf{IL}(p(m)) = |w| + \mathsf{IL}(q(0))$ for some $q \in Q_{\mathsf{St}}$ and a shortest positive path $p(m) \xrightarrow{w} q(0)$ from $p(m)$ to $q(0)$. We assume $w = u_1 v^i u_2$ for short u_1, v, u_2 as in the proof of Prop. 16. Hence if m is bigger than a small bound then $i > 0$. Let $\mathsf{per} = |\mathsf{effect}(v)|$. Then we have

$$
\begin{aligned}
&p(m - \mathsf{per}) \xrightarrow{u_1 v^{i-1} u_2} q(0),\\
&p(m - 2 \cdot \mathsf{per}) \xrightarrow{u_1 v^{i-2} u_2} q(0),\\
&\ldots,\\
&p(m - x \cdot \mathsf{per}) \xrightarrow{u_1 v^{i-x} u_2} q(0)
\end{aligned}
$$

for $x = (m - \mathsf{base}) \div \mathsf{per}$ where we put $\mathsf{base} = |u_1| + |u_2| + |v|$ to be safe, i.e. to guarantee that $u_1 v^{i-j} u_2$ is indeed enabled in $p(m - j \cdot \mathsf{per})$, for all $j = 1, 2, \ldots, x$. Since $p(m - j \cdot \mathsf{per}) \xrightarrow{u_1 v^{i-j} u_2} q(0)$ is a positive path, we have $\mathsf{IL}(p(m - j \cdot \mathsf{per})) \leq |u_1 v^{i-j} u_2| + \mathsf{IL}(q(0))$ by Prop. 9(4). Since $|u_1 v^{i-j} u_2| + \mathsf{IL}(q(0)) < |w| + \mathsf{IL}(q(0)) = \mathsf{IL}(p(m)$, we are done. $\square$

3.5 Eqlevel tuples

We introduce the eqlevel tuples illustrated in Fig. 6, assuming a given doca $\mathcal{A}$ as in (2), with the associated det-LTSs $\mathcal{T}(\mathcal{A})$ and $\mathcal{T}_{\mathsf{ext}}(\mathcal{A})$. A simple property of these tuples considerably simplifies the later analysis.

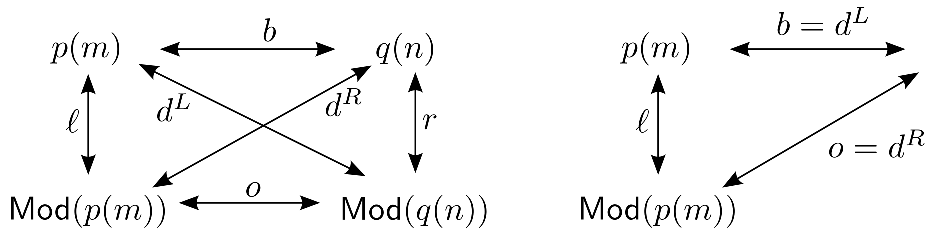


Figure 6: Eqlevel tuple $(b, \ell, r, o, d^L, d^R)$ associated to a pair $(p(m), q(n))$, and to a pair $(p(m), C)$ where $C \in Q_{\mathsf{Mod}}$

Definition 18. *Each pair $(p(m), q(n))$ of stable configurations in $\mathcal{T}(\mathcal{A})$ has the* associated eqlevel tuple $(b, \ell, r, o, d^L, d^R)$ *(of elements from $\mathbb{N} \cup \{\omega\}$) defined as follows:*

- $b = \mathsf{EqL}(p(m), q(n))$ *(Basic),*
- $\ell = \mathsf{IL}(p(m))$ *(Left),*
- $r = \mathsf{IL}(q(n))$ *(Right),*
- $o = \mathsf{EqL}(\mathsf{Mod}(p(m)), \mathsf{Mod}(q(n))$ *(mOd),*
- $d^L = \mathsf{EqL}(p(m), \mathsf{Mod}(q(n))$ *(Diagonal Left),*
- $d^R = \mathsf{EqL}(q(n), \mathsf{Mod}(p(m))$ *(Diagonal Right).*

Each pair $(p(m), C)$ where $C \in Q_{\mathsf{Mod}}$ and $p(m)$ is a stable configuration in $\mathcal{T}(\mathcal{A})$ has the associated eqlevel tuple $(b, \ell, r, o, d^L, d^R)$ *defined as follows:*

- $b = d^L = \mathsf{EqL}(p(m), C)$,
- $\ell = \mathsf{IL}(p(m))$,
- $r = \omega$,
- $o = d^R = \mathsf{EqL}(\mathsf{Mod}(p(m)), C)$.

We could similarly associate a tuple to $(C, q(n))$ but this is not needed in later reasoning. The following trivial fact yields an important corollary for the eqlevel tuples; it holds for general LTSs but we confine ourselves to the introduced det-LTSs.

Proposition 19. *Given states $s_1, s_2, \ldots, s_m$ in a det-LTS where $m \geq 2$ and $s_1 \overset{e_1}{\longleftrightarrow} s_2$, $s_2 \overset{e_2}{\longleftrightarrow} s_3$, $\ldots$, $s_{m-1} \overset{e_{m-1}}{\longleftrightarrow} s_m$, $s_m \overset{e_m}{\longleftrightarrow} s_1$, the minimum of $\{e_1, e_2, \ldots, e_m\}$ cannot be e_i for just one i.*

Proof. We assume by contradiction that $\min\{e_1, \ldots, e_m\} = e_i$ for just one $i \in \{1, \ldots, m\}$; w.l.o.g. we assume $i = 1$, and we note that $e_1 < \omega$ (since $m \geq 2$). Then we have $s_2 \sim_{e_1+1} s_3 \sim_{e_1+1} s_4 \cdots \sim_{e_1+1} s_m \sim_{e_1+1} s_1$ and hence $s_1 \sim_{e_1+1} s_2$ by transitivity and symmetry of $\sim_{e_1+1}$; this contradicts the assumption $s_1 \overset{e_1}{\longleftrightarrow} s_2$. ☐

Corollary 20. *In the "triangle" (b, ℓ, d^R), we always have $b = \ell$ or $b = d^R$ or $\ell = d^R$ (or $b = \ell = d^R$) as the minimum. Similarly for the "triangles" (d^R, r, o), (b, d^L, r), and (ℓ, d^L, o). In the "rectangle" (b, ℓ, r, o), the minimum is also achieved by at least two elements (concretely by $b = \ell$, $b = r$, $b = o$, $\ell = r$, $\ell = o$, or $r = o$).*

3.6 Paths in $\mathcal{T}(\mathcal{A}) \times \mathcal{T}(\mathcal{A})$

Since we are interested in comparing two states in a det-LTS $\mathcal{T}$, it is useful to define the product $\mathcal{T} \times \mathcal{T}$; the transitions in $\mathcal{T} \times \mathcal{T}$ are just the letter-synchronized pairs of transitions in $\mathcal{T}$. Eqlevel-decreasing paths in $\mathcal{T} \times \mathcal{T}$ will be of particular interest. A formal definition follows.

Definition 21. *Let $\mathcal{T} = (S_{\mathsf{St}}, S_\varepsilon, \Sigma, (\overset{a}{\mapsto})_{a\in\Sigma}, \overset{\varepsilon}{\mapsto})$ be a det-LTS. We define the det-LTS*

$$\mathcal{T} \times \mathcal{T} = (S_{\mathsf{St}} \times S_{\mathsf{St}}, S'_\varepsilon, \Sigma, (\overset{a}{\mapsto})_{a\in\Sigma}, \overset{\varepsilon}{\mapsto})$$

where $S'_\varepsilon = (S_{\mathsf{St}} \times S_\varepsilon) \cup (S_\varepsilon \times S_{\mathsf{St}}) \cup (S_\varepsilon \times S_\varepsilon)$ and the transitions are defined as follows:

1. *If $s, t \in S_{\mathsf{St}}$ and $s \overset{a}{\mapsto} s'$ and $t \overset{a}{\mapsto} t'$ (for $a \in \Sigma$) then $(s,t) \overset{a}{\mapsto} (s', t')$.*
2. *If $s \in S_{\mathsf{St}}$, $t \in S_\varepsilon$, and $t \overset{\varepsilon}{\mapsto} t'$ then $(s,t) \overset{\varepsilon}{\mapsto} (s, t')$.*
3. *If $s \in S_\varepsilon$, $t \in S_{\mathsf{St}}$, and $s \overset{\varepsilon}{\mapsto} s'$ then $(s,t) \overset{\varepsilon}{\mapsto} (s', t)$.*
4. *If $s \overset{\varepsilon}{\mapsto} s'$ and $t \overset{\varepsilon}{\mapsto} t'$ then $(s,t) \overset{\varepsilon}{\mapsto} (s', t')$.*

A path $(s_0, s'_0) \xrightarrow{a_1} (s_1, s'_1) \xrightarrow{a_2} (s_2, s'_2) \cdots \xrightarrow{a_z} (s_z, s'_z)$ *in $\mathcal{T} \times \mathcal{T}$ (where $(s_i, s'_i) \in S_{\mathsf{St}} \times S_{\mathsf{St}}$ by Def. 4) is* eqlevel-decreasing *if $\mathsf{EqL}(s_i, s'_i) > \mathsf{EqL}(s_{i+1}, s'_{i+1})$ for all $i \in \{0, 1, \ldots, z-1\}$.*

We can easily verify that $\mathcal{T} \times \mathcal{T}$ is indeed a det-LTS. We also note that in eqlevel-decreasing paths we must have $\mathsf{EqL}(s_{i+1}, s'_{i+1}) = \mathsf{EqL}(s_i, s'_i) - 1$, by Observation 1. We also observe:

Observation 22.

1. *Any subpath of an eqlevel-decreasing path in $\mathcal{T} \times \mathcal{T}$ is a shortest path from its start to its end.*
2. *Suppose the path $(s,t) \xrightarrow{w} (s', t')$ is eqlevel-decreasing. If $(s,t) \xrightarrow{v} (s'', t'')$ where $|v| < |w|$ then $\mathsf{EqL}(s'', t'') > \mathsf{EqL}(s', t')$.*

We now look at $\mathcal{T}(\mathcal{A}) \times \mathcal{T}(\mathcal{A})$ for a doca $\mathcal{A}$.

Definition 23. *We call $(p(m), q(n)) \xrightarrow{a} (p'(m'), q'(n'))$ a* reset step *(in $\mathcal{T}(\mathcal{A}) \times \mathcal{T}(\mathcal{A})$) if at least one of component-steps $p(m) \xrightarrow{a} p'(m')$, $q(n) \xrightarrow{a} q'(n')$ is a reset step in $\mathcal{T}(\mathcal{A})$. If precisely one of component-steps is a reset step then $(p(m), q(n)) \xrightarrow{a} (p'(m'), q'(n'))$ is a* one-side reset step, *if both component-steps are reset steps then $(p(m), q(n)) \xrightarrow{a} (p'(m'), q'(n'))$ is a* both-side reset step.

We note that one of m', n' is 0 when $(p(m), q(n)) \xrightarrow{a} (p'(m'), q'(n'))$ is a one-side reset step, and $m' = n' = 0$ when it is a both-side reset step.

Fig. 7 shows an example of a path $\mathcal{T}(\mathcal{A}) \times \mathcal{T}(\mathcal{A})$, projected to $\mathbb{N} \times \mathbb{N}$ (a pair $(p(m), q(n))$ is projected to (m, n)); the dotted lines represent one-side reset steps. Theorem 2 claims, in fact, that the eqlevel-decreasing paths in $\mathcal{T}(\mathcal{A}) \times \mathcal{T}(\mathcal{A})$ which start from pairs of zero configurations are short.

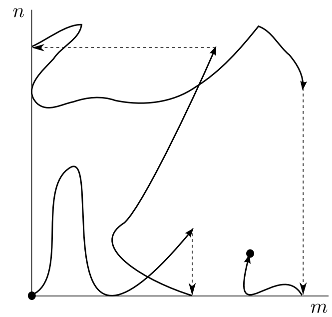


Figure 7: A path from $(p(0), q(0))$ in $\mathcal{T}(\mathcal{A}) \times \mathcal{T}(\mathcal{A})$ (with some one-side resets), projected to $\mathbb{N} \times \mathbb{N}$.

3.7 IL-equality lines

We assume a fixed doca $\mathcal{A}$, and consider the cases $\mathsf{IL}(p(m)) = \mathsf{IL}(q(n)) < \omega$ (i.e., $\ell = r < \omega$ in Fig. 6); we explore what we can say about the respective points $(m,n) \in \mathbb{N} \times \mathbb{N}$. By Convention after Prop. 16, each such case has the associated equalities $\mathsf{IL}(p(m)) = \rho \cdot m + \sigma + e$ and $\mathsf{IL}(q(n)) = \rho' \cdot n + \sigma' + e'$, and $\mathsf{IL}(p(m)) = \mathsf{IL}(q(n))$ thus implies $\rho \cdot m + \sigma + e = \rho' \cdot n + \sigma' + e'$.

Only in few cases we have $\rho = 0$ or $\rho' = 0$ (which is clear by Prop. 16 and Prop. 17(1)); in the other (many) cases we have $n = \frac{\rho}{\rho'} m + \frac{(\sigma - \sigma') + (e - e')}{\rho'}$ where $\frac{\rho}{\rho'} > 0$. This naturally leads to the following notions (illustrated in Fig. 8).

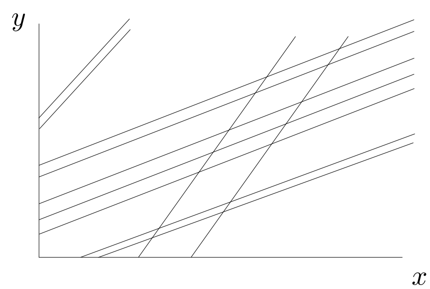


Figure 8: A sketch of IL-equality lines (in reality, lines contain only points with integer coordinates)

Definition 24. *A pair (μ, τ) of rational numbers is a* valid slope-shift pair *if there are some $p(m)$, $q(n)$ with the associated equalities $\mathsf{IL}(p(m)) = \rho \cdot m + \sigma + e$ and $\mathsf{IL}(q(n)) = \rho' \cdot n + \sigma' + e'$ such that $\rho \cdot m + \sigma + e = \rho' \cdot n + \sigma' + e'$, $\rho > 0$, $\rho' > 0$, $\mu = \frac{\rho}{\rho'}$, $\tau = \frac{(\sigma - \sigma') + (e - e')}{\rho'}$.*

Each valid slope-shift pair (μ, τ) defines an IL-equality line, *or just a* line *for short, namely the set $\{(x,y) \in \mathbb{N} \times \mathbb{N} \mid y = \mu \cdot x + \tau\}$.*

Any maximal set of parallel lines (having the same slope but various shifts) is a line-bunch. *(The maximality is taken w.r.t. set inclusion.) We say that $(x,y) \in \mathbb{N} \times \mathbb{N}$ is in a line-bunch H if (x,y) is in a line in H.*

Though each line contains at least one (m,n) such that $\mathsf{IL}(p(m)) = \mathsf{IL}(q(n)) < \omega$ for some p, q, the definition does not assume anything more specific about lines. The line-bunches can have various "gaps", and if a point (x,y) is not in a line-bunch H then it can still lie between two lines from H. The following proposition is easy to verify.

Proposition 25.

1. *There are only few lines, and thus also few line-bunches.*
   The set $\{(x,y) \in \mathbb{N} \times \mathbb{N} \mid (x,y) \in L_1 \cap L_2$ for two different lines $L_1, L_2\}$ is small.

2. *There are only few pairs $(p(m), q(n))$ where $\mathsf{IL}(p(m)) = \mathsf{IL}(q(n)) < \omega$ and (m, n) is not in a line.*

3.8 Eqlevel-decreasing line-climbing paths are short

We recall Fig. 4 which assumes a large gap $e_U - e_D$; to finish a proof of Theorem 14, we aim to show that all gaps in ZE are, in fact, small. In the next subsection (3.9) we show that a large gap $e_U - e_D$ would entail a long eqlevel-decreasing line-climbing path in $\mathcal{T}(\mathcal{A}) \times \mathcal{T}(\mathcal{A})$ (depicted in Fig. 9). In this subsection we show that all such paths are, in fact, short. Fig. 9 illustrates a line-climbing path from a pair projected to P_1 to a larger pair projected to P_2. The cyclicity and further structures in the figure will be discussed later.

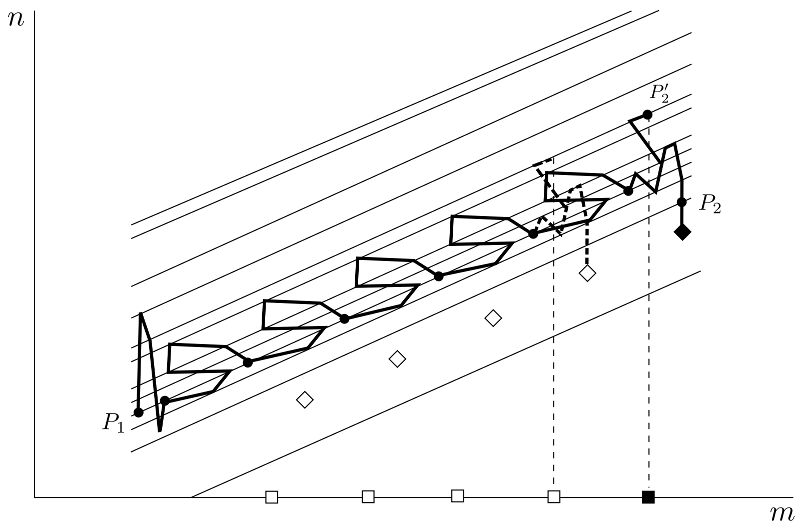


Figure 9: A line-climbing path (projections of all visited configuration-pairs are in IL-equality lines in one line-bunch)

Definition 26. *A* path in $\mathcal{T}(\mathcal{A}) \times \mathcal{T}(\mathcal{A})$ *is* positive *if each pair $(p(m), q(n))$ in the path satisfies $m > 0$, $n > 0$; this entails that there are no reset steps in the path.*

A positive path $(p_0(m_0), q_0(n_0)) \xrightarrow{a_1} (p_1(m_1), q_1(n_1)) \xrightarrow{a_2} \cdots \xrightarrow{a_z} (p_z(m_z), q_z(n_z))$ *is* line-climbing *if $m_0 < m_z$ and all (m_i, n_i), for $i = 0, 1, 2, \ldots, z$, are in one line-bunch.*

We do not require that (m_0, n_0) and (m_z, n_z) are in the same line, and we might have $n_z \leq n_0$; hence "line-climbing" might be understood as a shorthand for "(left-to-right) line-bunch climbing".

To get some intuition for what follows, imagine that Fig. 9 illustrates the projection of a "cyclic" line-climbing eqlevel-decreasing path from P_1 to P_2 which is followed by a simple step leading out of the respective line-bunch, namely to the black-diamond point. Cutting off the copies of the cycle in the path would give rise to the sequence of white-diamond points.

Fig. 9 also illustrates a similar path from P_1 to P_2' which is followed by another type of leaving the line-bunch, namely by a one-side reset step to the black-box point. Cutting off the copies of the cycle in the path would now give rise to the sequence of white-box points.

If the original path, including the line-bunch leaving step, is eqlevel-decreasing then the eqlevel of the "exit pair" (the black diamond or the black box) is less than the eqlevels of all "earlier exit pairs" (white diamonds or white boxes) (recall Observation 22(2)). The sequence of white-diamond (or white-box) points, finished by the black-diamond (or black-box) point, inspires the following definition.

Definition 27. *For $p, q \in Q_{\mathsf{St}}$, a* sequence *of pairs*

$$(p(m_0), q(n_0)), (p(m_1), q(n_1)), (p(m_2), q(n_2)), \ldots, (p(m_z), q(n_z))$$

where $z \geq 1$ is strange periodic *if the following conditions hold:*

1. $(m_i, n_i) = (m_0 + i \cdot c_1, n_0 + i \cdot c_2)$ *for some* $c_1, c_2 \in \mathbb{N}$ *and* $i = 0, 1, \ldots, z$*;*
2. $\mathsf{EqL}(p(m_i), q(n_i)) > \mathsf{EqL}(p(m_z), q(n_z))$ *for all* $i \in \{0, 1, \ldots, z{-}1\}$ *(hence* $c_1 > 0$ *or* $c_2 > 0$*);*
3. *the pairs* $(m_0, n_0), (m_1, n_1), \ldots, (m_z, n_z)$ *are not all in one* IL*-equality line.*

Prop. 25 implies that in any strange periodic sequence there are only few pairs $(p(m_i), q(n_i))$ such that $\mathsf{IL}(p(m_i)) = \mathsf{IL}(q(n_i)) < \omega$.

We now show that all strange periodic sequences are short, and then we derive that all line-climbing eqlevel-decreasing paths are short. (Fig. 9 suggests that such paths can be assumed to use a "cycle"; this will be established later by another use of Lemma 6.)

Proposition 28. *Strange periodic sequences are short.*

Proof. Let us assume a strange periodic sequence

$$(p(m_0), q(n_0)), (p(m_1), q(n_1)), (p(m_2), q(n_2)), \ldots, (p(m_z), q(n_z)) \tag{5}$$

as in Def. 27. Hence there are $c_1, c_2 \in \mathbb{N}$ such that $(m_i, n_i) = (m_0{+}i{\cdot}c_1, n_0{+}i{\cdot}c_2)$ for $i = 0, 1, \ldots, z$; moreover, $c_1 > 0$ or $c_2 > 0$, and the pairs in (5) are thus pairwise different.

For $i \in \{0, 1, \ldots, z\}$, by

$$(b_i, \ell_i, r_i, o_i, d_i^L, d_i^R) \text{ we denote the eqlevel tuple associated with } (p(m_i), q(n_i))$$

(recall Fig. 6 and Cor. 20). As we already noted, we have

$$\ell_i = r_i < \omega \text{ only for few } i \in \{0, 1, 2, \ldots, z\}. \tag{6}$$

We now explore certain "dense" periodic subsequences of (5). By a *periodic subsequence*, with the *period* $\mathsf{per} > 0$ and the *base* $b \geq 0$, we mean the sequence of pairs $(p(m_j), q(n_j))$ where j ranges over the *index set*

$$\mathcal{J} = \{z - x \cdot \mathsf{per}, z - (x{-}1) \cdot \mathsf{per}, z - (x{-}2) \cdot \mathsf{per}, \ldots, z - 2 \cdot \mathsf{per}, z - \mathsf{per}\}$$

for $x = (z{-}b) \div \mathsf{per}$. If both b and per are small (i.e., bounded by $\textsc{poly}(k)$ for a fixed polynomial $\textsc{poly}$ independent of the assumed doca $\mathcal{A}$ with k control states) then we say that this periodic *subsequence* is *dense*. We note that

if a dense subsequence is short then the whole sequence (5) is short (i.e., z is small).

By (2) in Def. 27 we have $b_i > b_z$ for all $i < z$, hence also $b_j > b_z$ for all $j \in \mathcal{J}$ where $\mathcal{J}$ is the index set of a periodic subsequence. Using Prop. 17(2), we now observe that there is a dense subsequence, with the index set $\mathcal{J}_1$, where $\ell_j \leq \ell_z$ for all $j \in \mathcal{J}_1$ (when $\ell_z < \omega$ and $c_1 > 0$ then we can even establish $\ell_j < \ell_z$). Similarly there is a dense subsequence, with the index set $\mathcal{J}_2$, where $r_j \leq r_z$ for all $j \in \mathcal{J}_2$. By using Prop. 10(1) we derive that there is also a dense subsequence, with the index set $\mathcal{J}_3$, where $o_j \leq o_z$ for all $j \in \mathcal{J}_3$. (Given d_1, d_2 guaranteed for p, q, m_z, n_z by Prop. 10(1), we can take $d_1 \cdot d_2$ as the period of the subsequence.)

Moreover, if $c_2 = 0$, and thus $q(n_i) = q(n_0)$ in all pairs in (5), then Prop. 11(1) implies that there is a dense subsequence, with the index set $\mathcal{J}_4$, where $d_j^R \leq d_z^R$ for all $j \in \mathcal{J}_4$.

We now perform a case analysis.

1. $c_1 > 0$, $c_2 = 0$ (the case $c_1 = 0$, $c_2 > 0$ is symmetric)

   Here we have $q(n_i) = q(n_0)$ in all pairs in (5). Considering the triangle $\{b_z, \ell_z, d_z^R\}$ (recall Fig. 6 and Cor. 20), we note that we must have $\ell_z \leq b_z < \omega$ or $d_z^R \leq b_z < \omega$. Hence there is a dense subsequence, indexed by $\mathcal{J}$, where $\ell_j \leq \ell_z \leq b_z < b_j$ for all $j \in \mathcal{J}$, or $d_j^R \leq d_z^R \leq b_z < b_j$ for all $j \in \mathcal{J}$. In both cases, Cor. 20 implies that $\ell_j = d_j^R < b_j$ for all $j \in \mathcal{J}$. Since each d_j^R belongs to the set $\{\, e \mid \mathsf{Mod}(p(m)) \xleftrightarrow{e} q(n_0) \text{ for some } m \,\}$, Prop. 11(2) implies that the set $\{d_j^R \mid j \in \mathcal{J}\} = \{\ell_j \mid j \in \mathcal{J}\}$ is small. Prop. 17(1) then implies that the set $\{p(m_0 + j \cdot c_1) \mid j \in \mathcal{J}\}$ is small; this implies that $\mathcal{J}$ is small and thus (5) is short.

2. $c_1 > 0$, $c_2 > 0$

   Looking at the rectangle $\{b_z, \ell_z, r_z, o_z\}$, we note that we have $\ell_z \leq b_z < \omega$ or $r_z \leq b_z < \omega$ or $o_z \leq b_z < \omega$. Hence there is a dense subsequence, indexed by $\mathcal{J}$, where $\ell_j \leq \ell_z \leq b_z < b_j$ for all $j \in \mathcal{J}$, or $r_j \leq r_z \leq b_z < b_j$ for all $j \in \mathcal{J}$, or $o_j \leq o_z \leq b_z < b_j$ for all $j \in \mathcal{J}$. In any case, Cor. 20 implies that for each $j \in \mathcal{J}$ we have $\ell_j = r_j < \omega$ or $\ell_j = o_j < \omega$ or $r_j = o_j < \omega$.

   We note that the set $\{(p(m_0 + j \cdot c_1), q(n_0 + j \cdot c_2)) \mid j \in \mathcal{J}, \ell_j = r_j < \omega\}$ is small by (6), and the set $\{(p(m_0 + j \cdot c_1), q(n_0 + j \cdot c_2)) \mid j \in \mathcal{J}, \ell_j = o_j < \omega \text{ or } r_j = o_j < \omega\}$ is small by Prop. 10(2) and Prop. 17(1). This implies that $\mathcal{J}$ is small and thus (5) is short.

□

Proposition 29. *Eqlevel-decreasing line-climbing paths are short.*

Proof. We consider an eqlevel-decreasing line-climbing path in a fixed line-bunch H, in the form

$$(p_0(m_0), q_0(n_0)) \xrightarrow{a_1} (p_1(m_1), q_1(n_1)) \xrightarrow{a_2} \cdots \xrightarrow{a_z} (p_z(m_z), q_z(n_z)) \tag{7}$$

as in Def. 26; we recall that the path is positive and $m_0 < m_z$. Moreover, we assume that (7) can not be prolonged by one step, by which we mean that one of the following conditions holds.

1. $\mathsf{EqL}(p_z(m_z), q_z(n_z)) = 0$.
2. Each eqlevel decreasing step $(p_z(m_z), q_z(n_z)) \xrightarrow{a} (p'(m'), q'(n'))$ is of one of the following types:
   (a) it is a (one-side or both-side) reset step,
   (b) it spoils the "one line-bunch property" ((m', n') is out of the line-bunch H),
   (c) $m_0 \geq m'$ (which entails $m_z = m_0{+}1$ and $m' = m_0$ when the step is simple).

E.g., $(p_0(m_0), q_0(n_0))$ might be projected to P_1 in Fig. 9; the projections P_2 and P_2' represent two possible end-pairs $(p_z(m_z), q_z(n_z))$ after which the line-bunch H is left by eqlevel decreasing steps.

We now note that the path (7) in $\mathcal{T}(\mathcal{A}) \times \mathcal{T}(\mathcal{A})$ can be alternatively presented as

$$((p_0, q_0, L_0), m_0) \xrightarrow{a_1} ((p_1, q_1, L_1), m_1) \xrightarrow{a_2} ((p_2, q_2, L_2), m_2) \xrightarrow{a_3} \cdots \xrightarrow{a_z} ((p_z, q_z, L_z), m_z) \tag{8}$$

where L_i denotes the (unique) IL-equality line in the line-bunch H which contains (m_i, n_i). This presentation looks like a path in $\mathcal{T}(\mathcal{B})$ for a doca $\mathcal{B}$ which has the triples (p, q, L) as the control states (where p, q are stable control states of $\mathcal{A}$ and L is a denotation of a line from the line-bunch H). We can think of such a doca $\mathcal{B}$ which has no reset control states and no zero rules and arises from $\mathcal{A}$ as follows:

> If $(p, a, 1, p', j_1)$ and $(q, a, 1, q', j_2)$ are (positive) rules of $\mathcal{A}$, where p', q' are stable, and L, L' are two lines from H defined by valid slope-shift pairs (μ, τ), (μ, τ'), respectively, and $j_2 - \mu \cdot j_1 = \tau' - \tau$
>
> then $((p, q, L), a, 1, (p', q', L'), j_1)$ is a (positive) rule of $\mathcal{B}$.

An equivalent formulation of the condition $j_2 - \mu \cdot j_1 = \tau' - \tau$ is to say that for all positive $m, n \in \mathbb{N}$ we have $(m, n) \in L$ iff $(m{+}j_1, n{+}j_2) \in L'$ (i.e., $n = \mu \cdot m + \tau$ iff $n{+}j_2 = \mu \cdot (m{+}j_1) + \tau'$).

For any tuple (p, q, L, a) there is obviously at most one tuple (p', q', L', j_1) such that $((p, q, L), a, 1, (p', q', L'), j_1)$ is a rule of $\mathcal{B}$; hence $\mathcal{B}$ is indeed a doca. The size of $\mathcal{B}$ (in particular the number of control states of $\mathcal{B}$) is small since the number of lines in H is small (recall Prop. 25(1)).

It is clear that any positive path in $\mathcal{T}(\mathcal{A}) \times \mathcal{T}(\mathcal{A})$ which visits only the pairs projected to the line-bunch H corresponds to a path in $\mathcal{T}(\mathcal{B})$; the paths (7) and (8) illustrate this correspondence.

By Observation 22(1), the path (7) is a shortest path from $(p_0(m_0), q_0(n_0))$ to $(p_z(m_z), q_z(n_z))$ in $\mathcal{T}(\mathcal{A}) \times \mathcal{T}(\mathcal{A})$. By Lemma 6, a shortest path from $((p_0, q_0, L_0), m_0)$ to $((p_z, q_z, L_z), m_z)$ in $\mathcal{T}(\mathcal{B})$

is of the form $((p_0,q_0,L_0),m_0) \xrightarrow{w} ((p_z,q_z,L_z),m_z)$ where $w = u_1 v^i u_2$ for some short u_1, v, u_2 (short w.r.t. the size of $\mathcal{B}$ which is small) and some $i \geq 0$; moreover, we can assume that the effect (the counter change) of the respective control state cycle $((p,q,L),..) \xrightarrow{v} ((p,q,L),..)$ is positive (since $m_0 < m_z$).

There is a slight problem that the path $((p_0,q_0,L_0),m_0) \xrightarrow{w} ((p_z,q_z,L_z),m_z)$ in $\mathcal{T}(\mathcal{B})$ might not correspond to a positive path from $(p_0(m_0), q_0(n_0))$ to $(p_z(m_z), q_z(n_z))$ in $\mathcal{T}(\mathcal{A}) \times \mathcal{T}(\mathcal{A})$ since $\mathcal{B}$ can go through a configuration $((p,q,L),m)$ where (μ,τ) is the slope-shift pair of L and $\mu{\cdot}m{+}\tau \leq 0$. Nevertheless u_1, v, u_2 are short, and this problem thus cannot arise when n_0 is larger than a small bound b. For showing that the path (7) is short, it suffices to show that its suffix starting in the first $(p_j(m_j), q_j(n_j))$ where n_j exceeds b is short. (The prefix before such $(p_j(m_j), q_j(n_j))$ is obviously short.)

We thus immediately assume that n_0 is larger than b, which then allows us to assume that $a_1 a_2 \ldots a_z$ in (7) is $w = u_1 v^i u_2$, as deduced from $\mathcal{T}(\mathcal{B})$. We now perform a case analysis.

1. $m_z = m_0{+}1$

   By applying Cor. 7 to the doca $\mathcal{B}$, we deduce that (7) is short.

2. $\mathsf{EqL}(p_z(m_z), q_z(n_z)) = 0$

   Path (7) is short since $i \leq |u_1| + |u_2| + |v|$. Otherwise by cutting off a copy of the cycle v, i.e. by performing $u_1 v^{i-1} u_2$ from $(p_0(m_0), q_0(n_0))$, we would reach $(p_z(m_z{-}d_1), q_z(n_z{-}d_2))$ where d_1 is the effect of the cycle $((p,q,L),..) \xrightarrow{v} ((p,q,L),..)$ and $d_2 = \mu \cdot d_1$ for the slope μ of L (i.e. of the line-bunch H). We would thus reach a pair with the zero eqlevel earlier (contradicting Observation 22(2)).

3. There is an eqlevel-decreasing both-side reset step $(p_z(m_z), q_z(n_z)) \xrightarrow{a} (p'(0), q'(0))$ where $p_z(m_z) \overset{a}{\mapsto} s(m) \overset{\varepsilon}{\mapsto} p'(0)$, $q_z(n_z) \overset{a}{\mapsto} s'(n) \overset{\varepsilon}{\mapsto} q'(0)$.

   Now $i \leq |u_1 u_2 v| + \mathsf{per}_s \cdot \mathsf{per}_{s'}$, since otherwise by cutting off $\mathsf{per}_s \cdot \mathsf{per}_{s'}$ copies of v we would reach $(p'(0), q'(0))$ earlier. Hence (7) is short in this case as well.

4. There is an eqlevel decreasing simple step $(p_z(m_z), q_z(n_z)) \xrightarrow{a} (p'(m'), q'(n'))$ (as from P_2 in Fig. 9).

   Then ("the diamond points in Fig. 9", i.e.) the sequence of pairs $(p'(m'_j), q'(n'_j))$ where

   $$(p_0(m_0), q_0(n_0)) \xrightarrow{u_1 v^j u_2 a} (p'(m'_j), q'(n'_j))$$

   and j ranges over $|u_1 u_2 v|, |u_1 u_2 v| + 1, |u_1 u_2 v| + 2, \ldots, i{-}1, i$ is obviously a strange periodic sequence (by recalling Observation 22(2)). Since this sequence is short (by Prop. 28), also (7) is short.

5. There is an eqlevel decreasing one-side reset step $(p_z(m_z), q_z(n_z)) \xrightarrow{a} (p'(m'), q'(0))$ (as from P'_2 in Fig. 9); we assume $q(n_z) \overset{a}{\mapsto} s(n) \overset{\varepsilon}{\mapsto} q'(0)$.

   Then ("a subsequence of box points in Fig. 9", namely) the sequence of pairs $(p'(m'_j), q'(0))$ where

   $$(p_0(m_0), q_0(n_0)) \xrightarrow{u_1 v^j u_2 a} (p'(m'_j), q'(0))$$

   and j ranges over $i - x \cdot \mathsf{per}_s, i - (x{-}1) \cdot \mathsf{per}_s, i - (x{-}2) \cdot \mathsf{per}_s, \ldots, i - 2 \cdot \mathsf{per}_s, i - \mathsf{per}_s, i$ where $x = (i{-}|u_1 u_2 v|) \div \mathsf{per}_s$ is obviously a strange periodic sequence. Since this sequence is short (by Prop. 28), also (7) is short.

□

3.9 Gaps in ZE are small

Assuming a doca $\mathcal{A}$, with the associated det-LTS $\mathcal{T}_{\mathsf{ext}}(\mathcal{A})$, by Def. 12 we have

$$\text{ZE} = \{e \in \mathbb{N} \mid \text{there are two stable zero configurations } C, C' \text{ in } \mathcal{T}_{\mathsf{ext}}(\mathcal{A}) \text{ s.t. } C \overset{e}{\longleftrightarrow} C'\}.$$

We assumed $0 \in \text{ZE}$ and we fixed an ordering $e_0 < e_1 < \cdots < e_f$ of ZE. We finally aim to contradict the existence of a large gap between $e_i = e_D$ and $e_{i+1} = e_U$ for some $i, 0 \leq i < f$ (recall Fig. 4); this will finish a proof of Theorem 14.

Before proving Lemma 31, we sketch the idea informally, using Fig. 10. Let us consider an eqlevel-decreasing path in $\mathcal{T}_{\mathsf{ext}}(\mathcal{A}) \times \mathcal{T}_{\mathsf{ext}}(\mathcal{A})$, like (9) below, which starts from a pair (C_0, C'_0) of stable zero configurations satisfying $\mathsf{EqL}(C_0, C'_0) = e_U$; let (C_j, C'_j) be the pair visited by our path after j steps. If both C_0, C'_0 are in Q_{Mod} (recall that $Q_{\mathsf{Mod}} = \{\mathsf{Mod}(p(m)) \mid p \in Q_{\mathsf{St}}, m \geq 0\}$) then also C_1, C'_1 are stable zero configurations (maybe in $\mathcal{T}(\mathcal{A})$), and thus $e_D = e_U - 1$; the gap is really small in this case. We thus further assume $C_0 \notin Q_{\mathsf{Mod}}$ (hence $C_0 = p(0)$ is in $\mathcal{T}(\mathcal{A})$); this also handles the case $C'_0 \notin Q_{\mathsf{Mod}}$ by symmetry.

We are now not primarily interested in studying how the concrete pairs (C_j, C'_j) can look like; we are interested in the tuples $(b_j, \ell_j, r_j, o_j, d^L_j, d^R_j)$ associated with (C_j, C'_j) by Def. 18 (recall Fig. 6). The dependence of this tuple on j is partly sketched in Fig. 10.

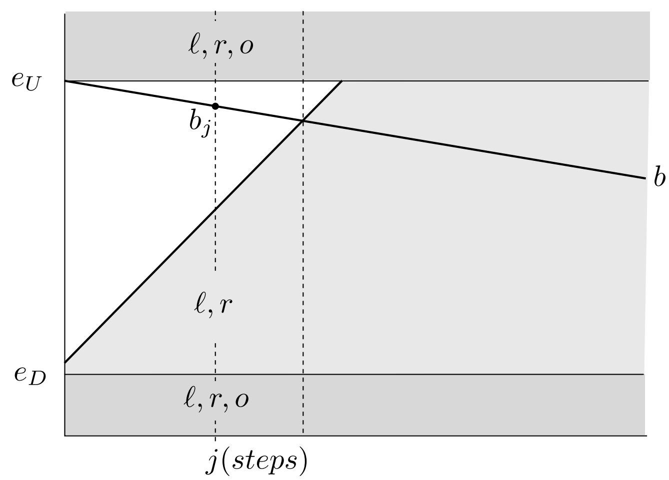


Figure 10: Constraints on b_j, ℓ_j, r_j, o_j after j steps of an eqlevel-decreasing path with $b_0 = e_U$

Since our path is eqlevel-decreasing (the eqlevel drops by 1 in each step), we know that $b_j = e_U - j$, which is depicted by a line (in the standard sense, having nothing to do with IL-equality lines) starting in point $(0, e_U)$ and having the slope -1. (For a better overall appearence, the vertical unit length in Fig. 10 is smaller than the horizontal one.)

Each o_j is either ω or an element of ZE (of E_3 after Def. 12); in particular, $o_j \geq e_U$ or $o_j \leq e_D$, which is depicted as a constraint in Fig. 10, using the horizontal lines at levels e_U and e_D.

We now recall Prop. 16 and the fact that each finite $\mathsf{IL}(q(0))$ is in ZE (in E_2 after Def. 12). Hence for each ℓ_j we have either $\ell_j \geq e_U$ or $\ell_j \leq e_D + \rho_{\text{M}} \cdot j + \sigma_{\text{M}}$ where ρ_{M} is the maximal number appearing as ρ in the fixed equalities $\mathsf{IL}(p(m)) = \rho \cdot m + \sigma + e$, and σ_{M} is the maximal number appearing there as σ. (We use the fact that the counter value is at most j in C_j, as well as in C'_j when C'_j is also in $\mathcal{T}(\mathcal{A})$, since we started from zero configurations.) We recall that both ρ_{M} and σ_{M} are small rational numbers. The above constraints on ℓ_j are also depicted in Fig. 10, using the horizontal line at level e_U and the line starting in $(0, e_D{+}\sigma_{\text{M}})$ and having the slope ρ_{M}. The same constraints hold for r_j.

We note that if the horizontal coordinate of the intersection of the "b-line" (with slope -1) and the "ℓ, r-line" (with the slope ρ_{M}) is small then $e_U - e_D$ is small. This is clear by noting that $b_j = e_U{-}j \leq e_D + \rho_{\text{M}} \cdot j + \sigma_{\text{M}}$ implies $e_U{-}e_D \leq (1{+}\rho_{\text{M}}) \cdot j + \sigma_{\text{M}}$.

In fact, we will show even something stronger, namely that the maximal prefix of our path in which b_j (for $j > 0$) is "solitary", i.e. $b_j \notin \{\ell_j, r_j, o_j\}$, is short. This will be based on Cor. 20, applied to the "rectangle" (b_j, ℓ_j, r_j, o_j). The previously established facts, like that about few possible values o_j, will entail that in a long b-solitary prefix we would "usually" have $\ell_j = r_j < \omega$, which in turn would entail a long line-climbing segment; this would contradict Prop. 29.

Definition 30. *A pair (C, C') of stable configurations in $\mathcal{T}_{\mathsf{ext}}(\mathcal{A})$ with the associated eqlevel tuple $(b, \ell, r, o, d^L, d^R)$ is* b-solitary *if $b \notin \{\ell, r, o\}$.*
A path in $\mathcal{T}_{\mathsf{ext}}(\mathcal{A}) \times \mathcal{T}_{\mathsf{ext}}(\mathcal{A})$ is b-solitary *if each configuration-pair in the path is b-solitary.*

We note that in a b-solitary pair (C, C') we must have that at least C is in $\mathcal{T}(\mathcal{A})$, by our choice in Def. 18.

Lemma 31. *All gaps $e_U - e_D$ in* ZE *are small.*

Proof. We assume some $e_D, e_U \in \mathrm{ZE}$ where $e_D < e_U$ and there is no $e \in \mathrm{ZE}$ such that $e_D < e < e_U$, and consider an eqlevel-decreasing path

$$(C_0, C'_0) \xrightarrow{a_1} (C_1, C'_1) \xrightarrow{a_2} (C_2, C'_2) \xrightarrow{a_3} \cdots \xrightarrow{a_z} (C_z, C'_z) \tag{9}$$

in $\mathcal{T}_{\mathsf{ext}}(\mathcal{A}) \times \mathcal{T}_{\mathsf{ext}}(\mathcal{A})$ where C_0, C'_0 are stable zero configurations, $C_0 \xleftrightarrow{e_U} C'_0$, and $C_z \xleftrightarrow{0} C'_z$. We thus have $\mathsf{EqL}(C_j, C'_j) = e_U - j$ for all $j \in \{0, 1, \ldots, z\}$.

Our aim is to show that $e_U - e_D$ is small. If $C_0 \in Q_{\mathsf{Mod}}$ and $C'_0 \in Q_{\mathsf{Mod}}$ then C_1, C'_1 are also (stable) zero configurations, and thus $\mathsf{EqL}(C_1, C'_1) = e_U - 1 \in \mathrm{ZE}$; we thus have $e_U - e_D = 1$.

We thus further assume that $C_0 \notin Q_{\mathsf{Mod}}$ (while $C'_0 \notin Q_{\mathsf{Mod}}$ is handled by symmetry). Let $(b_i, \ell_i, r_i, o_i, d_i^L, d_i^R)$ be the eqlevel tuple associated with (C_i, C'_i) $(i = 0, 1, \ldots, z)$, as in Def. 18; in the case $C'_i \in Q_{\mathsf{Mod}}$ we thus have $r_i = \omega$, $b_i = d_i^L$, $o_i = d_i^R$.

We now note that if there is some small $j > 0$ such that (C_j, C'_j) is not b-solitary then $e_U - e_D$ is small. This follows from the following two facts.

1. If $b_j = o_j$ then $b_j = e_U - j \le e_D$ (since o_i belongs to ZE for all i); hence $e_U - e_D \le j$.
2. If $b_j = \ell_j$ or $b_j = r_j$ then $b_j = e_U - j \le e_D + \rho_{\mathrm{M}} \cdot j + \sigma_{\mathrm{M}}$, and thus $e_U - e_D \le (1 + \rho_{\mathrm{M}}) \cdot j + \sigma_{\mathrm{M}}$, as was already discussed before Def. 30.

We now fix j so that

$$(C_1, C'_1) \xrightarrow{a_2} (C_2, C'_2) \xrightarrow{a_3} \cdots \xrightarrow{a_j} (C_j, C'_j) \tag{10}$$

is the maximal b-solitary prefix of the path (9) in which the first step is removed. We will show that j is small, by which the proof will be finished; we further assume $j \ge 1$.

The assumption $C_0 \notin Q_{\mathsf{Mod}}$ implies $C_1 \notin Q_{\mathsf{Mod}}$ (hence $C_1 = p(m)$ for some $p \in Q_{\mathsf{St}}$ and some $m \in \{0, 1\}$). Suppose $(C_1, C'_1) \xrightarrow{a_2} (C_2, C'_2) \xrightarrow{a_3} \cdots \xrightarrow{a_{j_1}} (C_{j_1}, C'_{j_1})$ is the maximal prefix of (10) such that $C'_{j_1} \in Q_{\mathsf{Mod}}$; we put $j_1 = 0$ if $C'_1 \notin Q_{\mathsf{Mod}}$. For all $i \in \{1, 2, \ldots j_1\}$ we have $b_i = e_U - i$, $r_i = \omega$, and $b_i \notin \{\ell_i, r_i, o_i\}$; hence $\ell_i = o_i < b_i$ (by Cor. 20). By Prop. 10(2) and Prop. 17(1), the set $\{C_i \mid 1 \le i \le j_1\}$ is small, which implies that the set $\{b_i \mid 1 \le i \le j_1\} = \{d_i^L \mid 1 \le i \le j_1\}$ is small, by Prop. 11(2). Since $b_{i_1} \neq b_{i_2}$ if $i_1 \neq i_2$, we get that j_1 is small. It is thus sufficient to show that the suffix

$$(C_{j_1+1}, C'_{j_1+1}) \xrightarrow{a_{j_1+2}} (C_{j_1+2}, C'_{j_1+2}) \xrightarrow{a_{j_1+3}} \cdots \xrightarrow{a_j} (C_j, C'_j) \tag{11}$$

of (10) is short. Let us rewrite (11) as

$$(p_0(m_0), q_0(n_0)) \xrightarrow{a'_1} (p_1(m_1), q_1(n_1)) \xrightarrow{a'_2} \cdots \xrightarrow{a'_{j'}} (p_{j'}(m_{j'}), q_{j'}(n_{j'})) \tag{12}$$

where $j' = j - (j_1 + 1)$, and $(p_i(m_i), q_i(n_i)) = (C_{j_1+1+i}, C'_{j_1+1+i})$, $a'_i = a_{j_1+1+i}$ for $i = 0, 1, \ldots, j'$. We note that $m_0 + n_0$ is small (since j_1 is small and C_0, C'_0 are zero configurations).

For simplicity, by $(b_i, \ell_i, r_i, o_i, d_i^L, d_i^R)$, where $0 \le i \le j'$, we further denote the eqlevel tuple associated with $(p_i(m_i), q_i(n_i))$ (not with (C_i, C'_i) anymore). Since the path (12) is eqlevel-decreasing, there is *no repeat*, i.e. $(p_{i_1}(m_{i_1}), (q_{i_1}(n_{i_1})) \neq (p_{i_2}(m_{i_2}), (q_{i_2}(n_{i_2}))$ if $i_1 \neq i_2$.

For each $i \in \{0, 1, \ldots, j'\}$, the pair $(p_i(m_i), q_i(n_i))$ is b-solitary, and thus

$$\min\{b_i, \ell_i, r_i, o_i\} \text{ is } r_i = o_i \text{ or } \ell_i = o_i \text{ or } \ell_i = r_i.$$

We now aim to show that

$$\text{there are only few } i \in \{0, 1, \ldots, j'\} \text{ for which we do not have } \ell_i = r_i < \omega. \tag{13}$$

To establish (13), it suffices to show that the sets $\{i \mid 0 \leq i \leq j', r_i = o_i < \omega\}$ and $\{i \mid 0 \leq i \leq j', \ell_i = o_i < \omega\}$ are small; by symmetry it suffices just to show that the former set is small.

We first note that the set

$$\{q_i(n_i) \mid 0 \leq i \leq j', r_i = o_i < \omega\}$$

is small by Prop. 10(2) and 17(1). Hence also the set

$$\{d_i^R \mid 0 \leq i \leq j', r_i = o_i < \omega\}$$

is small, by Prop. 11(2). The set

$$\{i \mid 0 \leq i \leq j', r_i = o_i < \omega, \min\{b_i, \ell_i, d_i^R\} = b_i = d_i^R\}$$

is thus also small (recall that $b_{i_1} \neq b_{i_2}$ if $i_1 \neq i_2$). The set

$$\{p_i(m_i) \mid 0 \leq i \leq j', r_i = o_i < \omega, \min\{b_i, \ell_i, d_i^R\} = \ell_i = d_i^R\}$$

is also small, by recalling Prop. 17(1). Since $\min\{b_i, \ell_i, d_i^R\}$ is $\ell_i = d_i^R$ or $b_i = d_i^R$ for all $i \in \{0, 1 \ldots, j'\}$ (recall that $b_i = \ell_i$ is excluded in b-solitary pairs), we get that both sets

$$\{q_i(n_i) \mid 0 \leq i \leq j', r_i = o_i < \omega\} \text{ and } \{p_i(m_i) \mid 0 \leq i \leq j', r_i = o_i < \omega\}$$

are small. Since there is no repeat in (12), we get that the set $\{i \mid 0 \leq i \leq j', r_i = o_i < \omega\}$ is small. We have thus established (13).

Let us now consider the *sum-increasing subsequence*

$$(p_{i_0}(m_{i_0}), q_{i_0}(n_{i_0})), (p_{i_1}(m_{i_1}), q_{i_1}(n_{i_1})), (p_{i_2}(m_{i_2}), q_{i_2}(n_{i_2})), \ldots \tag{14}$$

of the sequence of pairs in (12), where $0 = i_0 < i_1 < i_2 < \cdots$, and i_{h+1} is the first such that $m_{i_{h+1}} + n_{i_{h+1}}$ is bigger than $m_{i_h} + n_{i_h}$ (for $h = 0, 1, 2, \ldots$). If this subsequence is short then (12) is obviously short since we started with small $m_0 + n_0$ and $m_{i_{h+1}} + n_{i_{h+1}} \leq m_{i_h} + n_{i_h} + 2$ (and there is no repeat in (12)).

For $h = 0, 1, 2\ldots$ we now consider the subpaths of (12) starting in $(p_{i_h}(m_{i_h}), q_{i_h}(n_{i_h}))$ and finishing in $(p_{i_{h+1}}(m_{i_{h+1}}), q_{i_{h+1}}(n_{i_{h+1}}))$; we call them *segments*. A *segment* is called *unusual* if

- the segment visits a pair $(p(m), q(n))$ such that (m, n) is in no line-bunch, or is in the intersection of two different line-bunches, or satisfies $m = 0$ or $n = 0$, or
- the segment contains a step $(p(m), q(n)) \xrightarrow{a} (p'(m+j_1), q'(n+j_2))$ such that (m, n) and $(m+j_1, n+j_2)$ are in two different line-bunches.

Using (13) and Prop. 25 and the no-repeat property, we can easily verify that there are only few unusual segments.

Any other segment, called *usual*, is thus a positive path projected to one line-bunch; moreover, the concatenation of consecutive usual segments is also projected to one line-bunch. We note that if $(p_{i_h}(m_{i_h}), q_{i_h}(n_{i_h}))$ and $(p_{i_{h'}}(m_{i_{h'}}), q_{i_{h'}}(n_{i_{h'}}))$, for $h < h'$, are in the same line then $m_{i_h} < m_{i_{h'}}$. Since there are only few lines, less than some small b_1, and the lengths of eqlevel-decreasing line-climbing paths are less than some small b_2 by Prop. 29, we cannot have more than $\mathsf{b}_1 \cdot \mathsf{b}_2$ consecutive usual segments. This finally implies that (14) is short, and thus also (12) is short. Hence $e_U - e_D$ is small. $\square$

Now Lemma 13 and Lemma 31 give a proof of Theorem 14, and thus also of Theorem 2.

4 Additional remarks

The notions and their properties from the main proof also help to answer related questions. Here we only mention *regularity*. It is straightforward to verify that the language (the set of enabled traces) of a doca configuration $p(m)$ is non-regular iff we have $p(m) \xrightarrow{u} q_1(n) \xrightarrow{v} q_2(n{+}k) \xrightarrow{w} q'(0)$ where $q_1(n) \xrightarrow{vw} q'(0)$ is a positive path and $\mathsf{IL}(q'(0)) < \omega$. (In this case, from $p(m)$ we can reach $q(n')$) for some q and infinitely many n' where $\mathsf{IL}(q(n')) < \omega$.) It is then a routine (though a bit technical) to show that the regularity problem for doca is in NL (and NL-complete) as well.

Appendix (classical doca equivalence)

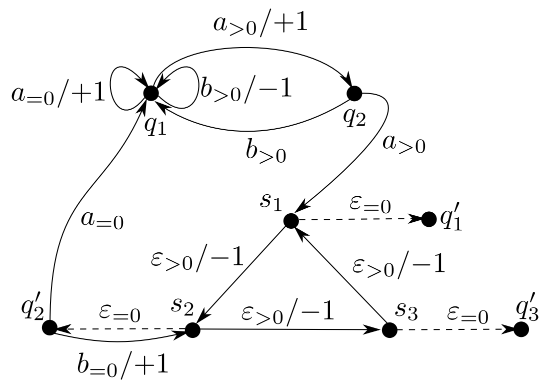


Figure 11: A classical doca

The aim of this Appendix is to sketch the ideas of a routine reduction of the standard doca language equivalence problem to our $\mathsf{Doca\text{-}Eq}$. A classical definition would define a *doca* as a tuple $\mathcal{A} = (Q, \Sigma, \delta, q_0, F)$ where Q is a finite set of *control states*, Σ is a finite *alphabet*, $\delta \subseteq Q \times (\Sigma \cup \{\varepsilon\}) \times \{0, 1\} \times Q \times \{-1, 0, 1\}$ is a *transition relation* satisfying the below given two conditions, $q_0 \in Q$ is the *initial state*, and $F \subseteq Q$ is the set of *accepting states*.

In this context, $\varepsilon \notin \Sigma$ is handled as a special symbol but it plays the role of the empty word in the semantics. The conditions for δ are the following.

1. For each triple (p, a, c), where $p \in Q$, $a \in \Sigma \cup \{\varepsilon\}$, $c \in \{0, 1\}$ there is at most one pair (q, j) such that $(p, a, c, q, j) \in \delta$; moreover, $j \neq -1$ if $c = 0$.

2. If $(p, \varepsilon, c, q, j) \in \delta$ then there are no $a \in \Sigma$, $q' \in Q$, $j' \in \{-1, 0, 1\}$ such that $(p, a, c, q', j') \in \delta$.

A *configuration* of $\mathcal{A}$ is a pair $(p, n) \in Q \times \mathbb{N}$; we write $p(n)$ instead of (p, n), as previously. We now define relations $\xrightarrow{w}$, $w \in \Sigma^*$, on $Q \times \mathbb{N}$ inductively as follows: $p(n) \xrightarrow{\varepsilon} p(n)$; if $(p, a, \mathrm{sgn}(n), q, j) \in \delta$ (where $a \in \Sigma \cup \{\varepsilon\}$) then $p(n) \xrightarrow{a} q(n{+}j)$ (here $\mathrm{sgn}(n) = 1$ if $n > 0$ and $\mathrm{sgn}(n) = 0$ if $n = 0$); if $p(n) \xrightarrow{u} p'(n')$ and $p'(n') \xrightarrow{v} p''(n'')$ then $p(n) \xrightarrow{uv} p''(n'')$. Since the symbol ε is handled as the empty word, we have $\varepsilon u = u\varepsilon = u$. We define the *language accepted by* $\mathcal{A}$ as

$$L(\mathcal{A}) = \{w \in \Sigma^* \mid q_0(0) \xrightarrow{w} q(n) \text{ for some } q \in F,\ n \in \mathbb{N}\}.$$

The *language equivalence problem* asks, given two doca $\mathcal{A}_1$, $\mathcal{A}_2$ if $L(\mathcal{A}_1) = L(\mathcal{A}_2)$.

We now sketch the ideas of reducing this problem to our problem $\mathsf{Doca\text{-}Eq}$. First we note that we can take the disjoint union $\mathcal{A}$ of $\mathcal{A}_1, \mathcal{A}_2$ and ask about the equality of languages of two different (initial) configurations. The doca $\mathcal{A}$, with k control states, can be routinely replaced by a doca $\mathcal{A}_{\mathsf{SC}}$ (with the "Shrinked Counter"), where a configuration $p(m)$ of $\mathcal{A}$ is represented by the configuration $p_i(j)$ of $\mathcal{A}_{\mathsf{SC}}$ where $i = m \bmod k$ and $j = (m \div k)$. The control state set of $\mathcal{A}_{\mathsf{SC}}$ is k-times bigger, to pay for shrinking the counter.

It is then easy to get rid of ε-rules which are not in ε-cycles, and to get rid of ε-cycles with nonnegative effects. Finally, the only ε-rules which remain are popping (decrementing the counter), and they are in cycles, which is exemplified by the states s_1, s_2, s_3 in Fig. 11. To each such state s in an ε-cycle we can add a control state q_s with the zero rule $(s, \varepsilon, 0, q_s, 0)$, to clearly separate the "reset control states" from the "stable ones"; this is illustrated by q'_1, q'_2, q'_3 in Fig. 11. The final step of the transformation to our reset-form doca (as in Fig. 1) is now obvious. In the example, all s_1, s_2, s_3 get the period 3, and we put $\mathsf{goto}_{s_2}(2) = q'_1$, $\mathsf{goto}_{s_3}(0) = q'_3$, etc. (In fact, using s_1 is sufficient in our special case since the non-ε incoming arcs of s_2, s_3 correspond to zero rules only.)

Trace equivalence coincides with language equivalence when all states are declared as accepting. A reduction from language equivalence to trace equivalence can be sketched as follows. For any triple (q, a, c) such that $q \in Q_{\mathsf{St}}$, $a \in \Sigma$, $c \in \{0, 1\}$ and there is no $(q, a, c, q', j) \in \delta$ we add the rule $(q, a, c, q_{\mathsf{sink}}, 0)$ where q_{sink} is an added "sink loop" state, with rules $(q_{\mathsf{sink}}, a, c, q_{\mathsf{sink}}, 0)$ for all $a \in \Sigma$ and $c \in \{0, 1\}$. We assume having arranged that all accepting control states are stable, and we now add the "loop" rules $(q, a_{\mathsf{acc}}, c, q, 0)$ for a special fresh letter a_{acc} and all $q \in F$, $c \in \{0, 1\}$ (so that $\mathsf{EqL}(p(m), q(n)) = 0$ when $p \in F$, $q \notin F$ or vice versa).

Since the reduction keeps the lengths of non-equivalence witnesses polynomially related, the analogues of Theorems 2 and 3 hold for the classical equivalence problem as well.

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