First-Order and Monadic Second-Order Model-Checking on Ordered Structures

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The model-checking problem on finite graphs

> $MC(\mathcal{L}, \mathcal{C})$
> $\quad\quad\quad Input:$ Graph $G \in \mathcal{C}$ and $\varphi \in \mathcal{L}(\{E\})$
> $\quad Parameter:$ $|\varphi|$
> $\quad\quad Problem:$ $G \models \varphi$?

The model-checking problem on the class of all graphs is
AW[*]-hard.

Fixed-parameter tractability

The model-checking problem $MC(\mathcal{L}, \mathcal{C})$ is *fixed-parameter tractable* if there is an algorithm that on input (G, φ) correctly decides whether $G \models \varphi$ in time $f(|\varphi|) \cdot \|G\|^{\mathcal{O}(1)}$ for some computable function f.

Classes of graphs with tractable (fpt) model-checking

Model checking on finite relational structures
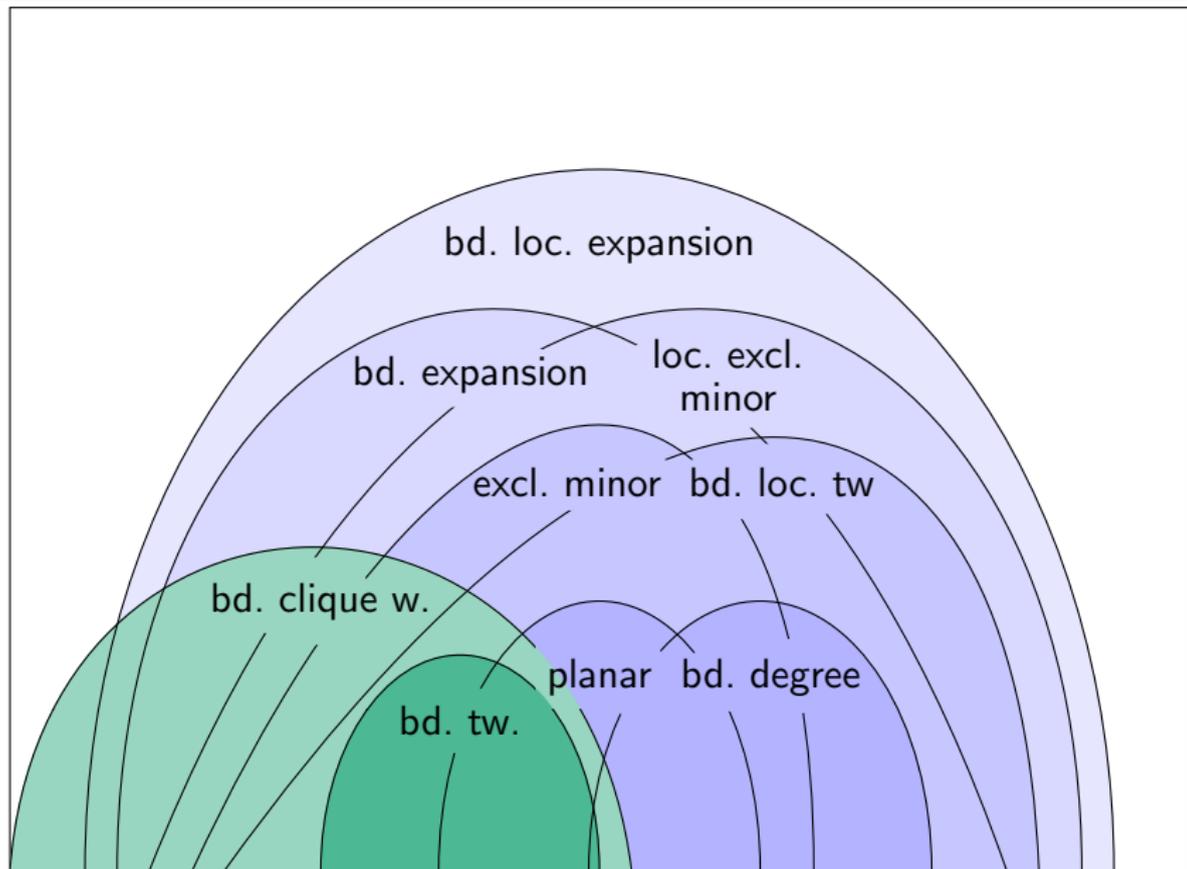
- Efficient model checking on a structure $\mathfrak{A}$ if the Gaifman graph $G(\mathfrak{A})$ has one of the above structural properties.
- On ordered structures, $G(\mathfrak{A})$ does not have any nice properties.
- On ordered structures, any algorithm based on locality of first-order logic is no longer efficient.

Choosing a bad successor

MC($\mathcal{L}[+1]$, $\mathcal{C}$)

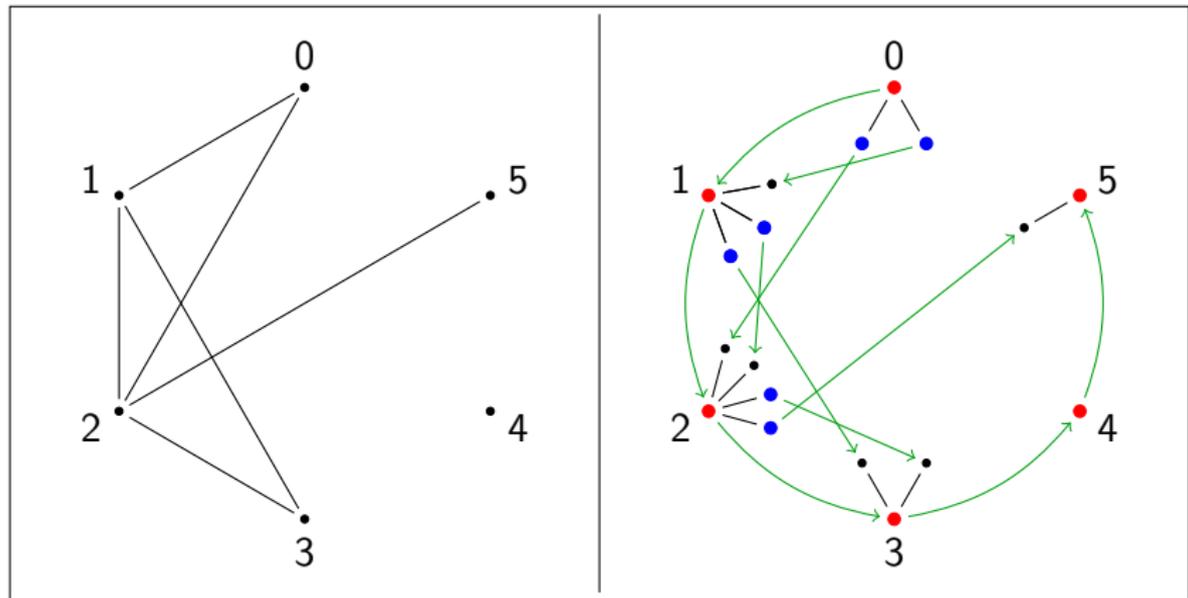|	
Input:	Graph $G \in \mathcal{C}$, S a successor relation on $V(G)$ and $\varphi \in \mathcal{L}(\{E, S\})$
Parameter:	$|\varphi|$
Problem:	$(G, S) \models \varphi$?

Theorem

Let $\mathcal{F}$ be the class of forests. Then MC(FO[+1], $\mathcal{F}$) is AW[*]-hard.

Choosing a bad successor

Given (G, φ) construct a forest F equipped with a successor relation S such that $\|F\| \in \|G\|^{\mathcal{O}(1)}$ and a sentence $\varphi' \in \mathrm{FO}(\{E, S\})$ such that
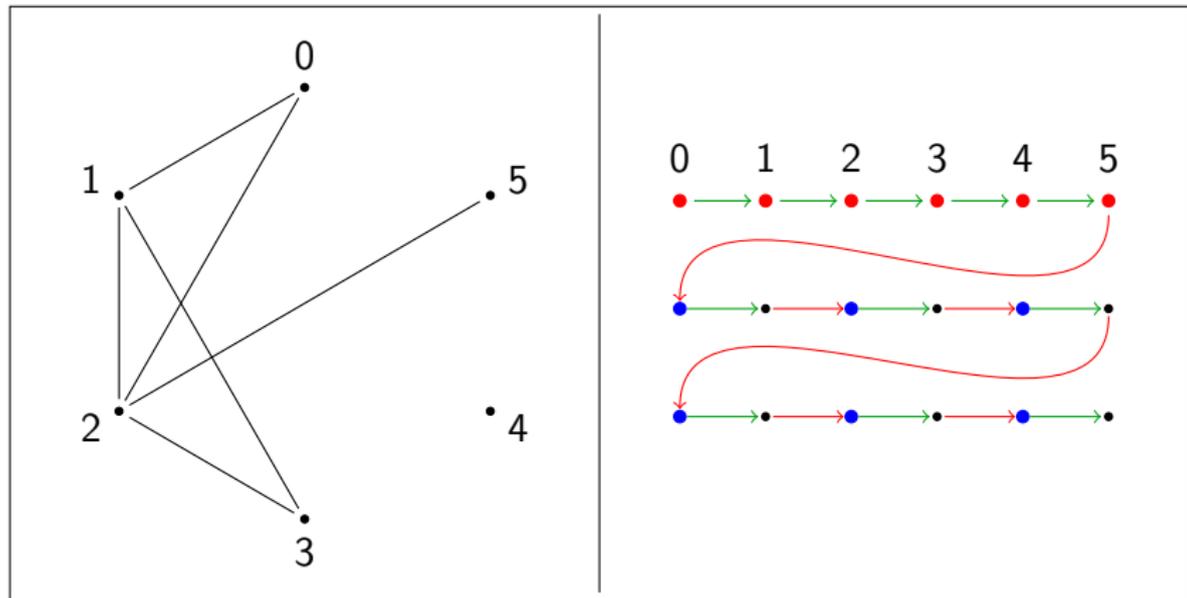
$$G \models \varphi \Leftrightarrow F \models \varphi'.$$

Choosing a bad successor

Given (G, φ) construct a forest F equipped with a successor relation S such that $\|F\| \in \|G\|^{\mathcal{O}(1)}$ and a sentence $\varphi' \in \mathrm{FO}(\{E, S\})$ such that

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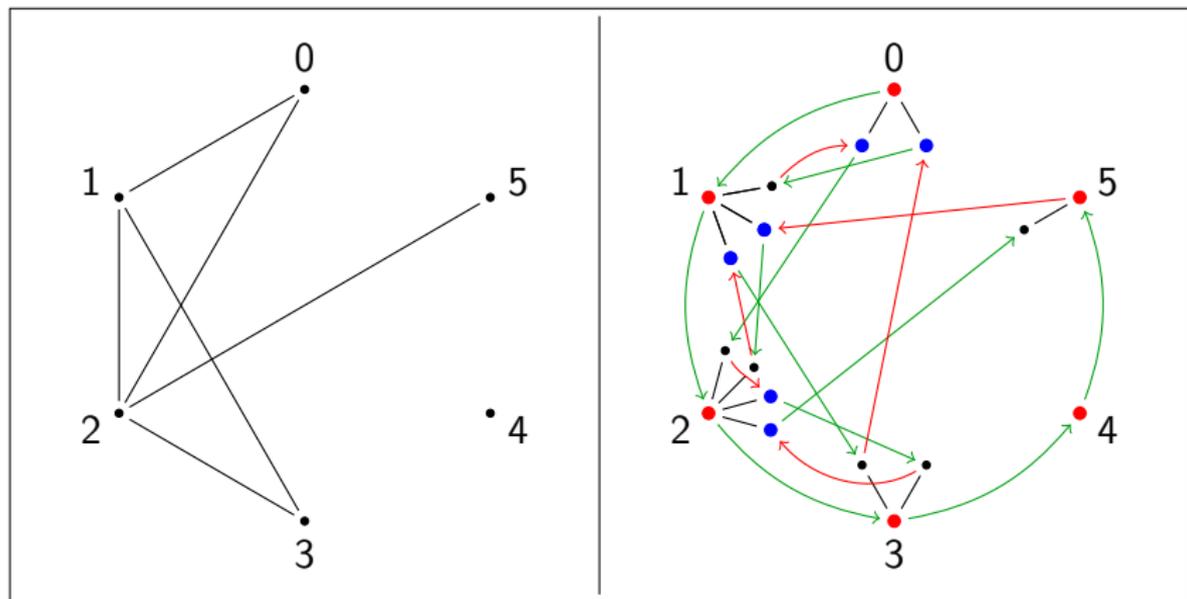
$$G \models \varphi \Leftrightarrow F \models \varphi'.$$

Successors on classes of bounded degree

- We require trees of unbounded rank!

Theorem (Seese)

FO model-checking on classes of bounded degree is fixed-parameter tractable.

Corollary

FO model-checking on classes of bounded degree with a fixed number of successor relations is fixed-parameter tractable.

Choosing a bad order

> ### $MC(\mathcal{L}[<], \mathcal{C})$
>
> | | |
> |---:|:---|
> | *Input:* | Graph $G \in \mathcal{C}$, $<$ a linear order on $V(G)$ and $\varphi \in \mathcal{L}(\{E, <\})$ |
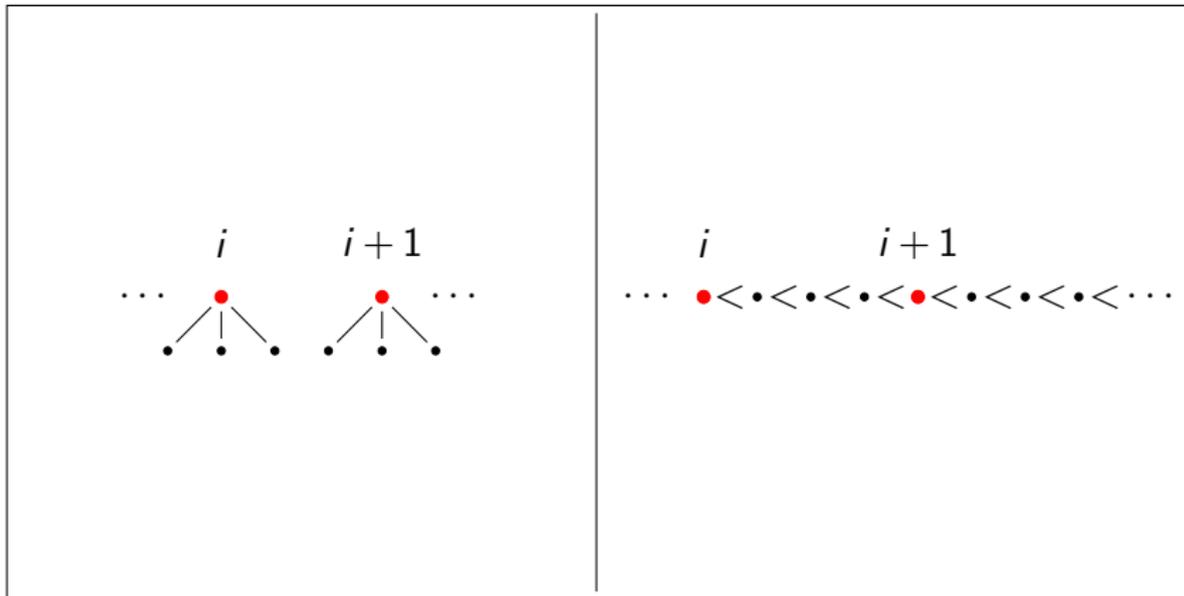> | *Parameter:* | $|\varphi|$ |
> | *Problem:* | $(G, <) \models \varphi$? |

Theorem

Let $\mathcal{S}$ be the class of successor structures with one unary predicate. Then $MC(FO[<], \mathcal{S})$ is AW[*]-hard.

Choosing a bad order

Replace (black) edges with help of the order.

Order-invariant logics

Definition

A formula φ over vocabulary $\tau = \sigma \cup \{<\}$ is order-invariant if for every σ-structure A and all linear orders $<_1, <_2$ over $V(A)$ we have $(A, <_1) \models \varphi \Leftrightarrow (A, <_2) \models \varphi$.

Similar for successor-invariance.

- FO[$<$-*inv*] and MSO[$<$-*inv*] for the set of all order-invariant FO and MSO formulas
- FO[$+1$-*inv*] and MSO[$+1$-*inv*] for the set of all successor-invariant FO and MSO formulas

- Structures in memory are implicitely ordered, yet the truth of a formula should not depend on the particular order.
- Order-invariant logics capture complexity classes (yet they have no effective syntax).

Expressive power of order-invariant logics

- MSO[$<$-*inv*] can express parity.
- There are order-invariant first-order sentences which are not equivalent to first-order sentences (Gurevich).
- There are successor-invariant first-order sentences which are not equivalent to first-order sentences (Rossman).
- On trees, FO[$<$-*inv*] collapses to FO (Benedikt and Segoufin, Niemistö).
- The expressive power of order-invariant logics is not well understood.
- Even if a formula is equivalent to a first-order formula, it might be much shorter than the equivalent formula and the transformation might not be effective.

Order-invariant MSO on classes of bounded tree-width

Theorem (Makowski, Chen and Flum)

If G is a graph of tree-width k then there is a successor relation S on $V(G)$ such that the Gaifman graph of (G, S) has tree-width at most $k + c$ for a small constant c and S is efficiently computable.

Theorem (Courcelle)

MSO model-checking on classes of bounded tree-width is fixed-parameter tractable.

Corollary

MSO[$<$-*inv*] model-checking on classes of bounded tree-width is fixed-parameter tractable.

Order-invariant MSO on classes of bounded clique-width

Theorem

There is an algorithm which, on input a graph G of clique-width at most k, computes a linear order $<$ on $V(G)$ and a clique-expression of width at most $g(k)$ generating the structure $(G, <)$, where g is a computable function.

Theorem (Courcelle, Makowski, Rotics)

MSO model-checking on classes of bounded clique-width is fixed-parameter tractable.

Corollary

Let $\mathcal{C}$ be a class of graphs of bounded clique-width. Then $MC(MSO[<\text{-}inv], \mathcal{C})$ is fixed parameter tractable.

Successor-invariant FO on planar graphs

Main Theorem

MC(FO[+1-*inv*], PLANAR) is fixed-parameter tractable.

Interpretation method

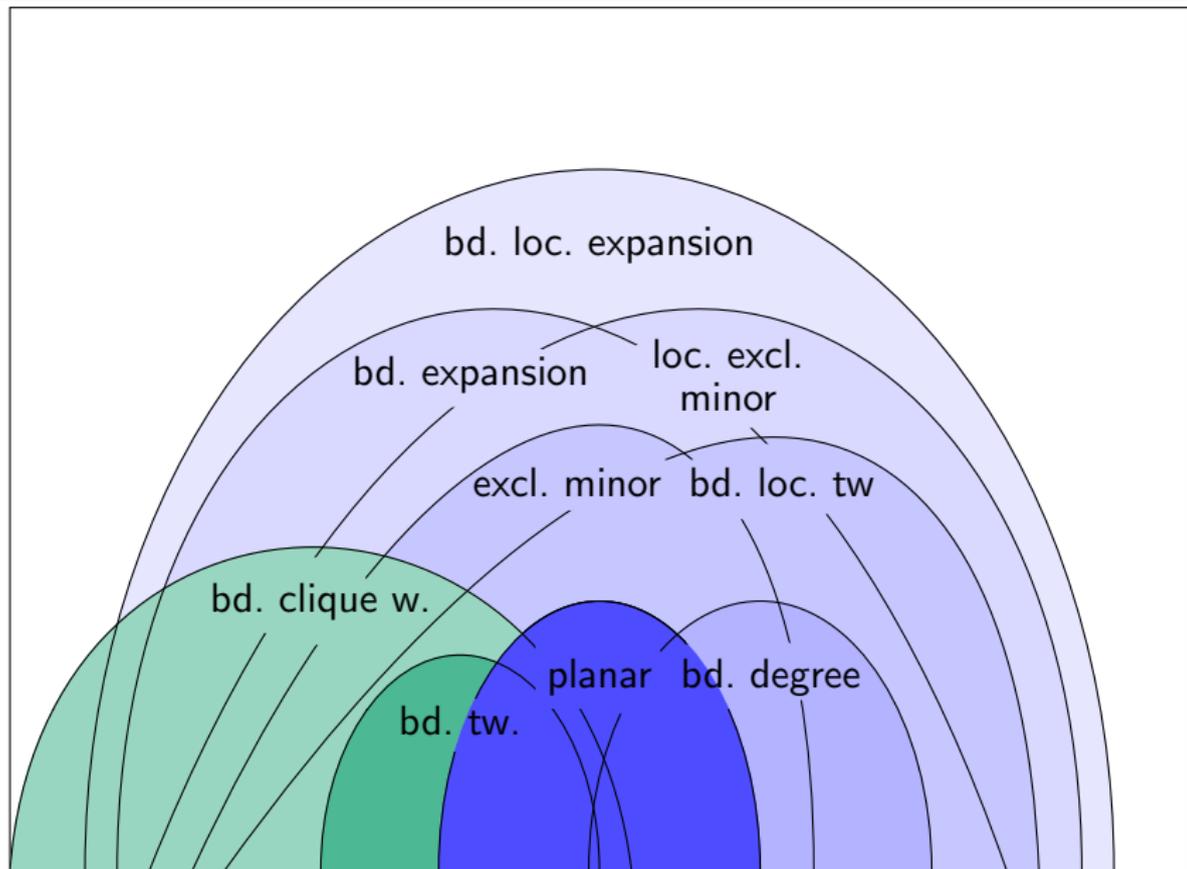## Interpretation method

Let $\mathcal{C}$ be a class of graphs, σ a signature and $\mathcal{D}$ a class of σ-structures with the following properties.

- There is a polynomial-time algorithm which, on input $G \in \mathcal{C}$, computes a σ-structure $H \in \mathcal{D}$ and formulas $\varphi_V(x), \varphi_E(x, y), \varphi_S(x, y) \in \mathrm{FO}(\sigma)$ such that $G' := (\varphi_V(H), \varphi_E(H))$ is isomorphic to G and $\varphi_S(H)$ defines the edge set of a successor relation on $V(G')$.
- $\mathrm{MC}(\mathrm{FO}, \mathcal{D})$ is fixed-parameter tractable.

Then $\mathrm{MC}(\mathrm{FO}[+1\text{-}inv], \mathcal{C})$ is fixed-parameter tractable.

Class $\mathcal{C}$ Class $\mathcal{D}$

$$G \xrightarrow{\quad\text{algorithmic encoding}\quad} H$$

$$\varphi \in \mathsf{FO}(\{E\}) \xrightarrow{\quad\text{interpretation}\quad} \mathcal{I}(\varphi) \in \mathsf{FO}(\sigma)$$
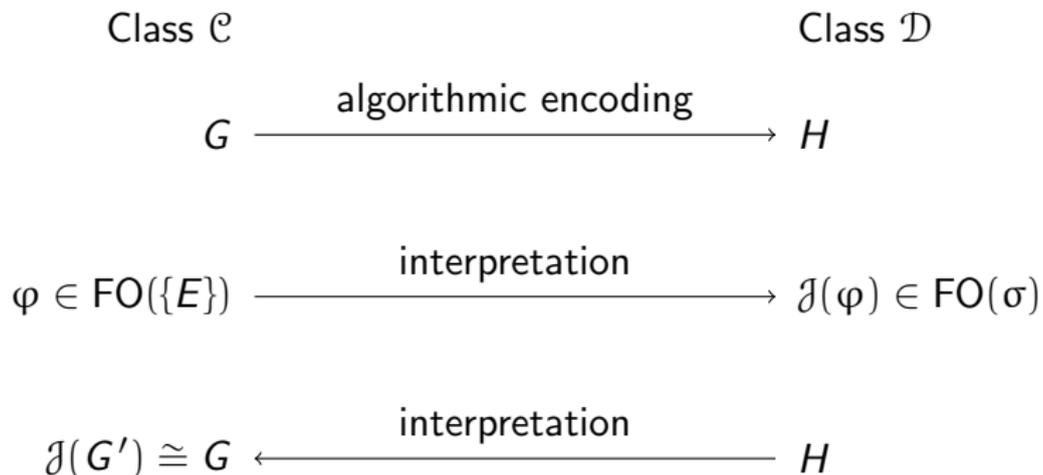
$$\mathcal{I}(G') \cong G \xleftarrow{\quad\text{interpretation}\quad} H$$

Local tree-width

Definition

- The *local tree-width* of a graph G is the function
  $ltw_G : \mathbb{N} \to \mathbb{N}$ with
  $ltw_G(r) := \max\{tw(G[N_r^G(v)]) : v \in V(G)\}$.

- $\mathcal{C}$ has *bounded local tree-width* if there is a computable
  non-decreasing function $f : \mathbb{N} \to \mathbb{N}$ such that $ltw_G(r) \leqslant f(r)$
  for all $G \in \mathcal{C}$ and $r \geqslant 0$.

Theorem (Frick and Grohe)

$MC(FO, \mathcal{C})$ is fixed-parameter tractable on any class $\mathcal{C}$ of bounded
local tree-width.

Lexicographic product

Definition

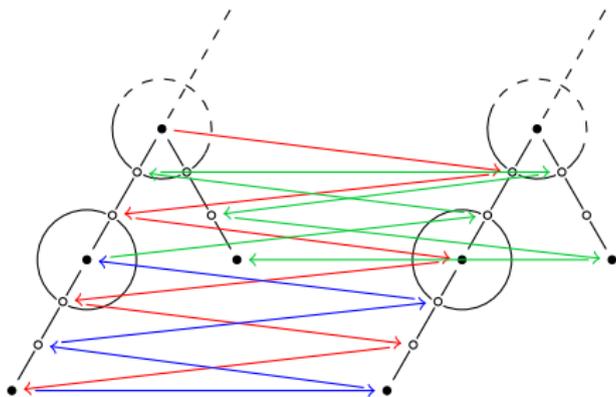Let G and H be graphs. $G \bullet H$ is the graph with vertex set

- $V := V(G) \times V(H)$ and
- edge set $E := \{\{(v, v'), (u, u')\} : \{v, u\} \in E(G),\ v', u' \in V(H)$ or $v = u$ and $\{v', u'\} \in E(H)\}$

For classes $\mathcal{C}$ of graphs: $\mathcal{C} \bullet H := \{G \bullet H : G \in \mathcal{C}\}$.
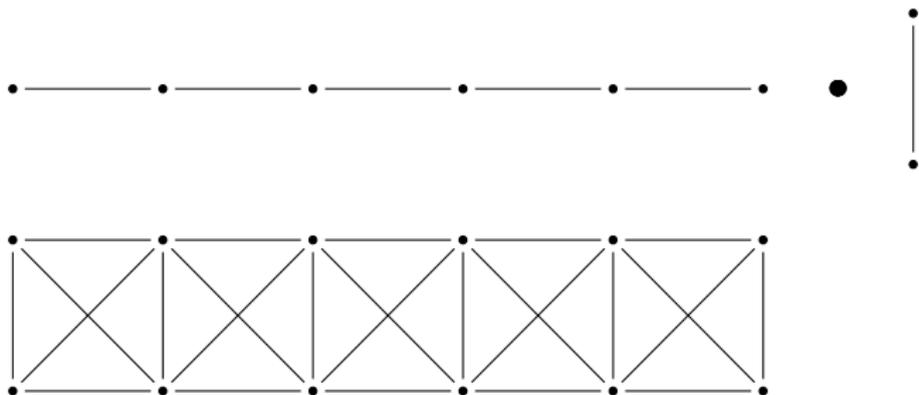
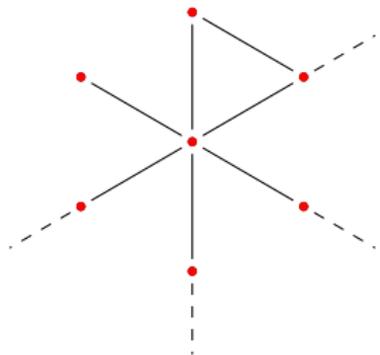Let G be a connected graph, T a spanning tree of G and $\mathfrak{C}(T)$ the circular extension of T. Then $\mathfrak{C}(T) \bullet K_2$ contains a Hamiltonian path H such that every subpath of H of length at least 7 contains a center vertex of the first copy.
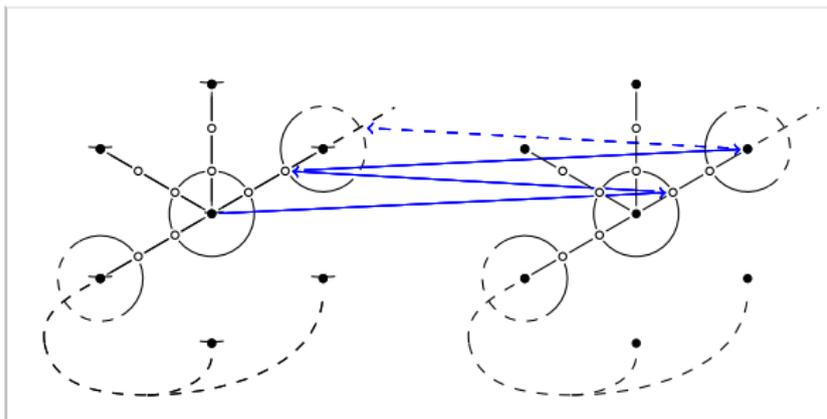
Construction

G Spanning extension T of $P(H) \bullet K_2$ TH

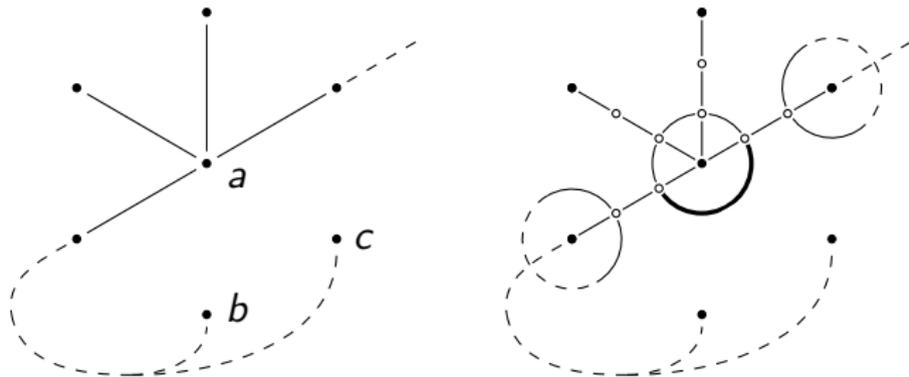# Construction has bounded local tree-width

Theorem (Robertson and Seymour)

The class of planar graphs has bounded local tree-width. More precisely, every planar graph of radius r has tree-width $\leqslant 3r + 1$.

Lemma

Let $\mathcal{C}$ be a class of graphs of bounded local tree-width and $k \geqslant 0$. Then $\mathcal{C} \bullet K_k$ has bounded local tree-width.

The r-neighborhood of any vertex of the construction is found as a minor of a $3r + 2$-neighborhood of $\mathfrak{C}(G) \bullet K_3$.