

Recognisability
and
Algebras of Infinite Trees

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Algebraic Language Theory

Recognisability

φ : free algebra \rightarrow finite algebra

$$L = \varphi^{-1}[P]$$

Which algebras?

finite words monoid/semigroup

infinite words ω -semigroup

finite trees clones, preclones, term algebras, forest algebras,...

infinite trees ?

ω -semigroups

$\langle S, S_\omega \rangle$ with associative operations

- $S \times S \rightarrow S$
- $S \times S_\omega \rightarrow S_\omega$
- $S^\omega \rightarrow S_\omega$

Example

$S := [2], S_\omega := [2]$ with products

- $S \times S \rightarrow S : (s, t) \mapsto \max \{s, t\}$
- $S \times S_\omega \rightarrow S_\omega : (s, u) \mapsto u$
- $S^\omega \rightarrow S_\omega : (s_n)_n \mapsto \limsup_{n \rightarrow \infty} s_n$

recognises the set of all ω -words containing infinitely many letters a .

($a \mapsto 1$ and $b \mapsto 0$)

Wilke algebras

Replace infinite product $S^\omega \rightarrow S_\omega$ by ω -power operation $S \rightarrow S^\omega$.

Theorem The infinite product is uniquely determined by the ω -power operation.

Proof Every infinite product $\pi(a_0, a_1, \dots)$ has a factorisation

$$\underbrace{(a_0 \cdots a_{k-1})}_s \cdot \underbrace{(a_k \cdots a_{l-1})}_u \cdot \underbrace{(a_l \cdots a_{m-1})}_u \cdots$$

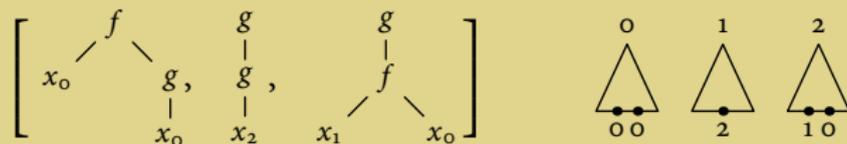
(Theorem of Ramsey). Hence,

$$\pi(a_0, a_1, \dots) = s \cdot u^\omega.$$

Algebras for infinite trees

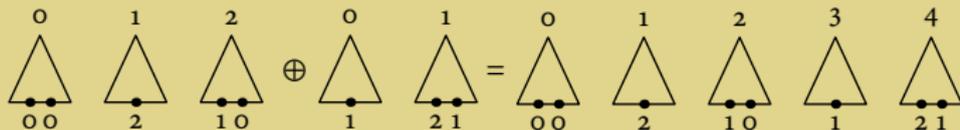
Term algebra

Elements: tuples of finite and infinite terms with variables x_0, x_1, \dots

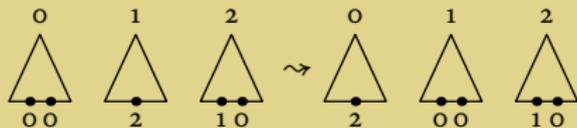


Operations

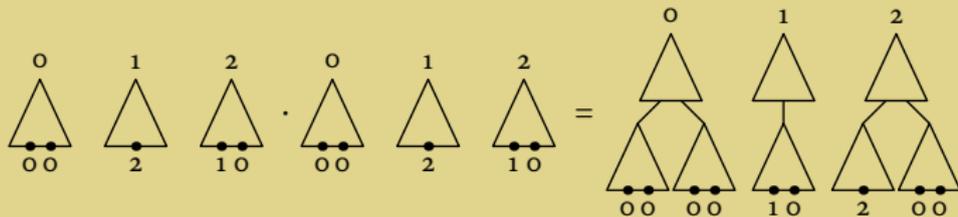
- horizontal product:** concatenation



- reordering**



- **vertical product:** substitution



- **infinite product**

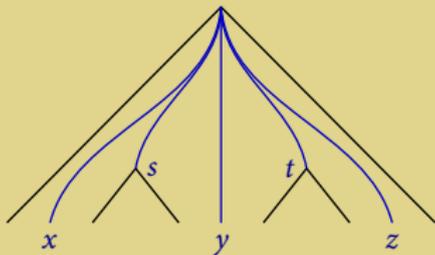
Hyperclones: homomorphic images of the term algebra

Automata to hyperclones

Path-hyperclones

hyperclones associated with an ω -semigroup $\langle S, S_\omega \rangle$

Elements: $\wp(S^n \times \wp(S_\omega))^{<\omega}$



$(s, t, \{x, y, z\})$

Example

$S := [2]$, $S_\omega := [2]$ with \max as product.

The corresponding hyperclone recognises the set of all trees containing at least one vertex with label a . ($a \mapsto 1$ and $b \mapsto 0$)

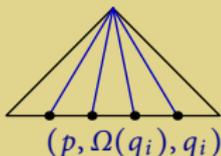
ω -semigroup for an automaton

$$S := Q \times D \times Q \cup \perp \quad S_\omega := Q \cup \perp$$

$$(p, d, q) \cdot (p', d', q') := \begin{cases} (p, \min\{d, d'\}, q') & \text{if } q = p' \\ \perp & \text{otherwise} \end{cases}$$

homomorphism

transition $(p, a, q_0, \dots, q_{n-1})$

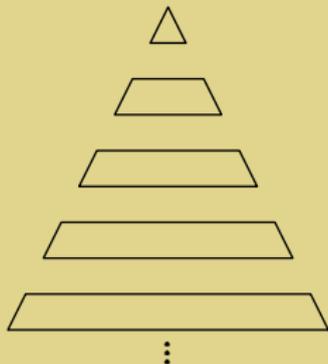


Theorem

The hyperclone associated with an automaton \mathcal{A} recognises $L(\mathcal{A})$.

Hyperclones to automata

Problem: To evaluate an infinite product



an automaton needs to access each “slice”.

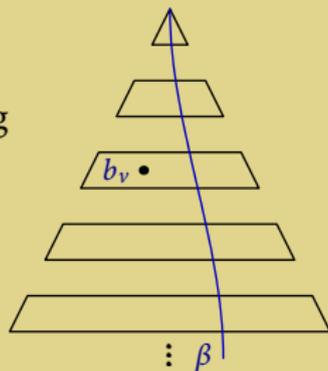
Idea: Compute infinite products one branch at a time.

Fix a branch β .

For each $v \notin \beta$, guess the value b_v of the subtree.

Insert these values and compute the remaining product.

$\text{Tr}(\beta)$: the values collected in this way



Definition

A hyperclone is **path-continuous** if $\pi(a_0, a_1, \dots)$ is determined by $\{ \text{Tr}(\beta) \mid \beta \text{ a branch} \}$.

Examples

(a) Every path-hyperclone is path-continuous.

(b) The following hyperclone is not path-continuous:

Elements: $[4]^n$

$$a \cdot \bar{b} := \max \{a, b_0, \dots, b_{n-1}\}$$

$$\pi(a_0, a_1, \dots) := \max (\{x\} \cup \{a_i \mid i < \omega\})$$

$$x := \begin{cases} 3 & \text{if there are infinitely many branches from which} \\ & \text{you can always reach a value } > 0 \\ 2 & \text{if there are finitely many such branches} \\ 0 & \text{if there are no such branches} \end{cases}$$

Theorem

Let L be a set of infinite trees. The following statements are equivalent:

- (1) L is recognised by a **tree automaton**.
- (2) L is recognised by a **finitary, path-continuous hyperclone**.
- (3) L is recognised by a **path-hyperclone** associated with a finite ω -semigroup.

Wilke algebras

Problem

To specify a hyperclone we need an infinite amount of data:

- infinitely many sorts
- infinite product

Solution

- We only need to consider finitely many sorts at a time.
- We can replace π by the ω -power ${}^\omega: a^\omega := \pi(a, a, \dots)$

ω -power

Obviously π determines ${}^\omega$.

Question: what about the converse?

Theorem

For finitary, path-continuous hyperclones, ${}^\omega$ determines π .

Proof idea

We have to evaluate $\pi(a_0, a_1, \dots)$ using only \cdot and ${}^\omega$.

Find a regular sequence a'_0, a'_1, \dots such that

$$\pi(a_0, a_1, \dots) = \pi(a'_0, a'_1, \dots).$$

We can represent a'_0, a'_1, \dots as $u \cdot v^\omega$ and set

$$\pi(a_0, a_1, \dots) := u \cdot v^\omega.$$

Path labellings

Additive labelling

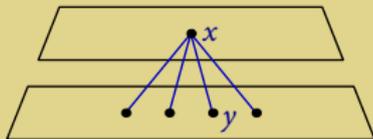
T tree, $\langle S, S_\omega \rangle$ ω -semigroup, $\lambda : \{ (x, y) \in T^2 \mid x < y \} \rightarrow S$

$$\lambda(x, z) = \lambda(x, y) \cdot \lambda(y, z) \quad \text{for } x < y < z$$

Labelling for a hyperclone

If y is the k -th successor of x define

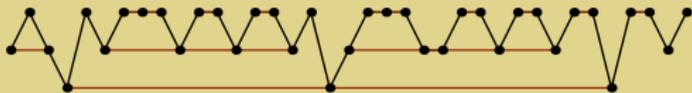
$$\lambda(x, y) := \{ a \cdot_k b \mid a \text{ the element at } x, b \text{ arbitrary} \}$$



Ramseyan splits

λ additive labelling of T , $\sigma : T \rightarrow [k]$

- $x \sqsubseteq_{\sigma} y$ iff $x \leq y$ and $\sigma(x) = \sigma(y) \leq \sigma(z)$, for all $x \leq z \leq y$



- σ is a **Ramseyan split** if

$$\lambda(u, v) \cdot \lambda(x, y) = \lambda(u, v) \quad \text{for all } u \sqsubseteq_{\sigma} x \sqsubset_{\sigma} y, u \sqsubset_{\sigma} v \text{ with } v, y \text{ comparable}$$

Theorem (Colcombet)

Every additive labelling λ has a Ramseyan split $\sigma : T \rightarrow [k]$ with $k \leq |S|$.

Theorem

Let λ be an additive labelling of T . There exists a prefix $T_0 \subseteq T$ of bounded height (in terms of S) with back-edges such that

$$\{ \lambda(\beta) \mid \beta \text{ a branch of } T_0 \} = \{ \lambda(\beta) \mid \beta \text{ a branch of } T \}.$$

Hyperclones and monadic second-order logic

Lemma

The class of languages recognised by finitary, path-continuous hyperclones is effectively closed under

- boolean operations,
- projection.

Theorem

For every MSO-formula φ , we can effectively construct a hyperclone recognising $L(\varphi)$.

Corollary (Rabin)

It is decidable whether an MSO-formula has a tree model.

Summary

- equivalence automata \leftrightarrow hyperclones
- Wilke algebras
- translation MSO \mapsto hyperclones

To do

- simplify definitions
- effective characterisations
- pseudo-varieties