INDISTINGUISHABILITY IN COUNTING LOGICS AND LINEAR PROGRAMMING RELAXATIONS

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Part I

LINEAR PROGRAMMING RELAXATIONS

Vertex cover

Problem:

Given an undirected graph $G = (V, E)$,
find the smallest number of vertices
that touches every edge.

Notation:

For a graph G, we write

$$\mathrm{vc}(G)$$

Linear programming relaxation

LP relaxation:

$$\begin{aligned}&\text{minimize } \textstyle\sum_{u \in V} x_u \\ &\text{subject to} \\ &\quad x_u + x_v \geq 1 \quad \text{for every } (u, v) \in E, \\ &\quad x_u \geq 0 \qquad\quad\ \text{for every } u \in V.\end{aligned}$$

Notation:

For a graph G, we write

$$\mathrm{vc}^{\mathbb{R}}(G)$$

Approximation

Approximation:

$$\mathrm{vc}^{\mathbb{R}}(G) \leq \mathrm{vc}(G) \leq 2 \cdot \mathrm{vc}^{\mathbb{R}}(G)$$

Integrality gap:

$$\sup_G \frac{\mathrm{vc}(G)}{\mathrm{vc}^{\mathbb{R}}(G)}$$

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Gap examples:

1. $\mathrm{vc}(K_{2n+1}) = 2n$,
2. $\mathrm{vc}^{\mathbb{R}}(K_{2n+1}) = \frac{1}{2}(2n+1)$.

LP tightenings

Add triangle inequalities:

minimize $\sum_{u \in V} x_u$

subject to

$$
\begin{aligned}
& x_u + x_v \geq 1 && \text{for every } (u, v) \in E, \\
& x_u \geq 0 && \text{for every } u \in V, \\
& x_u + x_v + x_w \geq 2 && \text{for every triangle } \{u, v, w\} \text{ in } G.
\end{aligned}
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Integrality gap:

Remains 2.

Gap examples:

Triangle-free graphs with small independence number.

Sherali-Adams hierarchy

Sherali-Adams hierarchy:

A systematic way of
generating all linear inequalities
that are valid over the integral hull.

Given a polytope:

$$P = \{x \in \mathbb{R}^n : Ax \geq b\},$$
$$P^{\mathbb{Z}} = \mathrm{convexhull}\{x \in \{0,1\}^n : Ax \geq b\}.$$

Produces explicit nested polytopes:

$$P = P^1 \supseteq P^2 \supseteq \cdots \supseteq P^{n-1} \supseteq P^n = P^{\mathbb{Z}}$$

Definition of P^k in four steps

Step 1: Lift

Multiply each $a_i^{\mathrm{T}} x \geq b_i$ by all multipliers of the form

$$\prod_{i \in I} x_i \prod_{j \in J} (1 - x_j)$$

where $I, J \subseteq [n]$, $|I \cup J| \leq k - 1$, and $I \cap J = \emptyset$.

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Comment:

Leaves an equivalent system of degree-k polynomials.
It's the lifting step.

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Step 2: Enforce 0-1

Expand the products and
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Comment:

Leaves degree-k multi-linear polynomials valid only on $\{0,1\}^n$.
It's the integrality step.

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Linearize each monomial $\prod_{i \in I} x_i$
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Comment:

Leaves a linear program $Q^k \subseteq \mathbb{R}^{n^k}$ on the y_I-variables.
It's the relaxation step.

Definition of P^k in four steps

Step 4: Project

Take all positive linear combinations
that cancel all y_I-variables with $|I| > 1$.

$$P^k := \{x \in \mathbb{R}^n : \exists y \in Q^k \text{ s.t. } y_{\{i\}} = x_i \text{ for every } i\}.$$

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Comment:

Leaves another linear program back on the x_i-variables in $\mathbb{R}^n$.
It's the projection step: from $\mathbb{R}^{n^k}$ to $\mathbb{R}^n$.

Solving P^k

Lift-and-project:

- Step 1: lift
- Step 2: enforce 0-1
- Step 3: relax
- Step 4: project

Note:

The polytope P^k is definable by
an LP on n^k variables and $m \cdot n^k$ inequalities.

Therefore:

Feasibility and optimization of linear functions over P^k
can be solved in time $m^{O(1)} n^{O(k)}$.

Part II

INDISTINGUISHABILITY IN COUNTING LOGICS

Counting quantifiers

Counting witnesses:

$\exists^{\geq i} x(\phi(x))$: there are at least i vertices x that satisfy $\phi(x)$.

Counting logic with k variables:

C^k: collection of formulas for which
all subformulas have at most k free variables.

C^k**-equivalence**:

$G \equiv_k^C H$: G and H satisfy the same sentences of C^k.

Combinatorial characterization of C^2-equivalence

Color-refinement:

1. color each vertex black,
2. color each vertex by number of neighbors in each color-class,
3. repeat 2 until color-classes don't split any more.

Notation:

$G \equiv^R H$: G and H produce the same coloring (up to order).

Theorem [Immerman and Lander]

$$G \equiv_2^C H \text{ if and only if } G \equiv^R H$$

LP characterization of color-refinement

The following are equivalent:

1. $G \cong H$,
2. there exists permutation matrix P such that $P^{\mathrm{T}} G P = H$,
3. there exists permutation matrix P such that $GP = PH$.

Define the LP relaxation of $\cong$:

$G \equiv^F H$: there exists doubly stochastic S such that $GS = SH$.

$$\begin{aligned} F \quad : \quad & GS = SH \\ & Se = e^{\mathrm{T}} S = e \\ & S \geq 0. \end{aligned}$$

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$$S \geq 0.$$

Theorem [Ramana, Scheinerman and Ullman]

$$G \equiv^R H \text{ if and only if } G \equiv^F H.$$

LP characterization of C^2-equivalence

Corollary:

$$G \equiv_2^C H \iff G \equiv^F H$$

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Next question:

Could $\equiv_3^C$-equivalence be related to
level-2 Sherali-Adams of $\equiv^F$?

Higher levels of Sherali-Adams

SA-levels of fractional isomorphism:

$$G \equiv_k^F H : \text{ the level-}k\text{ Sherali-Adams of } F \text{ is feasible.}$$

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Theorem [AA and Maneva 2012]:

$$G \equiv_k^F H \Longrightarrow G \equiv_k^C H \Longrightarrow G \equiv_{k-1}^F H.$$

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Theorem [AA and Maneva 2012]:

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Moreover:

1. This interleaving is strict for $k > 2$! [Grohe-Otto 2012]
2. A combined LP characterizes $\equiv_k^C$ exactly. [Grohe-Otto 2012]

Weak precusor for homomorphisms

Fractional homomorphisms vs. arc-consistency [Missing ref.]

$$G \to^F H \iff G \text{ is } H\text{-arc-consistent}$$

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LP-formulation [AA-Kolaitis-Vardi 2006]

$$G \rightarrow^F_k H \Longrightarrow G \leq_k^{\exists FO^+} H$$

Part III

APPLICATIONS

Two examples of applications

From combinatorial optimization to finite model theory:

New expressibility results

From finite model theory to combinatorial optimization:

New integrality gap constructions

Local LPs

Basic k-local LPs:

1. one variable $x_{\mathbf{u}}$ for each k-tuple $\mathbf{u} \in V^k$,
2. one inequality $\sum_{\mathbf{u} \in V^k} a_{\mathbf{u},\mathbf{v}} \cdot x_{\mathbf{u}} \geq b_{\mathbf{v}}$ for every k-tuple $\mathbf{v} \in V^k$,
3. coefficients $a_{\mathbf{u},\mathbf{v}}$ depend only on the type $\mathrm{atp}_G(\mathbf{u}, \mathbf{v})$,
4. coefficients $b_{\mathbf{v}}$ depend only on the type $\mathrm{atp}_G(\mathbf{v})$.

Generic k-local LPs:

Unions of generic basic k-local LPs
with coefficients $a_{t(\mathbf{x},\mathbf{y})}$ and $b_{t(\mathbf{y})}$ indexed
by complete atomic types $t(\mathbf{x}, \mathbf{y})$ and $t(\mathbf{y})$.

Instantiation of generic k-local LPs:

For a graph G, we write

$$P(G)$$

Example: metric polytope

Metric polytope: Given a graph $G = (V, E)$

$$
\begin{array}{ll}
\frac{1}{2}\sum_{uv\in E} x_{uv} \geq W & \\
x_{uv} = x_{vu} & \text{for every } u, v \in V \\
x_{uw} \leq x_{uv} + x_{vw} & \text{for every } u, v, w \in V \\
x_{uv} + x_{vw} + x_{uw} \leq 2 & \text{for every } u, v, w \in V \\
0 \leq x_{uv} \leq 1 & \text{for every } u, v \in V
\end{array}
$$

1. Objective function: basic 2-local LP
2. Symmetry constraint: two basic 2-local LPs
3. Triangle inequality: basic 3-local LP
4. Perimetric inequality: basic 3-local LP
5. Unit cube constraint: two basic 2-local LPs

Preservation of local LPs

Theorem [AA and Maneva]: Let P be a generic k-local LP.

If $G \equiv_k^F H$, then $P(G)$ is feasible iff $P(H)$ is feasible.

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'Just do it' proof:

1. Let $\{x_{\mathbf{u}}\}$ be a feasible solution for $P(G)$.
2. Let $\{X_{\mathbf{u},\mathbf{v}}\}$ be a feasible solution for $I(G,H)^k$.
3. Define:

$$y_{\mathbf{v}} := \sum_{\mathbf{u} \in G^k} X_{\mathbf{u},\mathbf{v}} \cdot x_{\mathbf{u}}.$$

4. Check that $\{y_{\mathbf{v}}\}$ is a feasible solution for $P(H)$.

More examples of local LPs

More examples:

1. maximum flows (2-local)
2. matchings on bipartite graphs (2-local)
3. fractional max-cut via the metric polytope (3-local)
4. fractional vertex cover (2-local)
5. level-r SA of k-local LPs are $O(kr)$-local LPs.

New expressibility results

Consider the max-flow LP on *st*-networks.
It is 2-local. It is integral.

Corollary:

$$G \equiv_3^C H \Rightarrow \mathrm{maxflow}(G) = \mathrm{maxflow}(H).$$

Corollary:

There exists a sentence in C^3 that,
over *st*-networks with n vertices, defines those
whose maximum flow is at least the out-degree of the source.

New expressibility results

Consider the metric polytope again.

Theorem [Barahona-Majoub 86]:

If G is a K_5 minor-free graph,
then the metric polytope of G is integral.

Corollary:

If G and H are K_5 minor-free,
then $G \equiv_4^C H \Rightarrow \mathrm{maxcut}(G) = \mathrm{maxcut}(H)$.

Corollary:

There exists a sentence in C^4 that,
over K_5 minor-free n-vertex graphs, defines those
whose max-cut is at least $n/4$.

New integrality gap constructions

Theorem [follows also from Schoenebeck 08]:

There exist n-vertex graphs G for which
$\Omega(n)$-levels of SA over the metric polytope
do not achieve $\mathrm{maxcut}^{\mathbb{R}}(G) = \mathrm{maxcut}(G)$.

New integrality gap constructions

(New) proof (sketch):

1. Start with the n-vertex CFI graphs $G \equiv^{C}_{\Omega(n)} H$ yet $G \not\cong H$.
2. In particular $(G, G) \equiv^{C}_{\Omega(n)} (G, H)$ yet $G \not\cong H$.
3. Apply the reduction from graph isomorphism to max-cut.
4. Get graphs $A \equiv^{C}_{\Omega(n)} B$ with $\mathrm{maxcut}(A) \neq \mathrm{maxcut}(B)$.
5. Apply transfer lemma: $A \equiv^{F}_{\Omega(n)} B$.
6. Apply preservation: $\mathrm{maxcut}^{\mathbb{R}}(A) = \mathrm{maxcut}^{\mathbb{R}}(B)$ over $P^{\Omega(n)}$.

Part IV

OPEN PROBLEMS

Get new integrality gaps

Challenging problem:

Prove that an integrality gap of $2 - o(1)$
resists $\Omega(n)$ SA-levels of vertex-cover
on n-vertex graphs.

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What would be enough?:

Find n-vertex graphs G and H such that:

1. $\mathrm{vc}(G) \geq (2 - o(1)) \cdot \mathrm{vc}(H)$
2. $G \equiv^{C}_{\Omega(n)} H$.

Get new inexpressibility results

Challenging problem:

Is perfect matching axiomatizable in $C^{O(1)}$?

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Is perfect matching axiomatizable in $C^{O(1)}$?

What would be enough for non-axiomatizability?:

Find G and H such that:

1. G has a perfect matching but H does not,
2. $G \equiv^{F}_{\omega(1)} H$.

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Challenging problem:

Is perfect matching axiomatizable in $C^{O(1)}$?

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1. G has a perfect matching but H does not,
2. $G \equiv^{F}_{\omega(1)} H$.

Important note:

In the bipartite case no such G and H exist
by [Blass, Gurevich and Shelah]

Final Slide

LET'S EXPLOIT THIS CONNECTION