Dependence and Independence

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Dependence and independence

$x = y$	y is identical to x
$=(\vec{x}, \vec{y})$	$\vec{y}$ depends on $\vec{x}$
$\vec{x} \perp \vec{y}$	$\vec{y}$ is independent of $\vec{x}$

Axioms of dependence

- If x depends on $\vec{y}$ and $\vec{y} \subseteq \vec{z}$, then x depends on $\vec{z}$.
- Whether x depends on $\vec{y}$ or not, is not sensitive to the order of the elements of $\vec{y}$.
- If x depends on $\vec{y}$ and each element of $\vec{y}$ depends on $\vec{z}$, then x depends on $\vec{z}$.

The main one is the third, called transitivity.

Axioms of independence

- $\vec{x}$ is always independent of $\emptyset$.
- If $\vec{x}$ is independent of $\vec{y}$, then $\vec{y}$ is independent of $\vec{x}$.
- Whether $\vec{x}$ is independent of $\vec{y}$ or not, is not sensitive to the order of the elements of $\vec{x}$, or of $\vec{y}$.
- If $\vec{x}$ is independent of $\vec{y}$ and $\vec{x}\vec{y}$ is independent of $\vec{z}$, then $\vec{x}$ is independent of $\vec{y}\vec{z}$.

The main one is the fourth, called Exchange Property.

Examples

- Linear dependence and independence in vector spaces.
- Algebraic dependence and independence in algebraically closed fields.
- Whitney (1933), Van der Waerden (1937): These concepts satisfy the same axioms!

More examples

- Discrete, disintegrated (x depends on $\vec{y}$ if $x \in \vec{y}$, and is otherwise independent of $\vec{y}$).
- Random variables, Bayesian networks, causality.
- Non-interference relation between two groups of concurrent processes sharing common resources.
- Functional and multivalued dependencies in databases.
- Logical (with Grädel, see below).
- Social choice theory (Arrow's Theorem).
- Non-locality phenomena in quantum physics (joint work with Abramsky).

Geometric model theory

- In models of "stable" theories one can define the concepts of dependence and independence so that the above axioms hold.
- Properties of the generated geometry determine the model up to isomorphism. (Shelah and others)
- The discrete case, vector spaces and algebraically closed fields are typical examples.

Logical dependence and independence (with Grädel)

- A team is a set of assignments (here 4):

x	y	z
2	1	3
0	0	1
1	2	0
2	2	3

- Alternatively, a set of "vectors" (here 3) of the same length.

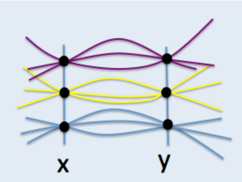
Semantics

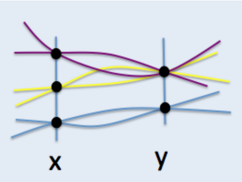
- If $\mathcal{M}$ is a model and X a team, then

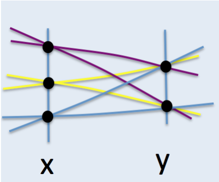
$$\mathcal{M} \models_X x = y \iff s(x) = s(y) \text{ for } s \in X.$$

$$\mathcal{M} \models_X =\!(\vec{x}, y) \iff s(\vec{x}) \mapsto s(y) \text{ is a function on } X.$$

$$\mathcal{M} \models_X \vec{x} \perp \vec{y} \iff \forall s, s' \in X \exists s'' \in X$$
$$(s''(\vec{x}) = s(\vec{x}) \wedge s''(\vec{y}) = s'(\vec{y})).$$







$x = y$ $=(x, y)$ $x \perp y$

Disjunction

$$\mathcal{M} \models_X \phi \vee \psi \iff \exists Y \exists Z (X = Y \cup Z \wedge \\ \mathcal{M} \models_Y \phi \wedge \\ \mathcal{M} \models_Z \psi)$$

$\mathcal{M} \models_X \forall x \phi \iff \mathcal{M} \models_Y \phi$, where

$$Y = \{s(a/x) \ : \ s \in X, \ a \in M\}$$

Model checking problem

Grädel: Model checking for $\mathcal{D}$ and $\mathcal{I}$ is NEXPTIME-complete.

Relativized independence

$$\vec{x} \perp_{\vec{y}} \vec{z}$$

Means: "$\vec{x}$ and $\vec{z}$ are independent, provided that $\vec{y}$ is kept fixed".

1. $\vec{z} \perp_{\vec{x}} \vec{y} \Rightarrow \vec{y}\vec{x} \perp_{\vec{x}} \vec{z}\vec{x}$ (Fixed Parameter Rule)
2. $\vec{x} \perp_{\vec{z}} \vec{y} \wedge \vec{u} \perp_{\vec{z}\vec{x}} \vec{y} \Rightarrow \vec{u} \perp_{\vec{z}} \vec{y}$. (First Transitivity Rule)
3. $\vec{y} \perp_{\vec{z}} \vec{y} \wedge \vec{z}\vec{x} \perp\ _{\vec{y}}\vec{u} \Rightarrow \vec{x} \perp_{\vec{z}} \vec{u}$ (Second Transitivity Rule)

The overall picture (with Kontinen. Galliani. Nurmi.)

$\mathcal{D} = \mathrm{FO} + {=}(\vec{x}, \vec{y})$: Downwards closed existential second order logic
Durand-Kontinen: Hierarchy results for $\mathcal{D}$

$\mathcal{I} = \mathcal{D} + \vec{x} \perp \vec{y}$: Existential second order logic, and hence $\vec{x} \perp_{\vec{y}} \vec{z}$ is definable in $\mathcal{I}$

$\mathcal{T} = \mathcal{I} +$ negation: Second order logic

Completeness Theorem (with Kontinen)

- First order consequences in $\mathcal{D}$ can be axiomatized, general consequences not.
- Natural deduction, disjunction elimination replaced by:

$$\dfrac{A \vee B \qquad \begin{matrix}[B] \\ \vdots \\ C\end{matrix}}{A \vee C}$$

- Dependence distribution rule.
- Dependence elimination rule.

Dependence Distribution Rule

1. Given ϵ, x, y and f.
2. If $\epsilon > 0$, then there is $\delta > 0$ depending only on ϵ such that if $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$.
3. Therefore, there is $\delta > 0$ depending only on ϵ such that if $\epsilon > 0$ and $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$.

Dependence Elimination Rule

1. Assume that for every x and every $\epsilon > 0$ there is $\delta > 0$ depending only on ϵ such that for all y, if $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$.
2. Therefore, for every x and every $\epsilon > 0$ there is $\delta > 0$ such that for all y, if $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$, and moreover, for another x' and $\epsilon' > 0$ there is $\delta' > 0$ such that for all y', if $|x' - y'| < \delta'$, then $|f(x') - f(y')| < \epsilon$ and if $\epsilon = \epsilon'$, then $\delta = \delta'$.

Application: Social choice

- Variables are "voters", values are preference relations.
- A priori, a voter can vote many kinds of preference relations.
- Social welfare function is also a variable, the value of which is a preference relation which depends on the voters preference relations, trying to form a "democratic" common opinion.
- Arrow's Theorem: If the social welfare function respects unanimous opinion, and is independent of irrelevant alternatives, then it is a dictatorship.

Application: Quantum physics (with Abramsky)

- Experimental model (team): Variables are inputs and outcomes of experiments.
- Dependencies express dependence of outcome on input.
- Independences express independence of outcomes from each other.

$$Y = \begin{array}{|ccccc|} x_1 & y_1 & \dots & x_n & y_n \\ \hline a_1^1 & b_1^1 & \dots & a_n^1 & b_n^1 \\ a_1^2 & b_1^2 & \dots & a_n^2 & b_n^2 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ a_1^m & b_1^m & \dots & a_n^m & b_n^m \end{array}$$

Hidden variables

A **hidden variable model** is of the form

$$Y = \begin{array}{|cccccc|} x_1 & y_1 & \dots & x_n & y_n & z \\ \hline a_1^1 & b_1^1 & \dots & a_n^1 & b_n^1 & \gamma^1 \\ a_1^2 & b_1^2 & \dots & a_n^2 & b_n^2 & \gamma^2 \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ a_1^m & b_1^m & \dots & a_n^m & b_n^m & \gamma^m \end{array}$$

where the γ^i are values of a hidden variable z.
A hidden variable model Y **satisfies** the experimental model X if X is the projection of Y to the domain of X.

Single-valuedness

A team X is said to support **single-valuedness of the hidden variable** z if z has only one value in the team.
We can express this with the formula

$$=(z).$$

Outcome-independence

An empirical team X is said to support **outcome - independence** if the following holds:

Suppose the team X has two measurement-outcome combinations s and s' with the same total input data $\vec{x}$ and the same hidden variable z, i.e. $s(\vec{x}) = s'(\vec{x})$ and $s(z) = s'(z)$. We demand that output $s(y_i)$ should occur as an output also if the outputs $s(\{y_j : j \neq i\})$ are changed to $s'(\{y_j : j \neq i\})$.

We can express output-independence with the formula

$$y_i \perp_{\vec{x},z} \{y_j : j \neq i\}.$$

- Einstein-Podolsky-Rosen result: There is an empirical model (team) which cannot be realized by any hidden variable models satisfying single-valuedness and outcome independence.

Summary

- There is a logic of dependence and independence, giving a unified treatment of concepts from mathematics, computer science, statistics, etc.
- It gives a new analysis of NP.
- There is a natural axiomatization, but limits to completeness.
- It can be applied to gain understanding in a variety of fields (e.g. social choice, quantum physics).

Thank you!