Applications of the First-order Integer Arithmetic to the Verification of Programs with Lists

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First Order Arithmetic of Natural Numbers

- $\langle \mathbb{N}, +, \cdot, 0, 1 \rangle$ is undecidable [Church '36]

- $\langle \mathbb{N}, +, |, 0, 1 \rangle$ is a subtheory of $\langle \mathbb{N}, +, \cdot, 0, 1 \rangle$: $f|g \leftrightarrow \exists x \; f \cdot x = g$
  - shown to be undecidable [Robinson '49]

- $\langle \mathbb{N}, +, |, 0, 1 \rangle^{\exists}$ is the *existential* fragment of $\langle \mathbb{N}, +, |, 0, 1 \rangle$
  - shown to be decidable [Lipshitz '78]

- $\langle \mathbb{N}, +, 0, 1 \rangle$ is decidable [Presburger '29]

Goal: better understand the boundary between decidable and undecidable subtheories of FO arithmetic.

Applications of FO Arithmetic of $\mathbb{N}$

Automated verification of systems with infinite state spaces:

- Automata with Integer Counters
  - reachability problems of counter automata with flat control structures and affine transition relations is definable in PA [Comon, Yurski '98]
  - reachability problems of counter automata with flat control structures and parameterized transition relations can be defined using more expressive decidable theories [Bozga, I '06]

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- Programs with Dynamic Linked Data Structures
  - verification of properties relative to the sizes of data structures
  - programs with lists [Bouajjani, Bozga, Habermehl, I, Moro, Vojnar '06]
  - programs with trees [Habermehl, I, Vojnar '06]
  - recent work on defining decidable classes of programs with lists

Outline

- Decidability within the Theory of Addition and Divisibility
  - Least Common Multiple Fragment
  - Left-Divides Fragment
- Reachability Problems for Programs with Lists
  - n-Cyclic Heaps, $n > 1$
  - Acyclic and 1-Cyclic Heaps
- Conclusions

Proving undecidability of $\langle \mathbb{N}, +, |, 0, 1 \rangle$

Hilbert's 10^{th} Problem:

10. **Determination of the solvability of a Diophantine equation.** Given a diophantine equation with any number of unknown quantities and with rational integral numerical coefficients: *To devise a process according to which it can be determined by a finite number of operations whether the equation is solvable in rational integers.*

Proving undecidability of $\langle \mathbb{N}, +, |, 0, 1 \rangle$

- the least common multiple relation is definable in $\langle \mathbb{N}, +, |, 0, 1 \rangle$:

$$[x, y] = z \iff \forall t \; x|t \wedge y|t \leftrightarrow z|t$$

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- the perfect square relation is definable in $\langle \mathbb{N}, +, |, 0, 1 \rangle$:

$$x^2 = y \iff y + x = [x, x + 1]$$

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- we have reduced Hilbert's 10^{th} Problem to satisfiability of $\langle \mathbb{N}, +, |, 0, 1 \rangle$
  - $\langle \mathbb{N}, +, [.], 0, 1 \rangle$ is undecidable
  - $\langle \mathbb{N}, +, |, 0, 1 \rangle$ with one universally quantified variable is undecidable

The Left-Divides Fragment of $\langle \mathbb{N}, +, |, 0, 1 \rangle$

Syntactic fragment of formulas with following *prenex form*:

$$\mathcal{L}_{|}^{(n)}\colon \mathbf{Q_1 z_1 \ldots Q_n z_n R_1 x_1 \ldots R_m x_m} \varphi(\vec{x}, \vec{z})$$

where each divisibility predicate is of the form:

$$f(\vec{z}) | g(\vec{x}, \vec{z})$$

and there are no implicit dependecies between $\vec{z}$ and $\vec{x}$.

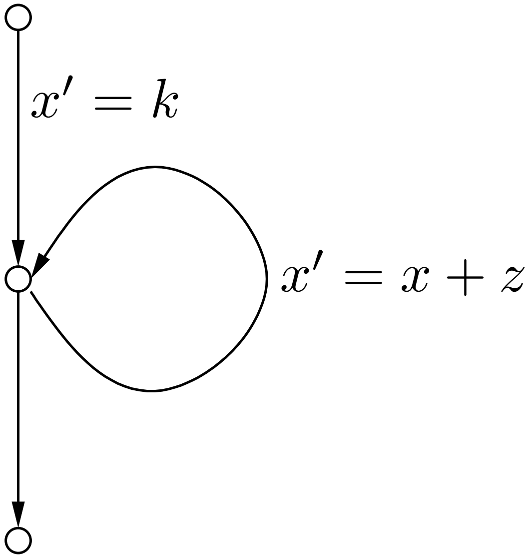
Good subject for research:

- is undecidable but has interesting decidable sub-fragments
- $\mathcal{L}_{|}^{(1)}$ is decidable

The Left-Divides Fragment of $\langle \mathbb{N}, +, |, 0, 1 \rangle$

Consider the following counter machine:

- x is the working variable
- z is system parameter
- $k \in \mathbb{Z}$ is a constant



$$\exists n \geq 0 \; [x = nz + k] \iff z | x - k \wedge (z \geq 0 \wedge x \geq k \;\vee\; z \leq 0 \wedge x \leq k)$$

Tools for Proving Decidability

The Chinese Remainder Theorem

We use a generalized version of the CRT:

$$\exists x \bigwedge_{i=1}^{K} m_i | (a_i x - r_i) \iff \bigwedge_{1 \leq i,j \leq K} (a_i m_j, a_j m_i) | (a_i r_j - a_j r_i) \ \wedge \ \bigwedge_{i=1}^{K} (a_i, m_i) | r_i$$

where (A, B) denotes the greatest common divisor of A and B

CRT eliminates existential quantifiers introducing g.c.d. subterms.

Taming the Greatest Common Divisor

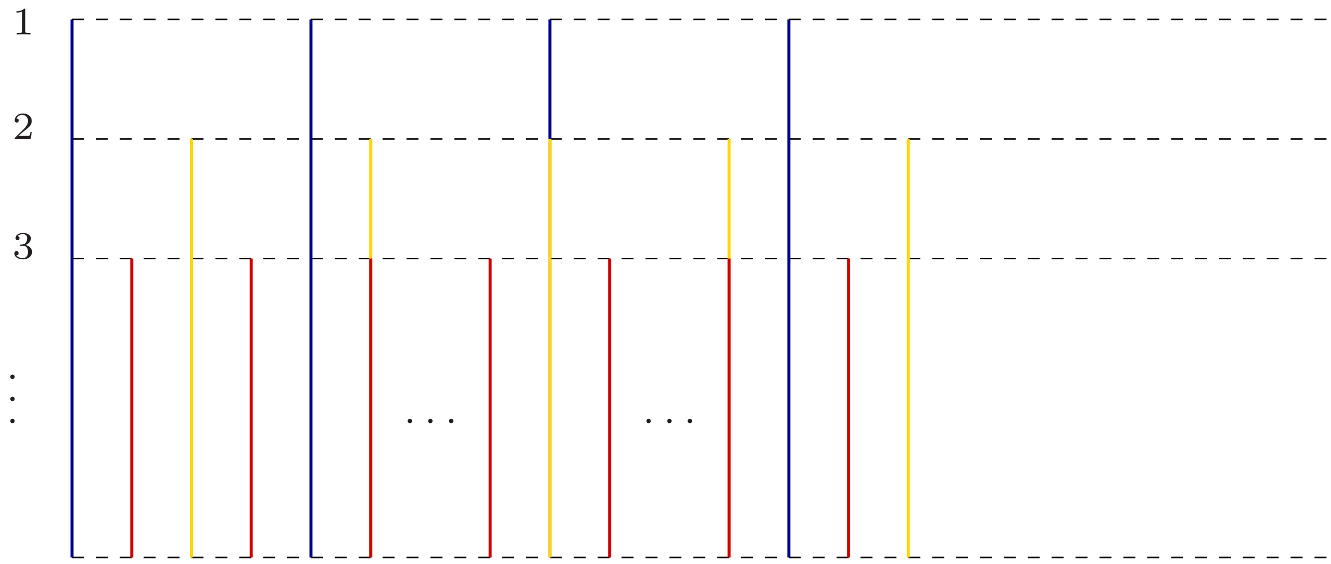
$$A(x) = ax + b, B(x) = cx + d$$

$$\exists p, q, r \in \mathbb{Z}\ \forall x\ .\ pA(x) = qB(x) + r$$

$$\forall x \Big[\big(A(x), B(x) \big) = \big(A(x), r \big) \Big]$$

$$\forall x \Big[\big(A(x), r \big) \mid F(x) \iff \bigvee_{i=0}^{r-1} A(x) \equiv i \mod r\ \wedge\ (i, r) | F(x) \Big]$$

A 25$ Conjecture of Erdös



N arithmetic progressions $\{a_i + b_i\mathbb{N}\}_{i=1}^{N}$ cover $\mathbb{N}$ $\iff$ they cover the set $1, 2, 3, \ldots, 2^N$. [R. Crittenden and C. Vanden Eynden '69]

Elimination of universal quantifiers:

$$\forall x \bigvee_{i=1}^{N} b_i | x - a_i \iff \bigwedge_{t=1}^{2^N} \bigvee_{i=1}^{N} b_i | t - a_i$$

A Generalized Version of Erdös' Lemma

N progressions $\{a_i + b_i\mathbb{N}\}_{i=1}^{N}$ cover a progression $a + b\mathbb{N}$ $\iff$ they cover the set $\{a, a+b, \ldots, a+b\cdot(2^N-1)\}$.

Elimination of universal quantifiers in presence of negation:

$$\forall x \bigvee_{i=1}^{N} b_i | x - a_i \quad \vee \quad \bigvee_{j=1}^{M} c_j \not| \; x - d_j \iff$$

$$\forall x \bigwedge_{j=1}^{M} c_j | x - d_j \quad \rightarrow \quad \bigvee_{i=1}^{N} b_i | x - a_i \iff$$

$$\underbrace{\neg\exists y \bigwedge_{j=1}^{M} c_j | y - d_j}_{\text{vacuity}} \vee \underbrace{\exists y \bigwedge_{j=1}^{M} c_j | y - d_j}_{\text{non-vacuity}} \wedge \underbrace{\bigwedge_{t=1}^{2^N} \bigvee_{i=1}^{N} b_i | \big([c_j]_{j=1}^{M} \cdot t + y\big) - a_i}_{\text{coverage}}$$

Reduction of $\mathcal{L}_{|}^{(1)}$ to Presburger Arithmetic

$$Q_1 x_1 \ldots Q_n x_n \bigvee_{i=1}^{N} \Big(\bigwedge_{j=1}^{M_i} f_{ij}(z) \mid f'_{ij}(\vec{x}, z) \wedge \bigwedge_{j=1}^{P_i} g_{ij}(z) \not| g'_{ij}(\vec{x}, z) \wedge \varphi_i(\vec{x}) \wedge \psi_i(z) \Big)$$

where $f_{ij}, f'_{ij}, g_{ij}, g'_{ij}$ are linear terms and φ_i, ψ_i are Presburger formulae

- eliminate quantified variables $x_n, x_{n-1}, \ldots, x_1$
- translate the solved form into Presburger Arithmetic

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Programs with Lists

Sequential non-recursive programs with singly-linked list data structures:

```
struct node {
        int data;
        struct node * next;
} * list;
```

A heap is a graph in which each node has at most one successor.

Safety: *whenever the control reaches a certain location, the heap of the program will satisfy a given property*

Termination: *the final location is reachable on every given input*

Syntax of Programs with Lists

- pointer variables $PVar = \{u, v, w, \ldots\}$ ranging over memory locations
- assignments:

  $u :=$ null $\mid u :=$ new $\mid u := w \mid u := w.$next $\mid u.$next $:=$ null $\mid u.$next $:= w$

- conditionals:

  $$\text{while} \mid \text{if } (u = v)\ \ldots$$

- assertions:

  $$\text{assert}(u = v)$$

- may use boolean combinations, e.g.

  $$\text{if } (u = v\ \wedge\ w = t) \ldots$$

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Destructive updating assignments

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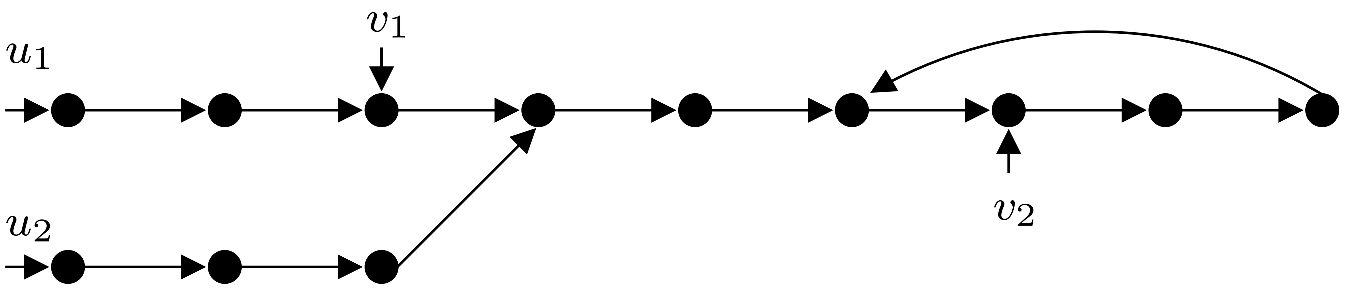
Flatness: no **while** or **if** occurs inside the body of a **while**

1-Selector Heaps



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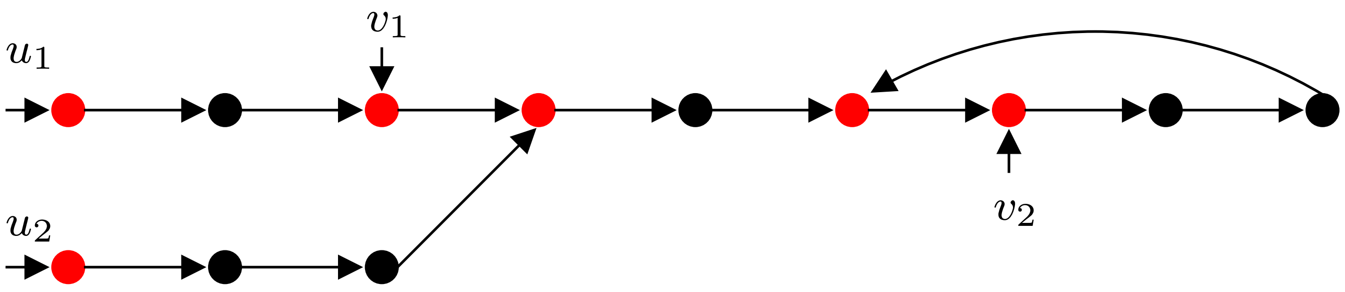

Cut points are either pointed to by a variable or have at least two predecessors

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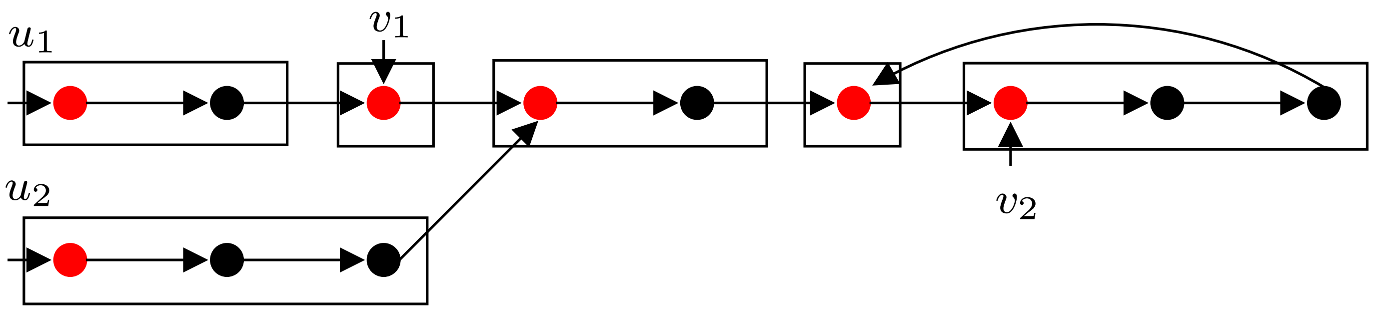
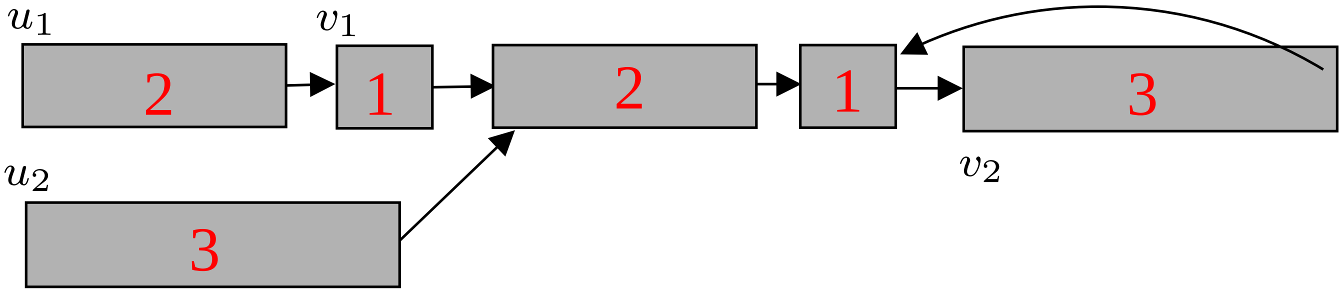


All nodes between two cut points belong to the same equivalence class

Parametric Heaps



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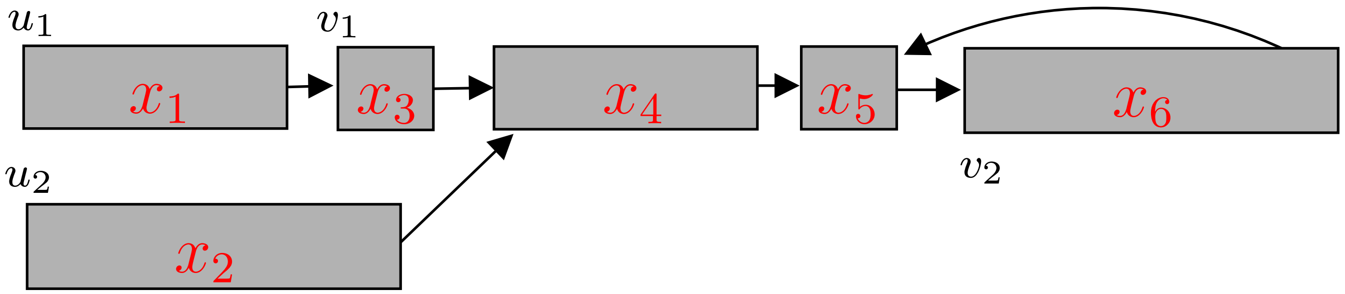

Symbolic representation of an infinite set of concrete heaps

Verification Problems for Programs with Lists

[**Safety**] Let P be a list program, $S = \{l_i \ : \ \mathsf{assert}(\varphi_i)\}_{i=1}^{k}$ a set of assertions occurring in P, and $H(\vec{x})$ a parametric heap. P is said to be safe w.r.t $H(\vec{x})$ and S iff for all $H(\vec{n}) \in H(\vec{x})$, and $1 \leq i \leq k$, φ_i is true whenever P, started on input $H(\vec{n})$, reaches l_i.

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[**Termination**] Let P be a list program, and $H(\vec{x})$ a parametric heap. P is said to terminate w.r.t $H(\vec{x})$ iff for all $H(\vec{n}) \in H(\vec{x})$, P started on input $H(\vec{n})$, has a finite execution.

Undecidability Results

n-Cyclic Heaps $n > 1$

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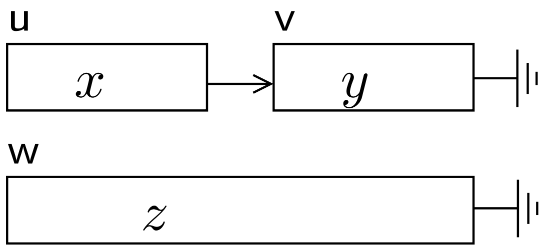
Flat Program for Addition $x + y = z$

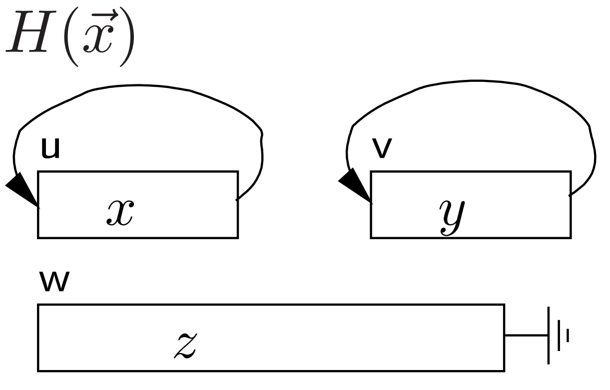
```
1: i := u;
2: j := w;
3: while (i ≠ null ∧ j ≠ null) do
4:          i := i.next;
5:          j := j.next;
6: od;
7: if (i = null ∧ j = null)
OK: ...
          else
ERR: ...
```

$H(\vec{x})$



Flat Program for Least Common Multiple $[x, y] = z$

```
1: i := u.next;
2: j := v.next;
3: k := w.next;
4: while (¬ (i = u ∧ j = v)) do
5:         i := i.next;
6:         j := j.next;
7:         k := k.next;
8: od;
9: if (k = null)
OK: ...
        else
ERR: ...
```



Reduction of complemented HTP to Safety/Termination

Given a Diophantine system S, build a program P and a parametric heap H, such that S doesn't have a solution iff P, started on any input from H, is safe (terminates)

$$\neg\left(\exists \vec{x} \;.\; \bigwedge_{i=1}^{n} P_i(\vec{x}) = 0\right) \iff \forall \vec{x} \;.\; \bigvee_{i=1}^{n} P_i(\vec{x}) \neq 0$$

$Prog(P_1)$;

if $Cond(P_1 = 0)$ then $Prog(P_2)$;

  if $Cond(P_2 = 0)$ then $Prog(P_3)$;

    ...

    assert(false); /* while (true) do od */

    ...

  fi

fi

Extensions of the Undecidability Results

Programs working on arbitrary heaps:

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Non-deterministic programs started with the empty heap:

- use a non-deterministic constructor to build heaps of a given shape

Decidability Results

Acyclic and 1-Cyclic Heaps

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Encode the program as a counter automaton $M = \langle \vec{x}, Q, q_0, \rightarrow \rangle$:

- $\vec{x} = \{x_1, x_2, \ldots, x_k\}$ is a set of counters
- Q is a set of control states, and $q_0 \in Q$ is the initial state
- $q \xrightarrow{R(\vec{x}, \vec{x}')} q'$, where $R(\vec{x}, \vec{x}')$ is an arithmetic formula

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Reachability relation definability:

- $R^n(\vec{x}, \vec{x}') =$ in/out relation after n iterations, and $R^* = \bigcup_{n \in \mathbb{N}} R^n$
- Is R^* definable in a logic decidable for satisfiability?
- A positive answer entails decidability of reachability problems

A Decidable Class of Counter Automata

- Flat control structure: no nested loops
- Transitions of the form $q \xrightarrow{g(\vec{x},\vec{y}) \wedge f(\vec{y},\vec{y}')} q'$:
  - $\vec{x}$ are parameters: no assignments to them
  - $\vec{y}$ are counters
  - guards $g(\vec{x}, \vec{y})$ are boolean combinations of linear/divisibility constraints
  - actions $f(\vec{y}, \vec{y}')$ are reset $(y = 0)$, copy $(y_i' = y_j)$ or increment $(y' = y + 1)$

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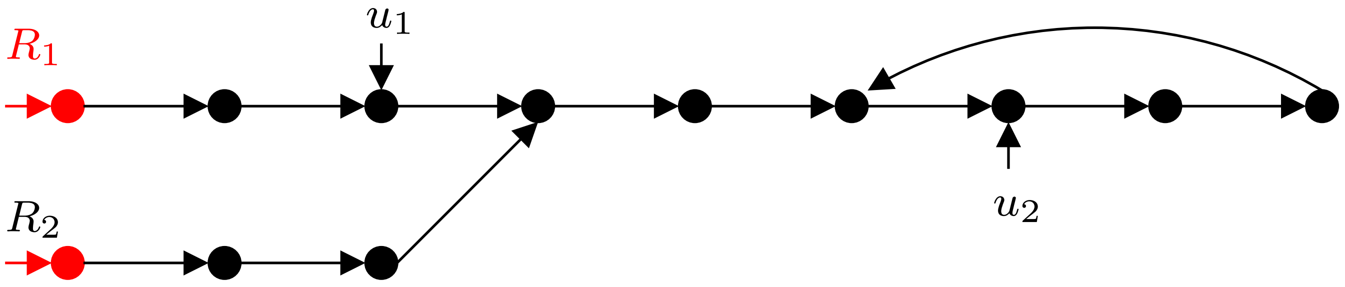
For this class R^* is definable in the fragment $\mathcal{L}_|^{(1)}$ of $\langle \mathbb{N}, +, |, 0, 1 \rangle$

Encoding Program States



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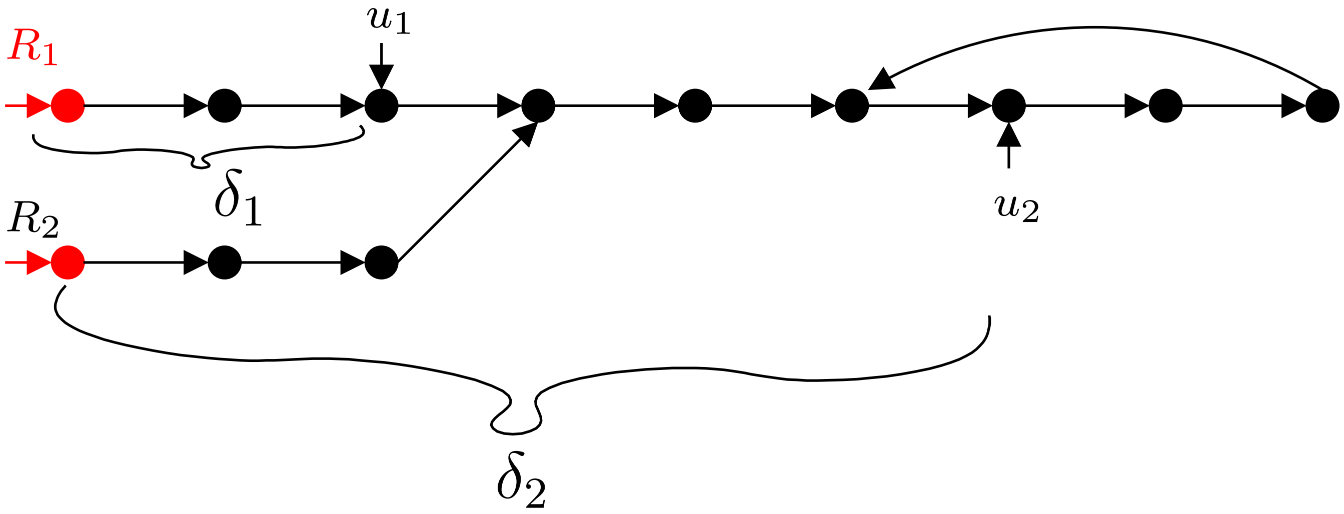
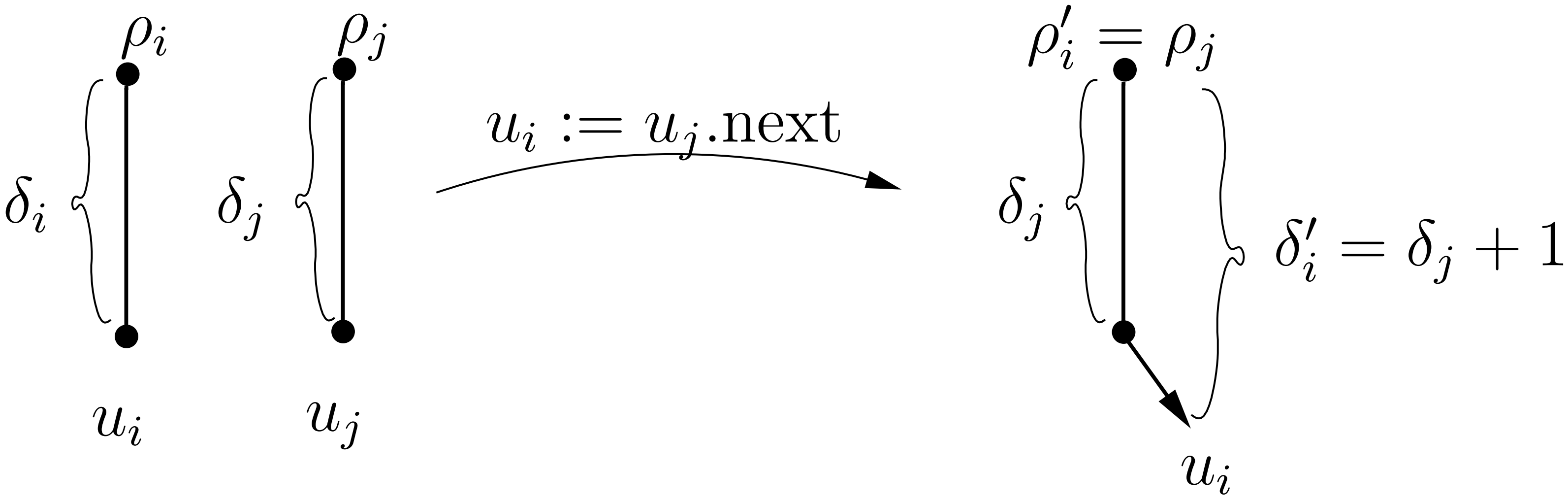


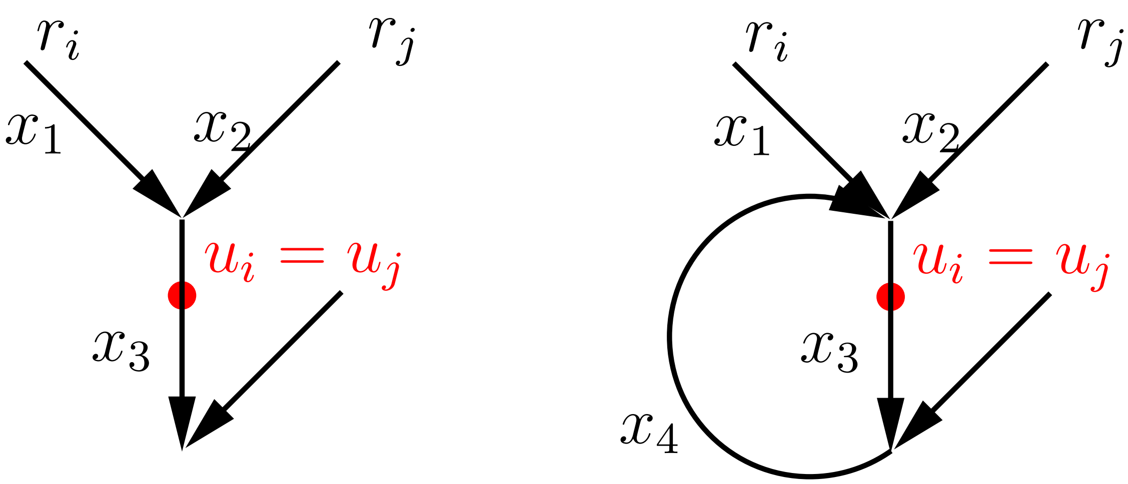
$$\{\langle R_1, \delta_1 \rangle, \langle R_2, \delta_2 \rangle\}$$

Encoding Assignments



The encoding preserves the control structure of the program

Encoding Guards



(a) (b)

If $u_i = \langle r_i, y_i \rangle$ and $u_j = \langle r_j, y_j \rangle$:

a) $0 \le y_i - x_1 \le x_3 \wedge 0 \le y_j - x_2 \le x_3 \wedge y_i - x_1 = y_j - x_2$

b) $0 \le y_i - x_1 \wedge 0 \le y_j - x_2 \wedge x_3 + x_4 \mid (y_i - x_1) - (y_j - x_2)$

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Conclusions

- Study of satisfiability problem within $\langle \mathbb{N}, +, |, 0, 1 \rangle$:
  - in general quantifier alternation leads to undecidability
  - fragments based on the separation of variables on the left/right of $|$ may be decidable
- Application of arithmetic to programs with singly-linked lists:
  - undecidability of safety/termination due to the complexity of the input
  - flat programs without destructive updates are undecidable if the input heap has more than one cycle
  - decidability established for acyclic and 1-cyclic heaps