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Infinitary methods in finite model theory

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Compactness in the finite

Definition: A set of FO sentences is *finitely consistent* if every finite subset has a model.

Compactness: Every finitely consistent set of FO sentences has a model (i.e. is consistent).

$$(\exists^n x)[x = x] \equiv \exists x_1 \ldots x_n \bigwedge \{x_i \neq x_j : 1 \leq i < j \leq n\}$$

$$\Phi_\infty = \{\exists^n x : n \geq 1\}$$

Failure: Each finite subset of Φ_∞ has a *finite* model, but Φ_∞ does not have a *finite* model.

Virtue of compactness

Definition: A first-order sentence ψ is *trivial* over finite models if it is eventually constant.

Fact: Every nontrivial first-order sentence θ has infinite models of both it and its negation.

Proof: There are arbitrarily large finite models of both θ and $\neg\theta$. Hence both $\{\theta\} \cup \Phi_\infty$ and $\{\neg\theta\} \cup \Phi_\infty$ are finitely satisfiable, so by compactness each have an infinite model.

Connectivity [Gaifman, Vardi, '85]

Theorem: Connectivity is not FO in the finite.

Proof: Take $\mathbf{G} \models \theta \Leftrightarrow \mathbf{G}$ is connected, $|\mathbf{G}| < \infty$, T =

- $(\forall x, y)[E(x, y) \rightarrow x \neq y \wedge E(y, x)]$ (simple)
- $(\forall x)(\exists^2 y)E(x, y) \wedge (\nexists^3 y)E(x, y)$ (two-regular)
- $\nexists x_{1\ldots n}\, [x_2 \neq x_n \wedge E(x_1, x_2) \wedge \ldots E(x_n, x_1)]$ (acyclic)

T is consistent with both θ and $\neg\theta$ (separately).
So by compactness we get $T \not\models \neg\theta$ and $T \not\models \theta$.
Models of T are unions of infinite chains, so T is uncountably categorical. $\therefore$ T is complete, >=<.

Gaifman graph

Definition: The *Gaifman graph* of a relational L-structure S is the <u>simple</u> graph over $|S|$ with

$$E = \{\langle a, b\rangle : a \neq b \;\&\; S \models R(\ldots a, \ldots b, \ldots)\ R \text{ in } L\}.$$

Idea: pair elements occurring jointly in tuple

Advantage: Can refer to graph notions such as distance $d(a, b)$ and degree in any L-structure.

Neighborhoods

Definition: The *r-ball* $B_r(a) = \{b : d(a, b) \leq r\}$.

The radius *r-neighborhood* of a is a structure:

$$N_r(a) = \langle B_r(a), R \cap [B_r(a)]^{arity(R)}, \ldots, a\rangle \text{ for all } R \text{ in } L.$$

The *component* of a is $N_\infty(a) = \bigcup\{N_r(a) : r > 0\}$.

<u>Tuples</u>: Define $d(\underline{a}, b) = \min \{d(a, b) : a \text{ in } \underline{a}\}$.

Extends $N_r(\underline{a})$ and $N_\infty(\underline{a})$ naturally for $|\underline{a}| > 0$.

Isomorphism locality

Theorem: [Hella, Libkin, Nurmonen, 1999]
Every first-order L-formula $\theta(\underline{x})$ is *Gaifman local*, i.e. there is a radius r such that for all relational L-structures S and tuples $\underline{a}$ and $\underline{b}$,

$$N_r(\underline{a}) \cong N_r(\underline{b}) \quad \Rightarrow \quad S \vDash \theta[\underline{a}] \leftrightarrow \theta[\underline{b}]$$

Proof: follows from Gaifman's theorem, 1982.

Internalize isomorphism

Proof: $\theta(x)$ not Gaifman local means for each r

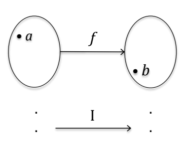
$$\mathbf{G}_r \vDash \theta[a] \wedge \neg\theta[b] \qquad \text{where } f : N_r(a) \cong N_r(b)$$

Take T = $\{\theta(a), \neg\theta(b), f : N_r(a) \cong N_r(b) : r \geq 0\}$.
T is finitely consistent. By compactness we get

$$(\mathbf{G}, f) \vDash \theta[a] \wedge \neg\theta[b] \qquad f : N_\infty(a) \cong N_\infty(b)$$

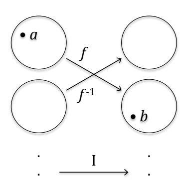
$(\mathbf{G}, a) \cong (\mathbf{G}, b)$

$f(a) = b \in N_\infty(a)$



$\mathbf{G} \vDash \theta[a] \Leftrightarrow \mathbf{G} \vDash \theta[b]$

$f(a) = b \notin N_\infty(a)$



Threshold locality

Theorem: [Fagin, Stockmeyer, Vardi, 1995]
Over degree d bounded structures, every first-order sentence φ is *Hanf threshold local*: it has a radius r and threshold t such that for all N,

$$|\{a \in |\mathbf{A}| : N_r(a) \cong N\}| \overset{t}{\triangleq} |\{b \in |\mathbf{B}| : N_r(b) \cong N\}|$$
$$\Downarrow$$
$$\mathbf{A} \vDash \varphi \Leftrightarrow \mathbf{B} \vDash \varphi$$

Proof: inspired by Hanf's lemma, 1965 (~~d~~, ~~t~~).

Use a model pair

$(\mathbf{G}_1, \mathbf{G}_2, R)$, $R \subseteq V_1 \times V_2$. Theory T says degree d &:

- $\langle V_1, E_1\rangle \vDash \varphi$ & $\langle V_2, E_2\rangle \vDash \neg\varphi$ (substitution)
- $\{R(x, y) \rightarrow N_r(x) \cong N_r(y) : r > 0\}$ (since size $\sim d^r$)
- $\{\forall^t x\, \exists^t y\, R(x, y) \wedge \forall^t y\, \exists^t x\, R(x, y) : t > 0\}$ (1 by 1)

If φ is not threshold local, then by compactness T has a model pair where $R(u, v) \rightarrow N_\infty(u) \cong N_\infty(v)$.
The isomorphisms form a finitely branching tree under inclusion (König's infinity lemma). $[u \sim v]$

Multiplicity of congruence classes

- Let $[e] = \{e' \in |\mathbf{G}| : N_\infty(e') \cong N_\infty(e)\}$. T implies pointed components occur equi-numerously.
- Let $[N_\infty(e)] = \{N \subseteq |\mathbf{G}| : N \cong N_\infty(e)\}$. Show the same for these un-pointed components.

If $m = |[e] \cap N_\infty(e)| < \infty$ then $|[N_\infty(e)]| = |[e]| \div m$.
If $m = \infty$, $\{d(e, e') : e \sim e' \in N_\infty(e)\}$ is unbounded, so the type $\{d(c_i, c_j) > n : c_i \sim e \sim c_j, i, j, n \in \omega\}$ is consistent. Again use compactness: $|[N_\infty(e)]| = \infty$.
Hence $\mathbf{G}_1 \equiv \mathbf{G}_2$, contradicting $\mathbf{G}_1 \vDash \varphi$ & $\mathbf{G}_2 \vDash \neg\varphi$.