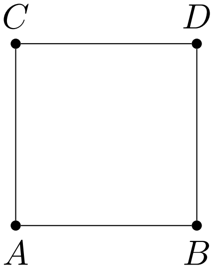
Lower bounds for the resolution width problem and existential pebble games

Christoph Berkholz

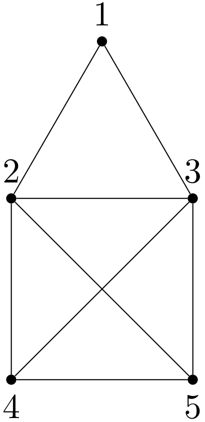
HU Berlin

FMT 2012

The homomorphism problem



$\mathcal{A}$



$\mathcal{B}$

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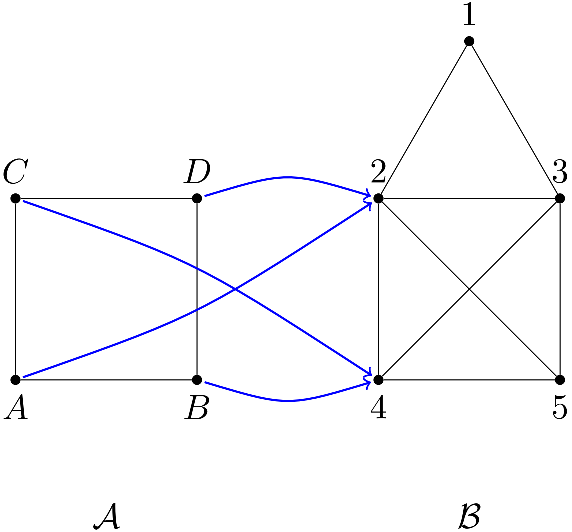
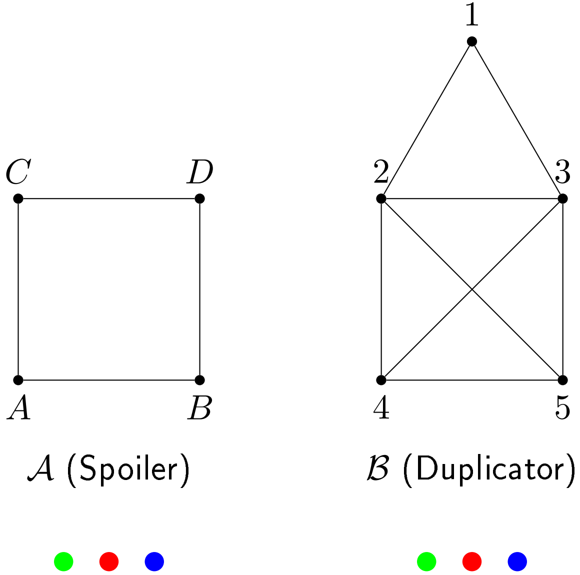


The existential k-pebble game



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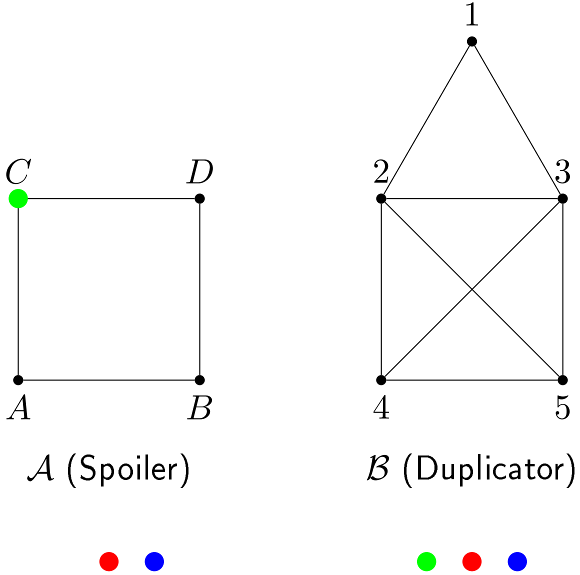

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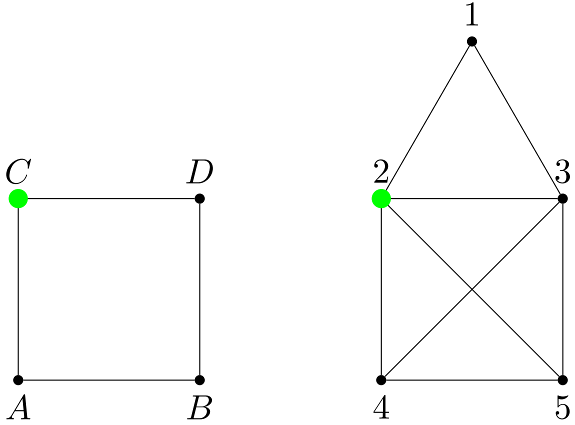

$\mathcal{A}$ (Spoiler) $\mathcal{B}$ (Duplicator)

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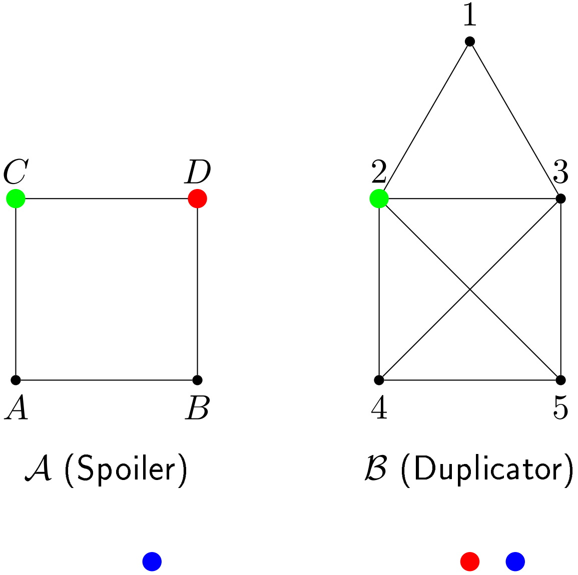

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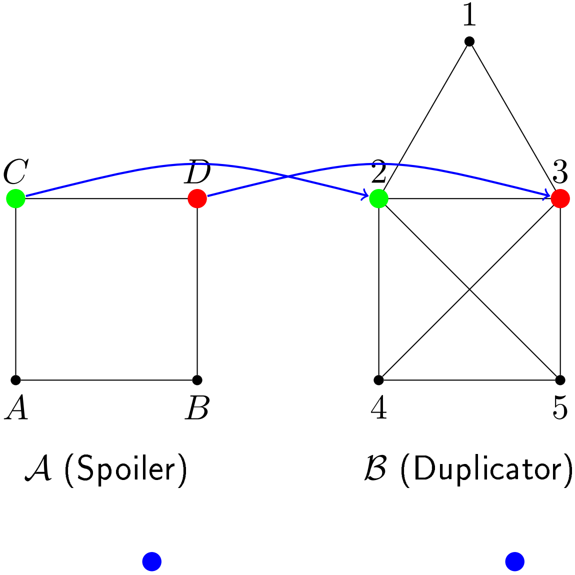

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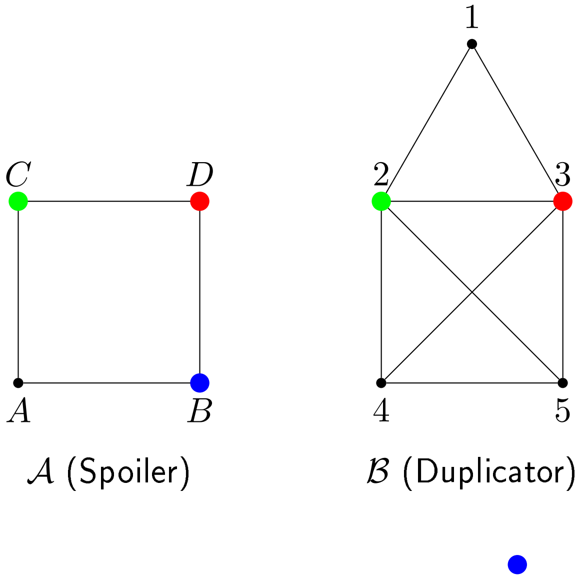

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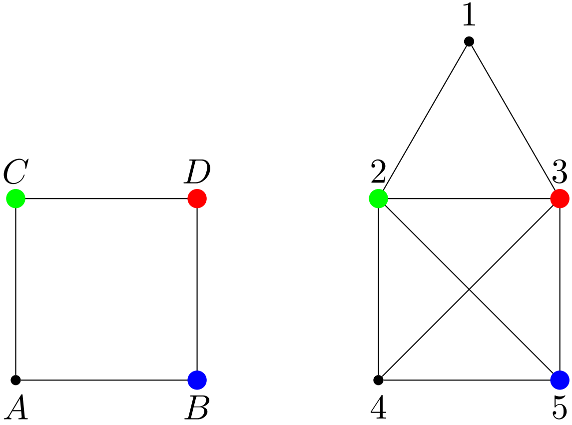

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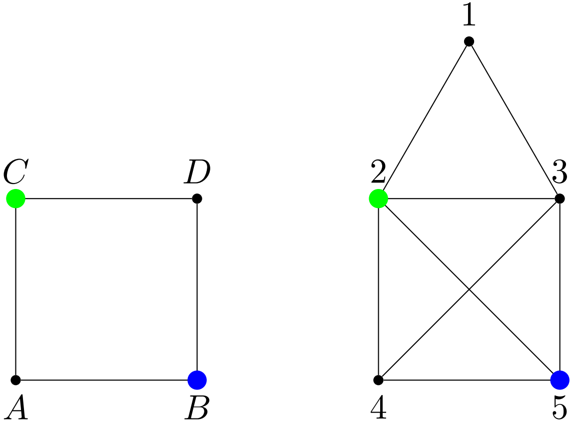

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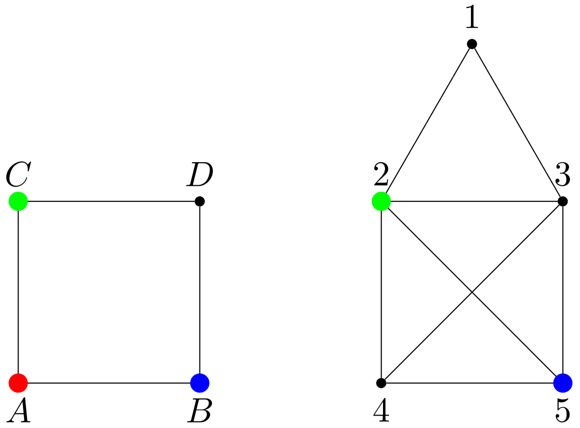

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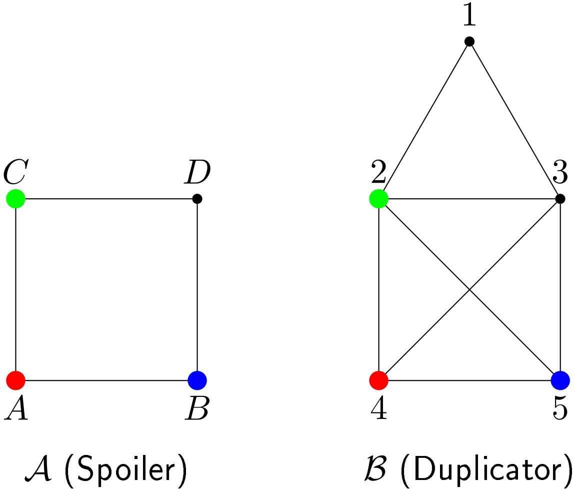

The existential k-pebble game

TFAE on finite relational $\mathcal{A}$, $\mathcal{B}$ (Kolaitis, Vardi '95; '00):

- Duplicator wins the ex. k-pebble game on $\mathcal{A}$ and $\mathcal{B}$.

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- For all $\varphi \in \exists$-pos FO^k: $\mathcal{A} \models \varphi \Longrightarrow \mathcal{B} \models \varphi$.

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Given two finite structures, does Duplicator win the existential k-pebble game on these structures?

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- k parameter: XP-complete. (B. '12)

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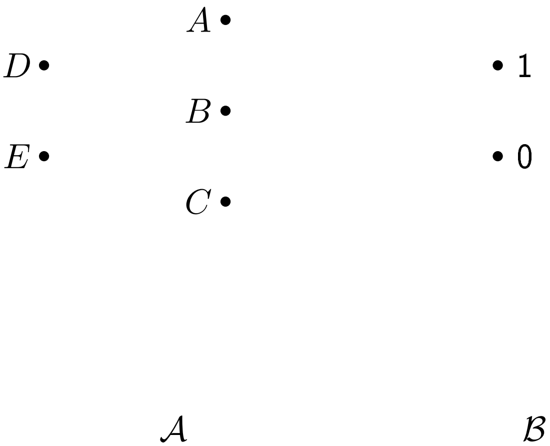
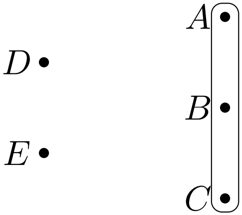


The Boolean existential k-pebble game

$$\Gamma = \{A \vee B \vee \overline{C},$$



$\{(A, B, C)\}$ $\qquad$ $\{0,1\}^3 \setminus \{(0,0,1)\}$

$\mathcal{A}(\Gamma)$ $\qquad$ $\mathcal{B}$

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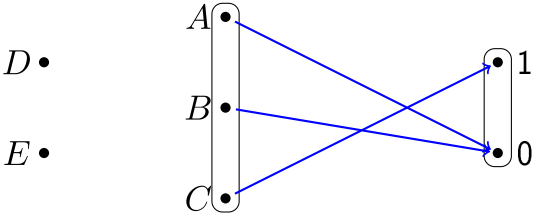

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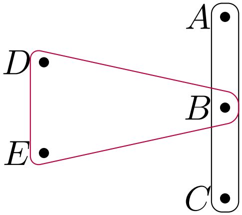

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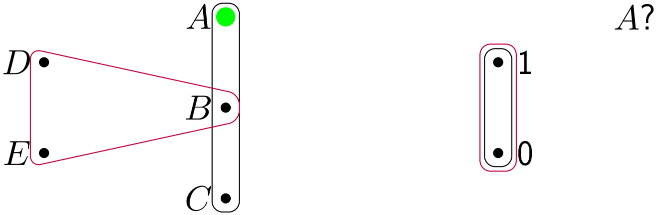

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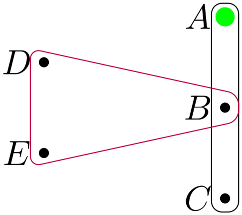

$A \mapsto 1$

$\{(A, B, C)\}$

$\{(B, D, E)\}$

$\{0,1\}^3 \setminus \{(0,0,1)\}$

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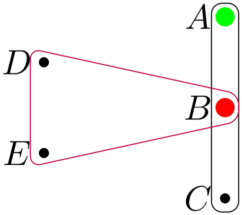

$A \mapsto 1$

$B?$

$\{(A, B, C)\}$

$\{(B, D, E)\}$

$\{0,1\}^3 \setminus \{(0,0,1)\}$

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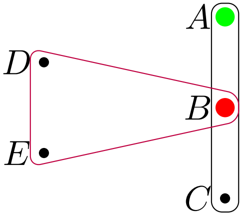
$\mathcal{A}(\Gamma)$ $\quad\quad$ $\mathcal{B}$

The Boolean existential k-pebble game

$$\Gamma = \{A \vee B \vee \overline{C}, \quad \overline{B} \vee D \vee \overline{E}, \quad \ldots\}$$



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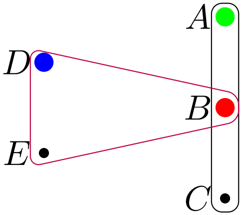

$A \mapsto 1$

$B \mapsto 1$

$D?$

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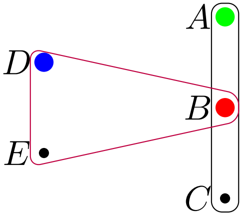

$A \mapsto 1$
$B \mapsto 1$
$D \mapsto 0$

$\{(A, B, C)\}$ $\{0,1\}^3 \setminus \{(0,0,1)\}$

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$\mathcal{A}(\Gamma)$ $\mathcal{B}$

Bounded width resolution

Spoiler wins the Boolean ex. $(k+1)$-pebble game on Γ
$\iff \Gamma$ has a width-k resolution refutation (Atserias, Dalmau '08)

$$\Gamma = \{D \vee \overline{E},\quad C \vee D \vee E,\quad A \vee B \vee \overline{C},\quad \overline{A} \vee D \vee E,\quad \overline{B} \vee \overline{C} \vee D,\quad \overline{D}\}$$

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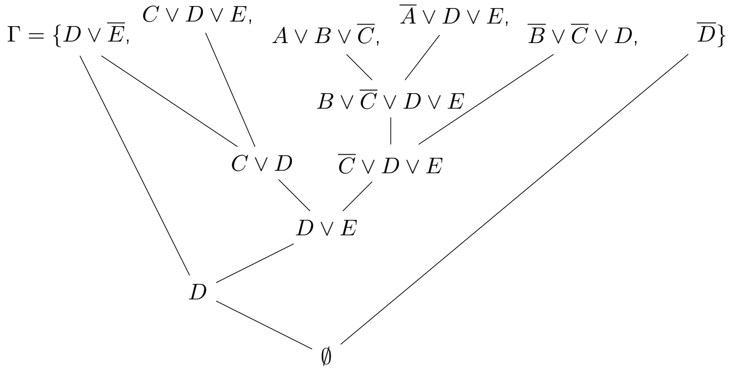

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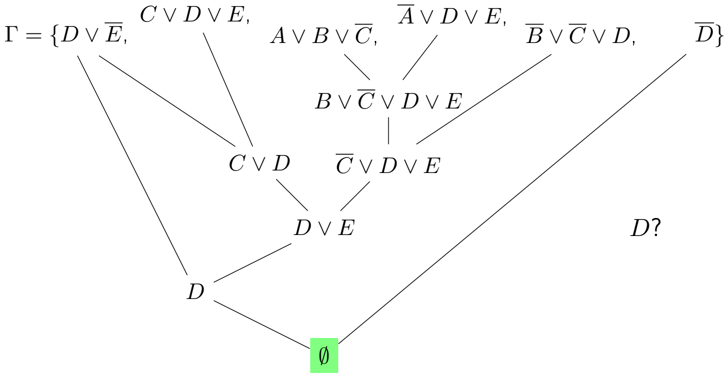

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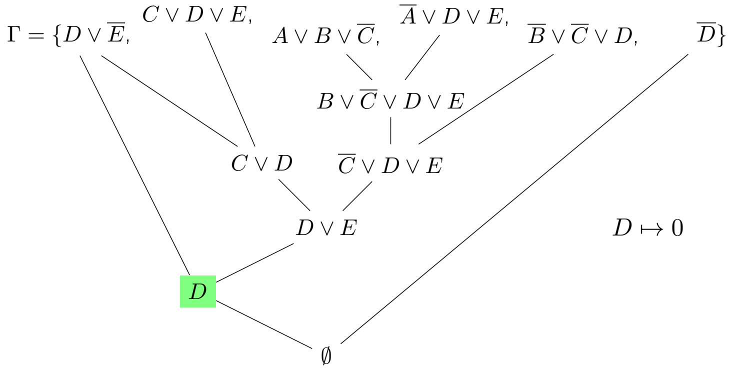

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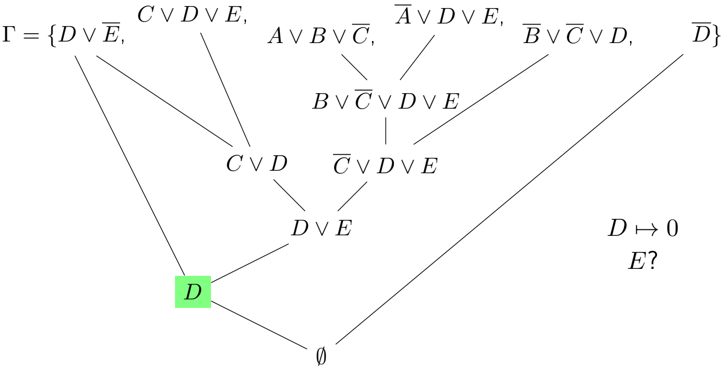

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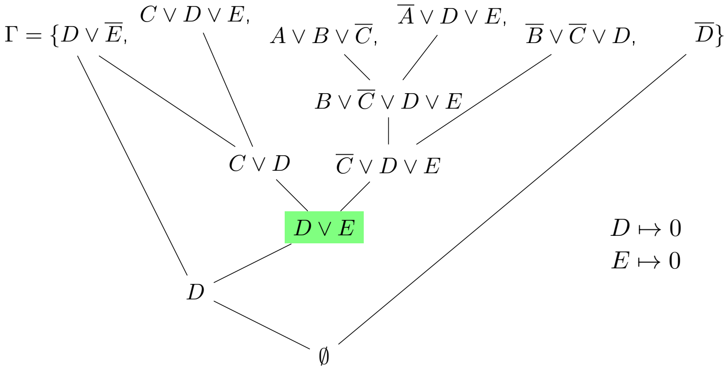

$D \mapsto 0$
$E \mapsto 0$

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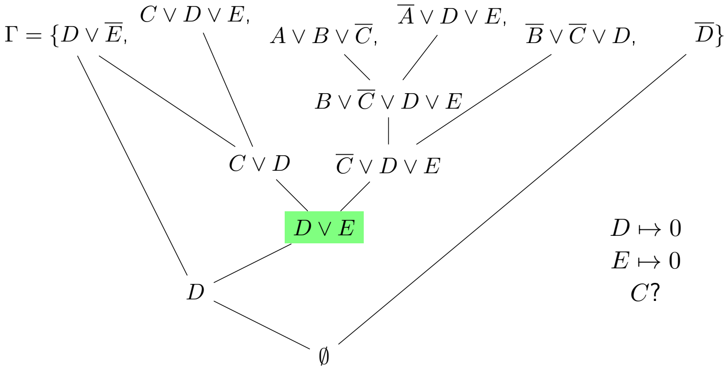

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$C?$

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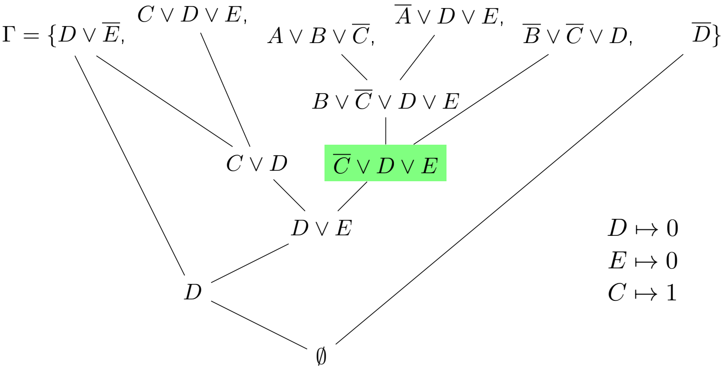

$D \mapsto 0$
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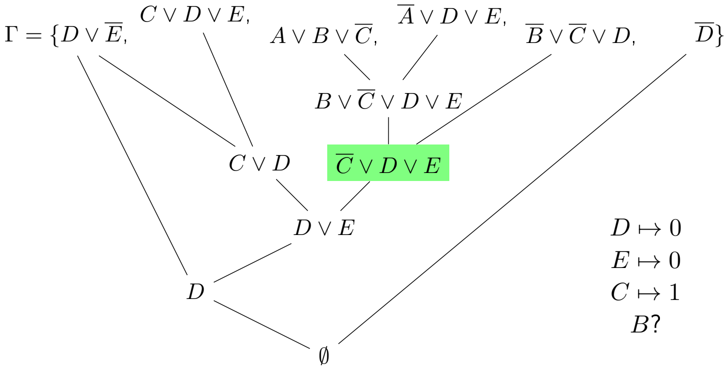

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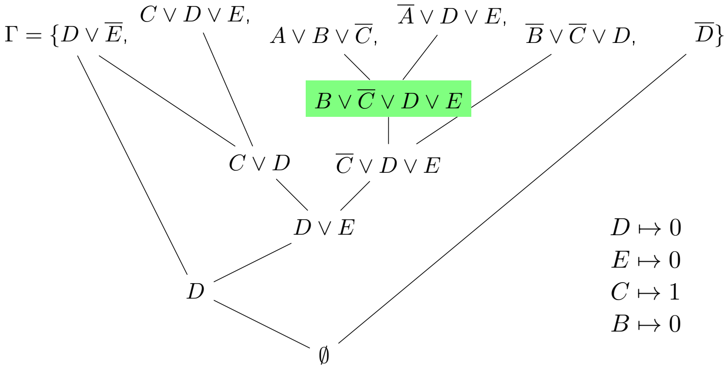

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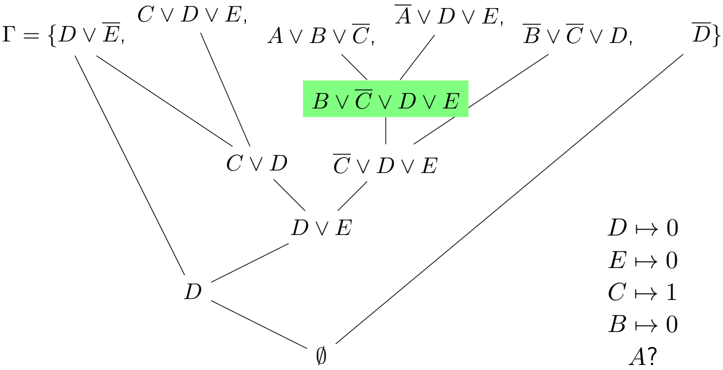

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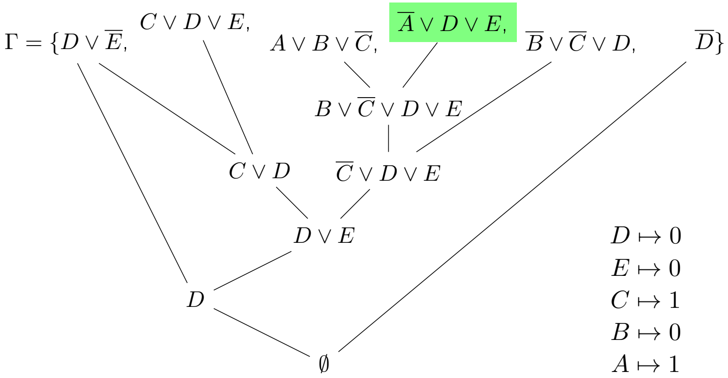

$D \mapsto 0$
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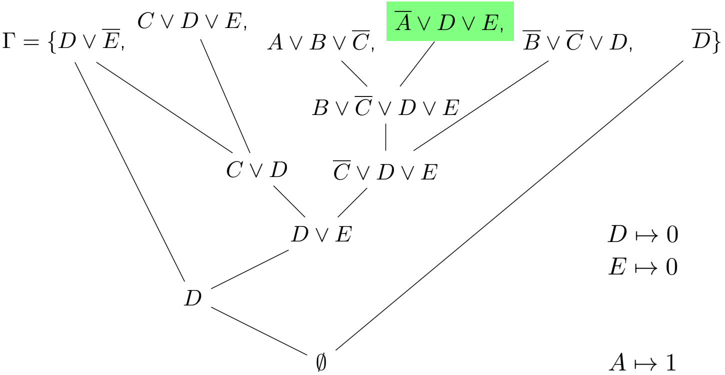
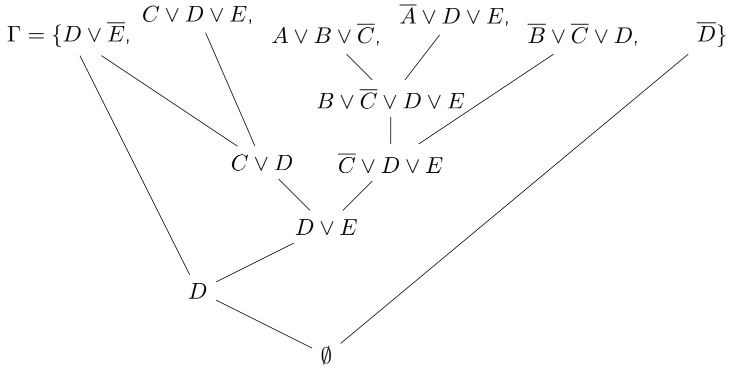


Bounded width resolution

Spoiler wins the regular Boolean ex. $(k+1)$-pebble game on Γ
$\iff$ Γ has a regular width-k resolution refutation (Hertel '08)



Complexity

(Regular) resolution width problem

Given a 3-CNF Γ, does there exist a (regular) resolution refutation of width k?

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Theorem (B. 12)

The resolution width problem is

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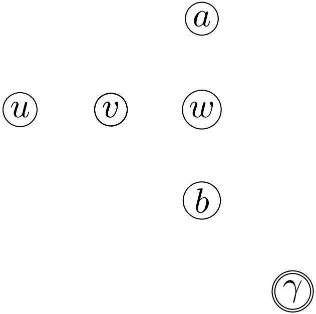
The *regular* *resolution width problem is*

- k *part of input:* PSPACE-*complete.*

The k-pebble KAI-game



- nodes X, goal $\gamma \in X$

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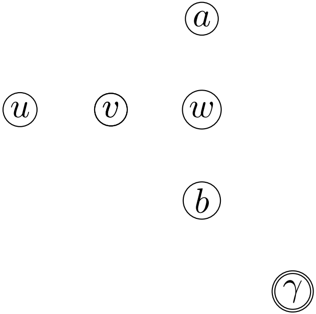

- nodes X, goal $\gamma \in X$
- k pebbles

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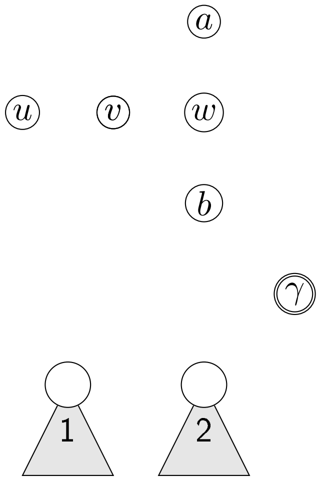
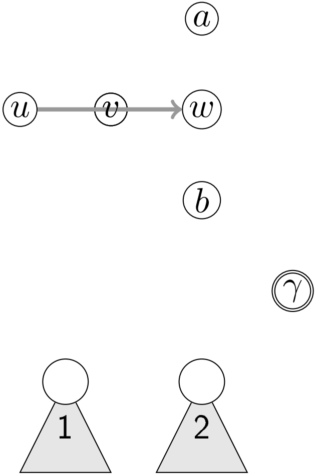


- nodes X, goal $\gamma \in X$
- k pebbles
- 2 players

The k-pebble KAI-game



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- 2 players
- rules (u, v, w)

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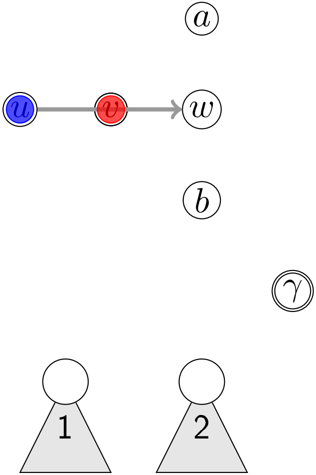

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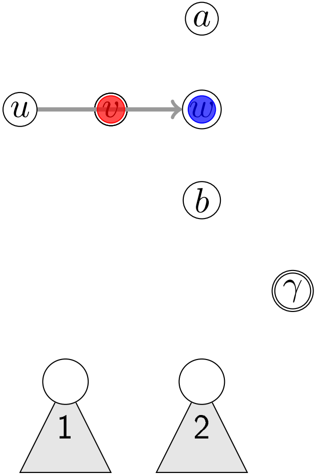
The k-pebble KAI-game



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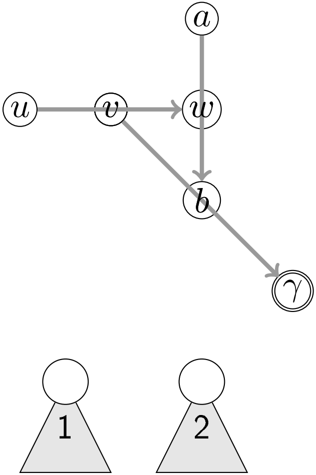

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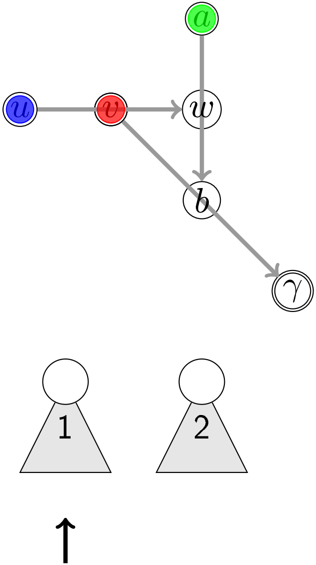

- nodes X, goal $\gamma \in X$
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- start position ● $\mapsto a$, ● $\mapsto v$, ● $\mapsto u$

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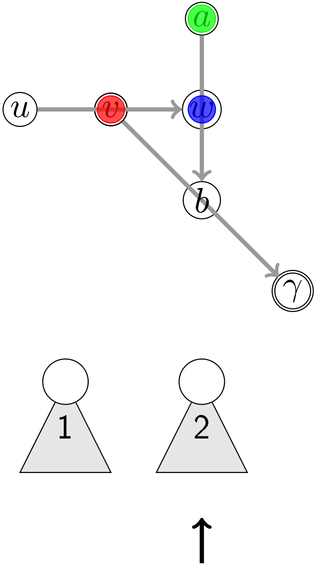

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- k pebbles (green) (blue) (red)
- 2 players
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- start position (green) $\mapsto a$, (red) $\mapsto v$, (blue) $\mapsto u$

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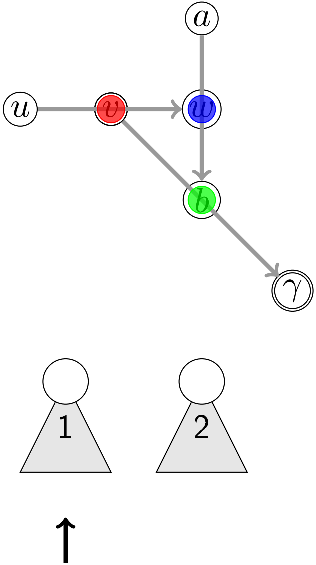

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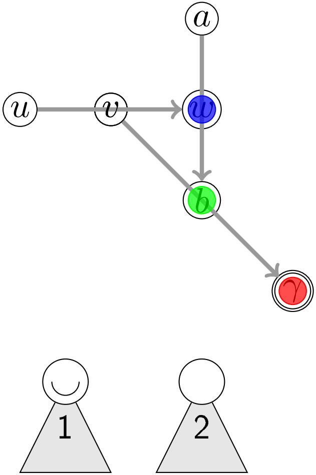

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(Acyclic) KAI-game problem

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The acyclic KAI-game problem is

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The reduction

We reduce the k-pebble KAI-game on I to the Boolean existential $(k+2)$-pebble game on $\Gamma(I)$.

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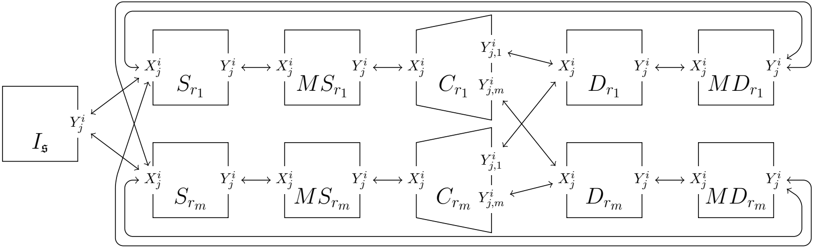

Thank you for your attention.