Definability of linear equation systems over groups and rings

A. Dawar, E. Grädel, B. Holm, E. Kopczynski, W. Pakusa

University of Cambridge, University of Warsaw, RWTH Aachen University

Workshop on Finite and Algorithmic Model Theory,
Les Houches, 14 May 2012

A logic for polynomial time

Atserias, Bulatov, Dawar $\mathsf{Slv}(\mathbf{G}) \notin \mathsf{FP{+}C}$

Dawar, Grohe, Holm, Laubner $\mathsf{FP{+}C} \lneq \mathsf{FP{+}rk} \leq \mathsf{PTIME}$

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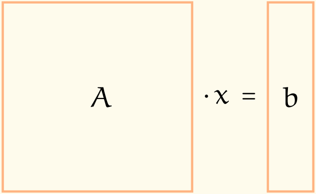
Matrix rank and linear equation systems

Fields $A \cdot x = b$ solvable iff $\mathrm{rk}(A) = \mathrm{rk}(A|b)$:

If $r = \mathrm{rk}(A)$, then $a_1 \cdot \mathbf{c}_1 + \cdots + a_r \cdot \mathbf{c}_r + a \cdot \mathbf{b} = \mathbf{0}$



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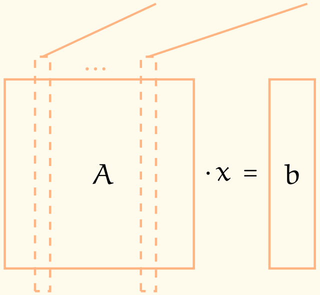

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Groups Undefined

Question: Is $\mathsf{Slv}(\mathbf{G}) \in \mathsf{FP{+}rk}$?

Definability of solvability problems

(1) Inter-definability: $\rightsquigarrow$ natural domain for Slv

Theorem

k-ideal rings $\overset{\text{FP-red.}}{\Longrightarrow}$ cyclic groups of prime power order.

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Normal form for $\mathsf{FO}+\mathsf{slv}_{\mathsf{F}}$.

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Theorem

Normal form for $\mathsf{FO}{+}\mathsf{slv}_{\mathsf{F}}$.

(3) Supra-definability: $\rightsquigarrow$ reducing Slv to a uniform problem

Theorem

$\mathsf{Slv}(\star), \mathsf{rk}(\mathbf{F}) \xRightarrow{\text{FO-red.}} \mathsf{GM}$ (group membership)

Inter-definability: a natural class for solvability

$\mathsf{Slv}(\mathbf{CG})$: Cyclic groups ($\mathbb{Z}_{p^e}$)

$\mathsf{Slv}(\mathbf{I}_k\mathbf{R})$: k-gen. ideal rings ($I \lhd R \Rightarrow I = \pi_1 R + \cdots + \pi_k R$)

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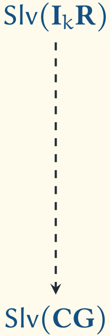

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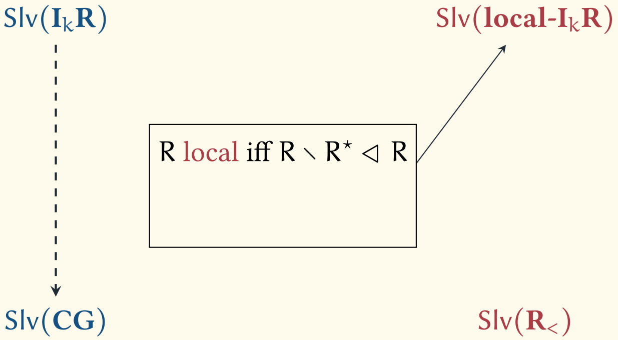

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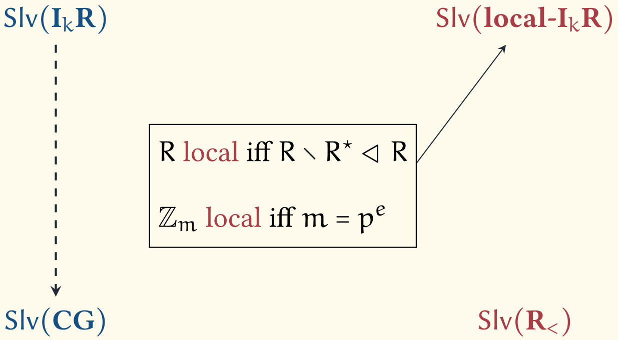

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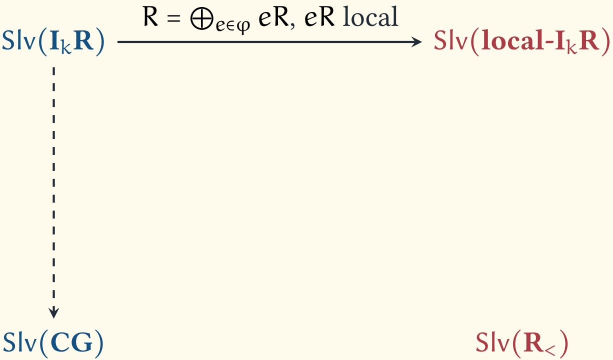

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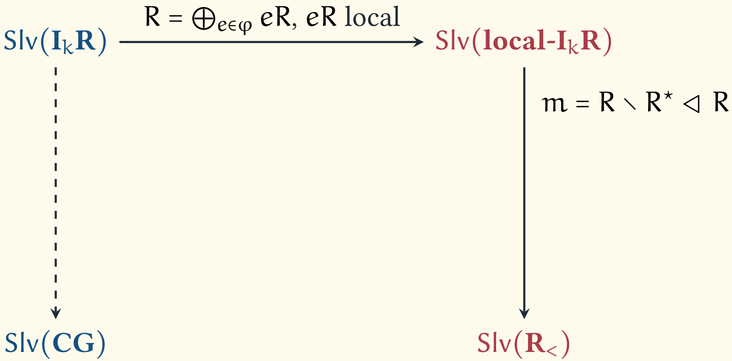

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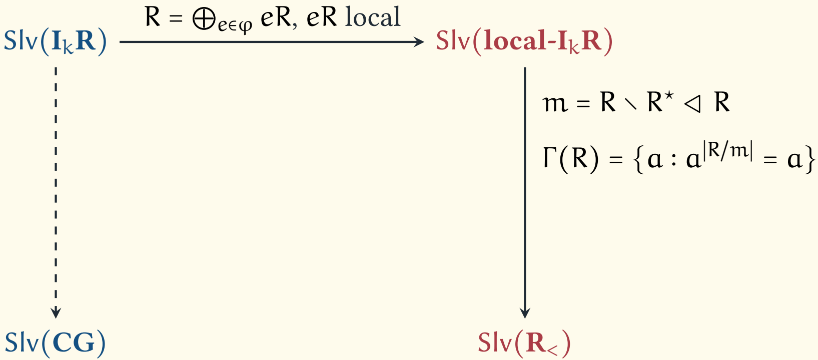

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$\mathsf{Slv}(\mathbf{I}_k\mathbf{R}) \xrightarrow{R = \bigoplus_{e \in \varphi} eR,\ eR \text{ local}} \mathsf{Slv}(\textbf{local-}\mathbf{I}_k\mathbf{R})$

$\mathsf{Slv}(\mathbf{I}_k\mathbf{R}) \dashrightarrow \mathsf{Slv}(\mathbf{CG})$

$\mathsf{Slv}(\textbf{local-}\mathbf{I}_k\mathbf{R}) \longrightarrow \mathsf{Slv}(\mathbf{R}_<)$, where:

$\mathfrak{m} = R \setminus R^\star \lhd R$

$\Gamma(R) = \{a : a^{|R/\mathfrak{m}|} = a\}$

$\Gamma(R) \xrightarrow{\sim} R/\mathfrak{m},\ r \mapsto r + \mathfrak{m}$

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$r \mapsto \sum \underbrace{a_{i_1 \cdots i_k}}_{\in \Gamma(R)} \pi_1^{i_1} \cdots \pi_k^{i_k}$

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$\mathsf{Slv}(\mathbf{R}_<) \longrightarrow \mathsf{Slv}(\mathbf{CG})$:

$$(R, +) = \oplus_{g \in \psi} \langle g \rangle$$

$$(A, \mathbf{b}) \mapsto (A_1, \mathbf{b_1}), \ldots, (A_k, \mathbf{b_k})$$

$\mathsf{Slv}(\mathbf{I}_k\mathbf{R}) \dashrightarrow \mathsf{Slv}(\mathbf{CG})$

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Theorem $\mathsf{Slv}(\mathbf{I}_k\mathbf{R}) \leq^{\mathsf{T}}_{\mathsf{FP}} \mathsf{Slv}(\mathbf{CG})$

Intra-definability: solvability as a logical operator

$$\mathsf{slv}(\bar{x}, \bar{y}, \bar{r}_i).\Big[\underbrace{\varphi_M(\bar{x}, \bar{y}, \bar{r})}_{\text{coefficient matrix}}, \underbrace{\varphi_b(\bar{x}, \bar{r})}_{\text{solution vector}}, \underbrace{(\varphi_R, \varphi_+, \varphi_\cdot)(\bar{r}_1, \bar{r}_2, \bar{r}_3)}_{\text{finite ring}}\Big]$$

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$$\Downarrow$$

FO+slv : First-order logic closed under solvability quantifier

$\mathrm{FO{+}slv}_F$: Solvability quantifier over a fixed finite field F

Intra-definability: solvability as a logical operator

Theorem

Every $\mathsf{FO}{+}\mathsf{slv}_{\mathsf{F}}$-formula is equivalent to a formula of the form

$$\mathsf{slv}(\bar{x}, \bar{y}).\big[\varphi_M(\bar{x}, \bar{y}), \mathbf{1}\big], \text{ with } \varphi_M \text{ quantifier-free.}$$

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Proof illustration: (negation)

$$\neg\mathsf{slv}(\bar{x}, \bar{y}).\big[\varphi, \mathbf{1}\big]$$

$$\text{Non-solvability} \equiv \neg\exists \mathbf{x} : \mathbf{M}\mathbf{x} = \mathbf{b} \overset{?}{\equiv} \exists \mathbf{y} : \mathbf{M}'\mathbf{y} = \mathbf{b}' \equiv \text{Solvability}$$

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Gaussian elimination implies:

$$\neg\exists\mathbf{x} : \mathbf{M}\mathbf{x} = \mathbf{b} \equiv \exists\mathbf{y} : \mathbf{y}(\mathbf{M}|\mathbf{b}) = (0, \ldots, 0|1).$$

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$$\mathsf{slv}(\bar{x},\bar{y}).\big[\varphi,\mathbf{1}\big] \wedge \mathsf{slv}(\bar{x},\bar{y}).\big[\psi,\mathbf{1}\big]$$

$$\boxed{\ \varphi\ } \cdot \mathbf{v_y} = \begin{bmatrix}1\\ \vdots \\ 1\end{bmatrix}$$

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Proof illustration: (universal quantification)

$$\forall z\,(\mathsf{slv}(\bar{x},\bar{y}).\left[\varphi(\bar{x},\bar{y},z),\mathbf{1}\right])$$

$$\boxed{\varphi(z_1)} \cdot \mathbf{v_y} = \begin{bmatrix}1\\ \vdots\\ 1\end{bmatrix}$$

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$$\boxed{\varphi(z_n)} \cdot \mathbf{v_y} = \begin{bmatrix}1\\ \vdots\\ 1\end{bmatrix}$$

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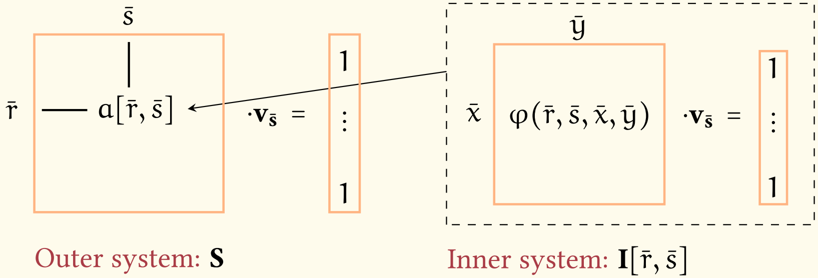
Proof illustration: (nesting of solvability)

$$\mathsf{slv}(\bar{r}, \bar{s}).\big[\mathsf{slv}(\bar{x}, \bar{y}).\big[\varphi(\bar{r}, \bar{s}, \bar{x}, \bar{y}), \mathbf{1}\big], \mathbf{1}\big]$$



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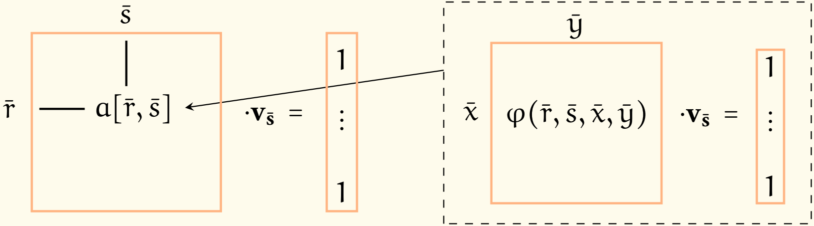

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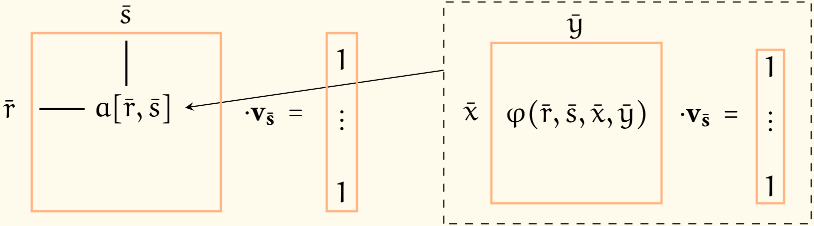

For $\bar{r}$: $\sum_{\bar{s}} \underbrace{a[\bar{r}, \bar{s}] \cdot v_{\bar{s}}} = 1$

Intra-definability: solvability as a logical operator

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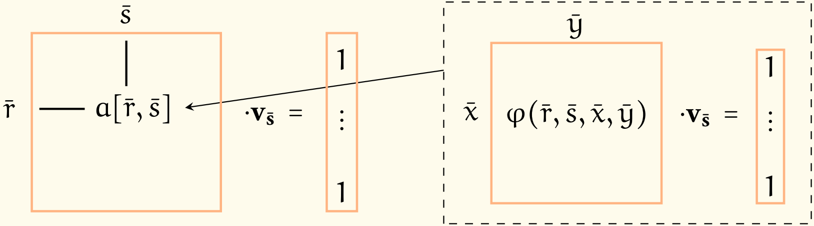

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$\rightsquigarrow$

For $\bar{r}$: $\sum_{\bar{s}} 1 \cdot v[\bar{r}, \bar{s}] = 1$

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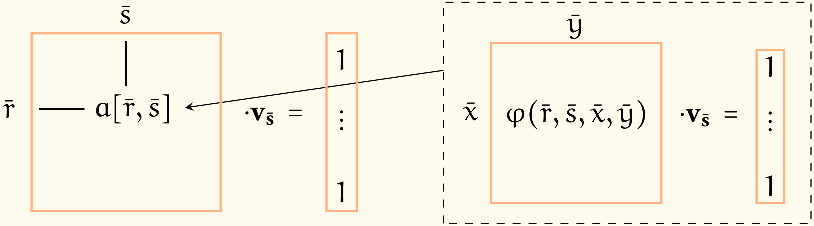

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Consistency conditions:

$v[\bar{r},\bar{s}] = 1 \;\Rightarrow\; a[\bar{r},\bar{s}] = 1$

$v[\bar{r},\bar{s}] \neq v[\bar{r}',\bar{s}] \;\Rightarrow\; a[\bar{r},\bar{s}] \neq a[\bar{r}',\bar{s}]$

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How to formalise: “If $v = 1$ then $A \cdot x = 1$ solvable”

Intra-definability: solvability as a logical operator

Proof illustration: (nesting of solvability)

$$\mathsf{slv}(\bar r, \bar s).\big[\mathsf{slv}(\bar x, \bar y).\big[\varphi(\bar r, \bar s, \bar x, \bar y), \mathbf{1}\big], \mathbf{1}\big]$$

For $\bar r$: $\sum_{\bar s} \underbrace{a[\bar r, \bar s] \cdot v_{\bar s}} = 1$

For $\bar r$: $\sum_{\bar s} 1 \cdot v[\bar r, \bar s] = 1$

Consistency conditions:

$v[\bar r, \bar s] = 1 \Rightarrow a[\bar r, \bar s] = 1$

$v[\bar r, \bar s] \neq v[\bar r', \bar s] \Rightarrow a[\bar r, \bar s] \neq a[\bar r', \bar s]$

How to formalise: "If $v = 1$ then $A \cdot x = 1$ solvable"

$$\left[\begin{array}{c|c} & -v+1 \\ A & \vdots \\ & -v+1 \end{array}\right] \cdot \mathbf{x} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

From solvability to group membership

Permutation group membership (GM)

Given: Permutations $\pi_1, \ldots, \pi_k$ and π on a set Ω

Question: Is $\pi \in \langle \pi_1, \ldots, \pi_k \rangle \leq S_\Omega$?

$$\mathsf{GM}, \#\mathsf{GM} \in \mathsf{PTIME}$$

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linear system is solvable

$$\iff$$

$$\mathbf{b} \in \langle \mathbf{c}_{\bar{s}} \cdot d : d \in D, \bar{s} \rangle \leq (D, +)^I$$

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Cayley's theorem: FO-definable embedding $\iota : (D, +) \to S_D$

$\rightsquigarrow$ FO-definable embedding $\iota : (D, +)^I \to S_{I \times D}$

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Outlook: FP + nice operator for GM

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Definability of polynomial-time problems from algebra