Algorithms for Omega-Regular Games with Imperfect Information *

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Abstract

We study observation-based strategies for two-player turn-based games on graphs with omega-regular objectives. An observation-based strategy relies on imperfect information about the history of a play, namely, on the past sequence of observations. Such games occur in the synthesis of a controller that does not see the private state of the plant. Our main results are twofold. First, we give a fixed-point algorithm for computing the set of states from which a player can win with a deterministic observation-based strategy for any omega-regular objective. The fixed point is computed in the lattice of antichains of state sets. This algorithm has the advantages of being directed by the objective and of avoiding an explicit subset construction on the game graph. Second, we give an algorithm for computing the set of states from which a player can win with probability 1 with a randomized observation-based strategy for a Büchi objective. This set is of interest because in the absence of perfect information, randomized strategies are more powerful than deterministic ones. We show that our algorithms are optimal by proving matching lower bounds.

1 Introduction

Two-player games on graphs play an important role in computer science. In particular, the *controller synthesis* problem asks, given a model for a plant, to construct a model for a controller such that the behaviors resulting from the parallel composition of the two models respects a given specification (e.g., are included in an ω-regular set). Controllers can be synthesized as winning strategies in a game graph whose vertices represent the plant states, and whose players represent the plant and the controller [18, 17]. Other applications of game graphs include realizability and compatibility checking, where the players represent parallel processes of a system, or its environment [1, 11, 6].

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Most results about two-player games played on graphs make the hypothesis of *perfect information*. In this setting, the controller knows, during its interaction with the plant, the exact state of the plant. In practice, this hypothesis is often not reasonable. For example, in the context of hybrid systems, the controller acquires information about the state of the plant using sensors with finite precision, which return imperfect information about the state. Similarly, if the players represent individual processes, then a process has only access to the public variables of the other processes, not to their private variables [19, 2].

Two-player games of *imperfect information* are considerably more complicated than games of perfect information. First, decision problems for imperfect-information games usually lie in higher complexity classes than their perfect-information counter-parts [19, 14, 2]. The algorithmic difference is often exponential, due to a subset construction that, similar to the determinization of finite automata, turns an imperfect-information game into an equivalent perfect-information game. Second, because of the determinization, no symbolic algorithms are known to solve imperfect-information games. This is in contrast to the perfect-information case, where (often) simple and elegant fixed-point algorithms exist [12, 8]. Third, in the context of imperfect information, deterministic strategies are sometimes insufficient. A game is *turn-based* if in every state one of the players chooses a successor state. In turn-based games of perfect information the set of winning states coincides with the set of states where the probability of winning is 1, and so deterministic strategies suffice to win (and thus also to win with probability 1). In contrast, in turn-based games of imperfect information the set of winning states is in general a strict subset of the set of states where the probability of winning is 1, and so randomized strategies are required to win with probability 1 (see Example 1). Fourth, winning strategies for imperfect-information games need memory even for simple objectives such as safety and reachability (see Example 2). This is again in contrast to the perfect-information case, where turn-based safety and reachability games can be won with memoryless strategies.

The contributions of this paper are twofold. First, we provide a symbolic fixed-point algorithm to compute winning states in games of imperfect information for arbitrary ω-regular objectives. The novelty is that our algorithm is symbolic; it does not carry out an explicit subset construction. Instead, we compute fixed points on the lattice of antichains of state sets. Antichains of state sets can be seen as a symbolic and compact representation for $\subseteq$-downward-closed sets of sets of states.[1] This solution extends our recent result [10] from safety objectives to all ω-regular objectives. To justify the correctness of the algorithm, we transform games of imperfect information into games of perfect information while preserving the existence of winning strategies for every objective. The reduction is only part of the proof, not part of the algorithm. For the special case of parity objectives, we obtain a symbolic EXPTIME algorithm for solving parity games of imperfect information. This is optimal, as the reachability problem for games of imperfect information is known to be EXPTIME-hard [19].

Second, we study randomized strategies and winning with probability 1 for imperfect-information games. To our knowledge, for these games no algorithms (symbolic or not) are present in the literature. Following [7], we refer to winning with probability 1 as *almost-sure* winning (*almost* winning, for short), in contrast to *sure* winning with deterministic strategies. We provide a symbolic EXPTIME algorithm to compute the set of almost-winning states for games of imperfect

[1] We recently used this symbolic representation of $\subseteq$-downward-closed sets of state sets to propose a new algorithm for solving the universality problem of nondeterministic finite automata. First experiments show a very promising performance; see [9] for details.

information with Büchi objectives (reachability objectives can be obtained as a special case, and for safety objectives almost winning and sure winning coincide). Our solution is again justified by a reduction to games of perfect information. However, for randomized strategies the reduction is different, and considerably more complicated. We prove our algorithm to be optimal, showing that computing the almost-winning states for reachability games of imperfect information is EXPTIME-hard. The problem of computing the almost-winning states for coBüchi objectives under imperfect information in EXPTIME remains an open problem.

The paper is organized as follows. Section 2 presents the definitions; Section 3 gives the algorithm for the case of sure winning with deterministic strategies; Section 4, for the case of almost winning with randomized strategies; and Section 5 provides the lower bounds.

Related work. In [17], Pnueli and Rosner study the synthesis of reactive modules. In their framework, there is no game graph; instead, the environment and the objective are specified using an LTL formula. In [14], Kupferman and Vardi extend these results in two directions: they consider CTL* objectives and imperfect information. Again, no game graph, but a specification formula is given to the synthesis procedure. We believe that our setting, where a game graph is given explicitly, is more suited to fully and uniformly understand the role of imperfect information. For example, Kupferman and Vardi claim that imperfect information comes at no cost, because if the specification is given as a CTL (or CTL*) formula, then the synthesis problem is complete for EXPTIME (resp. 2EXPTIME), just as in the perfect-information case. These hardness results, however, depend on the fact that the specification is given compactly as a formula. In our setting, with an explicit game graph, reachability games of perfect information are PTIME-complete, whereas reachability games of imperfect information are EXPTIME-complete [19]. None of the above papers provide symbolic solutions, and none of them consider randomized strategies.

It is known that for Partially Observable Markov Decision Processes (POMDPs) with boolean rewards and limit-average objectives the quantitative analysis (whether the value is greater than a specified threshold) is EXPTIME-complete [15]. However, almost winning is a qualitative question, and our hardness result for almost winning of imperfect-information games does not follow from the known results on POMDPs. We give in Section 5 a detailed proof of the hardness result of [19] for sure winning of imperfect-information games with reachability objectives, and we show that this proof can be extended to almost winning as well. To the best of our knowledge, this is the first hardness result that applies to the qualitative analysis of almost winning in imperfect-information games.

A class of *semiperfect*-information games, where one player has imperfect information and the other player has perfect information, is studied in [4]. That class is simpler than the games studied here; it can be solved in NP $\cap$ coNP for parity objectives.

2 Definitions

A *game structure* (*of imperfect information*) is a tuple $G = \langle L, l_0, \Sigma, \Delta, \mathcal{O}, \gamma \rangle$, where L is a finite set of states, $l_0 \in L$ is the initial state, Σ is a finite alphabet, $\Delta \subseteq L \times \Sigma \times L$ is a set of labeled transitions, $\mathcal{O}$ is a finite set of observations, and $\gamma : \mathcal{O} \to 2^L \backslash \emptyset$ maps each observation to the set of states that it represents. We require the following two properties on G: (i) for all $\ell \in L$ and all $\sigma \in \Sigma$, there exists $\ell' \in L$ such that $(\ell, \sigma, \ell') \in \Delta$; and (ii) the set $\{\gamma(o) \mid o \in \mathcal{O}\}$ partitions L. We say that G is a game structure of *perfect information* if $\mathcal{O} = L$ and $\gamma(\ell) = \{\ell\}$ for all $\ell \in L$.

We often omit $(\mathcal{O}, \gamma)$ in the description of games of perfect information. For $\sigma \in \Sigma$ and $s \subseteq L$, let $\mathsf{Post}^G_\sigma(s) = \{\ell' \in L \mid \exists \ell \in s : (\ell, \sigma, \ell') \in \Delta\}$.

Plays. In a game structure, in each turn, Player 1 chooses a letter in Σ, and Player 2 resolves nondeterminism by choosing the successor state. A *play* in G is an infinite sequence $\pi = \ell_0\sigma_0\ell_1 \ldots \sigma_{n-1}\ell_n\sigma_n \ldots$ such that (i) $\ell_0 = l_0$, and (ii) for all $i \geq 0$, we have $(\ell_i, \sigma_i, \ell_{i+1}) \in \Delta$. The *prefix up to* ℓ_n of the play π is denoted by $\pi(n)$; its *length* is $|\pi(n)| = n + 1$; and its *last element* is $\mathsf{Last}(\pi(n)) = \ell_n$. The *observation sequence* of π is the unique infinite sequence $\gamma^{-1}(\pi) = o_0\sigma_0 o_1 \ldots \sigma_{n-1} o_n \sigma_n \ldots$ such that for all $i \geq 0$, we have $\ell_i \in \gamma(o_i)$. Similarly, the *observation sequence* of $\pi(n)$ is the prefix up to o_n of $\gamma^{-1}(\pi)$. The set of infinite plays in G is denoted $\mathsf{Plays}(G)$, and the set of corresponding finite prefixes is denoted $\mathsf{Prefs}(G)$. A state $\ell \in L$ is *reachable* in G if there exists a prefix $\rho \in \mathsf{Prefs}(G)$ such that $\mathsf{Last}(\rho) = \ell$. For a prefix $\rho \in \mathsf{Prefs}(G)$, the *cone* $\mathsf{Cone}(\rho) = \{\ \pi \in \mathsf{Plays}(G) \mid \rho \text{ is a prefix of } \pi\ \}$ is the set of plays that extend ρ. The *knowledge* associated with a finite observation sequence $\tau = o_0\sigma_0 o_1\sigma_1 \ldots \sigma_{n-1} o_n$ is the set $\mathsf{K}(\tau)$ of states in which a play can be after this sequence of observations, that is, $\mathsf{K}(\tau) = \{\mathsf{Last}(\rho) \mid \rho \in \mathsf{Prefs}(G) \text{ and } \gamma^{-1}(\rho) = \tau\}$.

Lemma 1 *Let $G = \langle L, l_0, \Sigma, \Delta, \mathcal{O}, \gamma\rangle$ be a game structure of imperfect information. For $\sigma \in \Sigma$, $\ell \in L$, and $\rho, \rho' \in \mathsf{Prefs}(G)$ with $\rho' = \rho \cdot \sigma \cdot \ell$, let $o_\ell \in \mathcal{O}$ be the unique observation such that $\ell \in \gamma(o_\ell)$. Then $\mathsf{K}(\gamma^{-1}(\rho')) = \mathsf{Post}^G_\sigma(\mathsf{K}(\gamma^{-1}(\rho))) \cap \gamma(o_\ell)$.*

Strategies. A *deterministic strategy* in G for Player 1 is a function $\alpha : \mathsf{Prefs}(G) \to \Sigma$. For a finite set A, a probability distribution on A is a function $\kappa : A \to [0, 1]$ such that $\sum_{a\in A} \kappa(a) = 1$. We denote the set of probability distributions on A by $\mathcal{D}(A)$. Given a distribution $\kappa \in \mathcal{D}(A)$, let $\mathsf{Supp}(\kappa) = \{a \in A \mid \kappa(a) > 0\}$ be the *support* of κ. A *randomized strategy* in G for Player 1 is a function $\alpha : \mathsf{Prefs}(G) \to \mathcal{D}(\Sigma)$. A (deterministic or randomized) strategy α for Player 1 is *observation-based* if for all prefixes $\rho, \rho' \in \mathsf{Prefs}(G)$, if $\gamma^{-1}(\rho) = \gamma^{-1}(\rho')$, then $\alpha(\rho) = \alpha(\rho')$. In the sequel, we are interested in the existence of observation-based strategies for Player 1. A *deterministic strategy* in G for Player 2 is a function $\beta : \mathsf{Prefs}(G) \times \Sigma \to L$ such that for all $\rho \in \mathsf{Prefs}(G)$ and all $\sigma \in \Sigma$, we have $(\mathsf{Last}(\rho), \sigma, \beta(\rho, \sigma)) \in \Delta$. A *randomized strategy* in G for Player 2 is a function $\beta : \mathsf{Prefs}(G) \times \Sigma \to \mathcal{D}(L)$ such that for all $\rho \in \mathsf{Prefs}(G)$, all $\sigma \in \Sigma$, and all $\ell \in \mathsf{Supp}(\beta(\rho, \sigma))$, we have $(\mathsf{Last}(\rho), \sigma, \ell) \in \Delta$. We denote by $\mathcal{A}_G$, $\mathcal{A}^O_G$, and $\mathcal{B}_G$ the set of all Player-1 strategies, the set of all observation-based Player-1 strategies, and the set of all Player-2 strategies in G, respectively. All results of this paper can be proved also if strategies depend on state sequences only, and not on the past moves of a play.

The *outcome* of two deterministic strategies α (for Player 1) and β (for Player 2) in G is the play $\pi = \ell_0\sigma_0\ell_1 \ldots \sigma_{n-1}\ell_n\sigma_n \ldots \in \mathsf{Plays}(G)$ such that for all $i \geq 0$, we have $\sigma_i = \alpha(\pi(i))$ and $\ell_{i+1} = \beta(\pi(i), \sigma_i)$. This play is denoted $\mathsf{outcome}(G, \alpha, \beta)$. The *outcome* of two randomized strategies α (for Player 1) and β (for Player 2) in G is the set of plays $\pi = \ell_0\sigma_0\ell_1 \ldots \sigma_{n-1}\ell_n\sigma_n \ldots \in \mathsf{Plays}(G)$ such that for all $i \geq 0$, we have $\alpha(\pi(i))(\sigma_i) > 0$ and $\beta(\pi(i), \sigma_i)(\ell_{i+1}) > 0$. This set is denoted $\mathsf{outcome}(G, \alpha, \beta)$. The *outcome set* of the deterministic (resp. randomized) strategy α for Player 1 in G is the set $\mathsf{Outcome}_i(G, \alpha)$ of plays π such that there exists a deterministic (resp. randomized) strategy β for Player 2 with $\pi = \mathsf{outcome}(G, \alpha, \beta)$ (resp. $\pi \in \mathsf{outcome}(G, \alpha, \beta)$). The outcome sets for Player 2 are defined symmetrically.

Objectives. An *objective* for G is a set ϕ of infinite sequences of observations and input letters, that is, $\phi \subseteq (\mathcal{O} \times \Sigma)^\omega$. A play $\pi = \ell_0\sigma_0\ell_1 \ldots \sigma_{n-1}\ell_n\sigma_n \ldots \in \mathsf{Plays}(G)$ *satisfies* the objective ϕ, denoted $\pi \models \phi$, if $\gamma^{-1}(\pi) \in \phi$. Objectives are generally Borel measurable: a Borel objective is a Borel set in the Cantor topology on $(\mathcal{O} \times \Sigma)^\omega$ [13]. We specifically consider reachability, safety, Büchi, coBüchi, and parity objectives, all of them Borel measurable. The parity objectives are a canonical form to express all ω-regular objectives [21]. For a play $\pi = \ell_0\sigma_0\ell_1 \dots$, we write $\mathsf{Inf}(\pi)$ for the set of observations that appear infinitely often in $\gamma^{-1}(\pi)$, that is, $\mathsf{Inf}(\pi) = \{o \in \mathcal{O} \mid \ell_i \in \gamma(o)$ for infinitely many i's$\}$.

- *Reachability and safety objectives.* Given a set $\mathcal{T} \subseteq \mathcal{O}$ of target observations, the *reachability* objective $\mathsf{Reach}(\mathcal{T})$ requires that an observation in $\mathcal{T}$ be visited at least once, that is, $\mathsf{Reach}(\mathcal{T}) = \{ \ell_0\sigma_0\ell_1\sigma_1 \dots \in \mathsf{Plays}(G) \mid \exists k \geq 0 \cdot \exists o \in \mathcal{T} : \ell_k \in \gamma(o) \}$. Dually, the *safety* objective $\mathsf{Safe}(\mathcal{T})$ requires that only observations in $\mathcal{T}$ be visited. Formally, $\mathsf{Safe}(\mathcal{T}) = \{ \ell_0\sigma_0\ell_1\sigma_1 \dots \in \mathsf{Plays}(G) \mid \forall k \geq 0 \cdot \exists o \in \mathcal{T} : \ell_k \in \gamma(o) \}$.

- *Büchi and coBüchi objectives.* The *Büchi* objective $\mathsf{Buchi}(\mathcal{T})$ requires that an observation in $\mathcal{T}$ be visited infinitely often, that is, $\mathsf{Buchi}(\mathcal{T}) = \{ \pi \mid \mathsf{Inf}(\pi) \cap \mathcal{T} \neq \emptyset \}$. Dually, the *coBüchi* objective $\mathsf{coBuchi}(\mathcal{T})$ requires that only observations in $\mathcal{T}$ be visited infinitely often. Formally, $\mathsf{coBuchi}(\mathcal{T}) = \{ \pi \mid \mathsf{Inf}(\pi) \subseteq \mathcal{T} \}$.

- *Parity objectives.* For $d \in \mathbb{N}$, let $p : \mathcal{O} \to \{ 0, 1, \dots, d \}$ be a *priority function*, which maps each observation to a nonnegative integer priority. The *parity* objective $\mathsf{Parity}(p)$ requires that the minimum priority that appears infinitely often be even. Formally, $\mathsf{Parity}(p) = \{ \pi \mid \min\{ p(o) \mid o \in \mathsf{Inf}(\pi) \}$ is even $\}$.

Observe that by definition, for all objectives ϕ, if $\pi \models \phi$ and $\gamma^{-1}(\pi) = \gamma^{-1}(\pi')$, then $\pi' \models \phi$.

Sure winning and almost winning. A strategy λ_i for Player i in G is *sure winning* for an objective ϕ if for all $\pi \in \mathsf{Outcome}_i(G, \lambda_i)$, we have $\pi \models \phi$. Given a game structure G and a state ℓ of G, we write G_ℓ for the game structure that results from G by changing the initial state to ℓ, that is, if $G = \langle L, l_0, \Sigma, \Delta, \mathcal{O}, \gamma \rangle$, then $G_\ell = \langle L, \ell, \Sigma, \Delta, \mathcal{O}, \gamma \rangle$. An *event* is a measurable set of plays, and given strategies α and β for the two players, the probabilities of events are uniquely defined [22]. For a Borel objective ϕ, we denote by $\mathrm{Pr}_\ell^{\alpha,\beta}(\phi)$ the probability that ϕ is satisfied in the game G_ℓ given the strategies α and β. A strategy α for Player 1 in G is *almost winning* for the objective ϕ if for all randomized strategies β for Player 2, we have $\mathrm{Pr}_{l_0}^{\alpha,\beta}(\phi) = 1$. The set of *sure-winning* (resp. *almost-winning*) *states* of a game structure G for the objective ϕ is the set of states ℓ such that Player 1 has a deterministic sure-winning (resp. randomized almost-winning) observation-based strategy in G_ℓ for the objective ϕ.

Theorem 1 (Determinacy) [16] *For all perfect-information game structures G and all Borel objectives ϕ, either there exists a deterministic sure-winning strategy for Player 1 for the objective ϕ, or there exists a deterministic sure-winning strategy for Player 2 for the complementary objective $\mathsf{Plays}(G) \setminus \phi$.*

Notice that deterministic strategies suffice for *sure* winning a game: given a randomized strategy α for Player 1, let α^D be the deterministic strategy such that for all $\rho \in \mathsf{Prefs}(G)$, the strategy $\alpha^D(\rho)$ chooses an input letter from $\mathsf{Supp}(\alpha(\rho))$. Then $\mathsf{Outcome}_1(G, \alpha^D) \subseteq \mathsf{Outcome}_1(G, \alpha)$, and thus, if α is sure winning, then so is α^D. The result also holds for observation-based strategies. However, for almost winning, randomized strategies are more powerful than deterministic strategies as shown by Example 1.

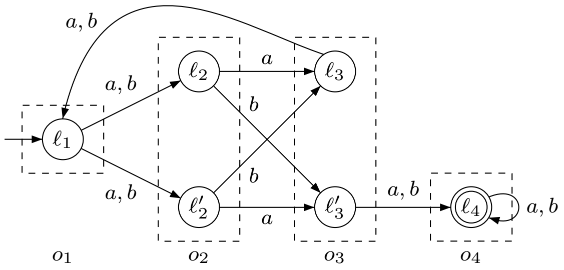

Figure 1: Game structure G.

Example 1 *Consider the game structure shown in Figure 1. The observations o_1, o_2, o_3, o_4 are such that $\gamma(o_1) = \{\ell_1\}$, $\gamma(o_2) = \{\ell_2, \ell_2'\}$, $\gamma(o_3) = \{\ell_3, \ell_3'\}$, and $\gamma(o_4) = \{\ell_4\}$. The transitions are shown as labeled edges in the figure, and the initial state is ℓ_1. The objective of Player 1 is $\mathsf{Reach}(\{o_4\})$, to reach state ℓ_4. We argue that the game is not sure winning for Player 1. Let α be any deterministic strategy for Player 1. Consider the deterministic strategy β for Player 2 as follows: for all $\rho \in \mathsf{Prefs}(G)$ such that $\mathsf{Last}(\rho) \in \gamma(o_2)$, if $\alpha(\rho) = a$, then in the previous round β chooses the state ℓ_2, and if $\alpha(\rho) = b$, then in the previous round β chooses the state ℓ_2'. Given α and β, the play $\mathsf{outcome}(G, \alpha, \beta)$ never reaches ℓ_4. Similarly, Player 2 has no sure winning strategy for the dual objective $\mathsf{Safe}(\{o_1, o_2, o_3\})$. Hence the game is not determined. However, the game G is almost winning for Player 1. Consider the randomized strategy that plays a and b uniformly at random at all states. Every time the game visits observation o_2, for any strategy for Player 2, the game visits ℓ_3 and ℓ_3' with probability $\frac{1}{2}$, and hence also reaches ℓ_4 with probability $\frac{1}{2}$. It follows that against all Player 2 strategies the play eventually reaches ℓ_4 with probability 1.*

Spoiling strategies. To spoil a strategy of Player 1 (for sure-winning), Player 2 does not need the full memory of the history of the play, he only needs counting strategies. We say that a deterministic strategy $\beta : \mathsf{Prefs}(G) \times \Sigma \to L$ for Player 2 is *counting* if for all prefixes $\rho, \rho' \in \mathsf{Prefs}(G)$ such that $|\rho| = |\rho'|$ and $\mathsf{Last}(\rho) = \mathsf{Last}(\rho')$, and for all $\sigma \in \Sigma$, we have $\beta(\rho, \sigma) = \beta(\rho', \sigma)$. Let $\mathcal{B}_G^c$ be the set of counting strategies for Player 2. The memory needed by a counting strategy is only the number of turns that have been played. This type of strategy is sufficient to spoil the non-winning strategies of Player 1.

Proposition 1 *Let G be a game structure of imperfect information and ϕ be an objective. There exists an observation-based strategy $\alpha^o \in \mathcal{A}_G^O$ such that for all $\beta \in \mathcal{B}_G$ we have $\mathsf{outcome}(G, \alpha^o, \beta) \in \phi$ if and only if there exists an observation-based strategy $\alpha^o \in \mathcal{A}_G^O$ such that for all counting strategies $\beta^c \in \mathcal{B}_G^c$ we have $\mathsf{outcome}(G, \alpha^o, \beta^c) \in \phi$.*

Proof. We prove the equivalent statement that: $\forall \alpha^o \in \mathcal{A}_G^o \cdot \exists \beta \in \mathcal{B}_G : \mathsf{outcome}(G, \alpha^o, \beta) \notin \phi$ iff $\forall \alpha^o \in \mathcal{A}_G^o \cdot \exists \beta^c \in \mathcal{B}_G^c : \mathsf{outcome}(G, \alpha^o, \beta^c) \notin \phi$. The right implication ($\leftarrow$) is trivial. For the left implication ($\rightarrow$), let $\alpha^o \in \mathcal{A}_G^o$ be an arbitrary observation-based strategy for Player 1 in G. Let $\beta \in \mathcal{B}_G$ be a strategy for Player 2 such that $\mathsf{outcome}(G, \alpha^o, \beta) \notin \phi$. Let $\mathsf{outcome}(G, \alpha^o, \beta) = \ell_0 \sigma_0 \ell_1 \dots \sigma_{n-1} \ell_n \sigma_n \dots$ and define a counting strategy β^c for Player 2 such that $\forall \rho \in \mathsf{Prefs}(G) \cdot \forall \sigma \in \Sigma$: if $\mathsf{Last}(\rho) = \ell_{n-1}$ and $\sigma = \sigma_{n-1}$ for $n = |\rho|$, then $\beta^c(\rho, \sigma) = \ell_n$, and otherwise $\beta^c(\rho, \sigma)$ is fixed arbitrarily in the set $\mathsf{Post}_\sigma^G(\mathsf{Last}(\rho))$. Clearly, β^c is a counting strategy and we have $\mathsf{outcome}(G, \alpha^o, \beta) = \mathsf{outcome}(G, \alpha^o, \beta^c)$ and thus $\mathsf{outcome}(G, \alpha^o, \beta^c) \notin \phi$. ∎

Remarks. First, the hypothesis that the observations form a partition of the state space can be weakened to a covering of the state space, where observations can overlap [10]. In that case, Player 2 chooses both the next state of the game ℓ and the next observation o such that $\ell \in \gamma(o)$. The definitions related to plays, strategies, and objectives are adapted accordingly. Such a game structure G with overlapping observations can be encoded by an equivalent game structure G' of imperfect information, whose state space is the set of pairs (ℓ, o) such that $\ell \in \gamma(o)$. The set of labeled transitions Δ' of G' is defined by $\Delta' = \{ ((\ell, o), \sigma, (\ell', o')) \mid (\ell, \sigma, \ell') \in \Delta \}$ and $\gamma'^{-1}(\ell, o) = o$. The games G and G' are equivalent in the sense that for every Borel objective ϕ, there exists a sure (resp. almost) winning strategy for Player i in G for ϕ if and only if there exists such a winning strategy for Player i in G' for ϕ.

Second, it is essential that the objective is expressed in terms of the observations. Indeed, the games of imperfect information with a nonobservable winning condition are more complicated to solve. For instance, the universality problem for Büchi automata can be reduced to such games, but the construction that we propose in Section 3 cannot be used. More involved constructions à la Safra are needed [20].

3 Sure Winning

First, we show that a game structure G of imperfect information can be encoded by a game structure G^{K} of perfect information such that for every objective ϕ, there exists a deterministic observation-based sure-winning strategy for Player 1 in G for ϕ if and only if there exists a deterministic sure-winning strategy for Player 1 in G^{K} for ϕ. We obtain G^{K} using a subset construction similar to Reif's construction for safety objectives [19]. Each state in G^{K} is a set of states of G which represents the knowledge of Player 1. In the worst case, the size of G^{K} is exponentially larger than the size of G. Second, we present a fixed-point algorithm based on antichains of set of states [10], whose correctness relies on the subset construction, but avoids the explicit construction of G^{K}.

3.1 Subset construction for sure winning

Subset construction. Given a game structure of imperfect information $G = \langle L, l_0, \Sigma, \Delta, \mathcal{O}, \gamma \rangle$, we define the *knowledge-based subset construction* of G as the following game structure of perfect information:

$$G^{\mathsf{K}} = \langle \mathcal{L}, \{l_0\}, \Sigma, \Delta^{\mathsf{K}} \rangle,$$

where $\mathcal{L} = 2^L \backslash \{\emptyset\}$, and $(s_1, \sigma, s_2) \in \Delta^{\mathsf{K}}$ iff there exists an observation $o \in \mathcal{O}$ such that $s_2 = \mathsf{Post}^G_\sigma(s_1) \cap \gamma(o)$ and $s_2 \neq \emptyset$. Notice that for all $s \in \mathcal{L}$ and all $\sigma \in \Sigma$, there exists a set $s' \in \mathcal{L}$ such that $(s, \sigma, s') \in \Delta^{\mathsf{K}}$.

A (deterministic or randomized) strategy in G^{K} is called a *knowledge-based* strategy. To distinguish between a general strategy in G, an observation-based strategy in G, and a knowledge-based strategy in G^{K}, we often use the notations α, α^o, and α^{K}, respectively.

Lemma 2 *For all sets $s \in \mathcal{L}$ that are reachable in G^{K}, and all observations $o \in \mathcal{O}$, either $s \subseteq \gamma(o)$ or $s \cap \gamma(o) = \emptyset$.*

Proof. First, the property holds for $s = \{l_0\}$, the initial state in G^{K} as it is a singleton. Second, we show that the property holds for any successor s' of any state s in G^{K}. Assume that $(s, \sigma, s') \in \Delta^{\mathsf{K}}$.

Then we know that $s' = \mathsf{Post}^G_\sigma(s) \cap \gamma(o)$ for some $o \in \mathcal{O}$. Hence, $s' \subseteq \gamma(o)$ and $s' \cap \gamma(o') = \emptyset$ for all $o' \neq o$ since the set $\{\gamma(o) \mid o \in \mathcal{O}\}$ partitions L. ■

By an abuse of notation, we define the *observation sequence* of a play $\pi = s_0\sigma_0 s_1 \dots \sigma_{n-1} s_n \sigma_n \dots \in \mathsf{Plays}(G^{\mathsf{K}})$ as the infinite sequence $\gamma^{-1}(\pi) = o_0\sigma_0 o_1 \dots \sigma_{n-1} o_n \sigma_n \dots$ of observations such that for all $i \geq 0$, we have $s_i \subseteq \gamma(o_i)$. This sequence is unique by Lemma 2. The play π *satisfies* an objective $\phi \subseteq (\mathcal{O} \times \Sigma)^\omega$ if $\gamma^{-1}(\pi) \in \phi$.

The correctness of the subset construction G^{K} is established by the following two lemmas which generalize the result of [19] for safety objective to any kind of objective. For Lemma 4, the proof of [19] is not sufficient, since violation of a safety objective can be witnessed by a finite prefix of play, while general objectives need an infinite witness.

Lemma 3 *If Player* 1 *has a deterministic sure-winning strategy in* G^{K} *for an objective* ϕ*, then he has a deterministic observation-based sure-winning strategy in* G *for* ϕ*.*

Proof. Let α^{K} be a deterministic sure-winning strategy for Player 1 in G^{K} with the objective ϕ. Define α^o a strategy for Player 1 in G as follows: for every $\rho \in \mathsf{Prefs}(G)$, let $\alpha^o(\rho) = \alpha^{\mathsf{K}}(\rho^{\mathsf{K}})$ where ρ^{K} is defined from $\rho = \ell_0\sigma_0\ell_1 \dots \sigma_{n-1}\ell_n$ by $\rho^{\mathsf{K}} = s_0\sigma_0 s_1 \dots \sigma_{n-1} s_n$ where $s_i = \mathsf{K}(\gamma^{-1}(\ell_0\sigma_0\ell_1 \dots \sigma_{i-1}\ell_i))$ for each $0 \leq i \leq n$. Clearly, α^o is a deterministic observation-based strategy as $\gamma^{-1}(\rho) = \gamma^{-1}(\rho')$ implies $\rho^{\mathsf{K}} = \rho'^{\mathsf{K}}$.

By contradiction, assume that α^o is not a sure-winning strategy for Player 1 in G with the objective ϕ. Then there exists a play $\pi \in \mathsf{Outcome}_1(G, \alpha^o)$ such that $\pi \not\models \phi$. Let $\pi = \ell_0\sigma_0\ell_1\sigma_1 \dots$ and consider the infinite sequence $\pi^{\mathsf{K}} = s_0\sigma_0 s_1\sigma_1 \dots$ where $s_i = \mathsf{K}(\gamma^{-1}(\pi(i)))$ for each $i \geq 0$. We show that $\pi^{\mathsf{K}} \in \mathsf{Outcome}_1(G^{\mathsf{K}}, \alpha^{\mathsf{K}})$. First, we have $s_0 = \mathsf{K}(\gamma^{-1}(\pi(0))) = \mathsf{K}(\gamma^{-1}(\ell_0)) = \{\ell_0\}$. Second, for any $i \geq 0$, we have $s_i = \mathsf{K}(\gamma^{-1}(\pi(i)))$ and by Lemma 1 we have $s_{i+1} = \mathsf{Post}^G_{\sigma_i}(s_i) \cap \gamma(o)$ where o is such that $\ell_{i+1} \in \gamma(o)$ and so $(s_i, \sigma_i, s_{i+1}) \in \Delta^{\mathsf{K}}$. Third, by definition of α^o, we have $\sigma_i = \alpha^o(\pi(i)) = \alpha^{\mathsf{K}}(\pi^{\mathsf{K}}(i))$. This entails that $\pi^{\mathsf{K}} \in \mathsf{Outcome}_1(G^{\mathsf{K}}, \alpha^{\mathsf{K}})$.

Now, observe that trivially $\ell_i \in \mathsf{K}(\gamma^{-1}(\pi(i)))$ for any $i \geq 0$, that is $\ell_i \in s_i$ and so $s_i \cap \gamma(o_i) \neq \emptyset$ where o_i is the unique observation such that $\ell_i \in \gamma(o_i)$. Hence, by Lemma 2, we have $s_i \subseteq \gamma(o_i)$. Consequently, $\gamma^{-1}(\pi^{\mathsf{K}}) = \gamma^{-1}(\pi)$ and thus $\pi^{\mathsf{K}} \not\models \phi$ which contradicts the fact that α^{K} is a sure-winning strategy for Player 1 in G^{K} with the objective ϕ. Therefore, α^o is a sure-winning strategy for Player 1 in G with the objective ϕ. ■

Lemma 4 *If Player* 1 *has a deterministic observation-based sure-winning strategy in* G *for an objective* ϕ*, then Player* 1 *has a deterministic sure-winning strategy in* G^{K} *for* ϕ*.*

Proof. First, it is easy to show by induction that for every finite prefix of play $\rho^{\mathsf{K}} = s_0\sigma_0 s_1 \dots \sigma_{n-1} s_n$ in $\mathsf{Prefs}(G^{\mathsf{K}})$, there exists a prefix of play $\rho = \ell_0\sigma_0\ell_1 \dots \sigma_{n-1}\ell_n$ in $\mathsf{Prefs}(G)$ that *generates* ρ^{K}, that is such that $s_i = \mathsf{K}(\gamma^{-1}(\ell_0\sigma_0\ell_1 \dots \sigma_{i-1}\ell_i))$ for each $0 \leq i \leq n$; and for all such prefix of play ρ' that generates ρ^{K}, we have $\gamma^{-1}(\rho) = \gamma^{-1}(\rho')$ (by Lemma 1).

Now, let α^o be a deterministic observation-based sure-winning strategy for Player 1 in G that is sure-winning for ϕ. We construct a deterministic strategy α^{K} for Player 1 in G^{K} as follows: for every $\rho^{\mathsf{K}} \in \mathsf{Prefs}(G^{\mathsf{K}})$, let $\alpha^{\mathsf{K}}(\rho^{\mathsf{K}}) = \alpha^o(\rho)$ where ρ generates ρ^{K}. By the above remark, α^{K} is well-defined (it is independent of the choice of ρ since α^o is observation-based).

By contradiction, assume that α^{K} is not sure-winning for Player 1 in G^{K} with objective ϕ. Then, there exists a play $\pi^{\mathsf{K}} \in \mathsf{Outcome}_1(G^{\mathsf{K}}, \alpha^{\mathsf{K}})$ with $\pi^{\mathsf{K}} \not\models \phi$.

We construct the DAG $D = \langle V, E\rangle$ where $V = \{(\ell, i) \mid \ell \in \mathsf{Last}(\pi^{\mathsf{K}}(i))\}$ and $E = \{((\ell, i), (\ell', i+1)) \mid (\ell, \sigma_i, \ell') \in \Delta\}$. By definition of G^{K}, for all $i \geq 0$, we have $\mathsf{Last}(\pi^{\mathsf{K}}(i)) \neq \emptyset$ and for all $\ell \in \mathsf{Last}(\pi^{\mathsf{K}}(i))$, there is a path in D from $(\ell_0, 0)$ to (ℓ, i). Therefore, V is infinite and by König's Lemma, there exists an infinite path $(\ell_0, 0)(\ell_1, 1)\dots$ in D and thus a play $\pi = \ell_0\sigma_0\ell_1\sigma_1\dots$ in G such that $\pi \in \mathsf{Outcome}_1(G, \alpha^o)$ and $\pi \not\models \phi$. This is in contradiction with the assumption that α^o is sure-winning in G for ϕ. Hence α^{K} is sure-winning for Player 1 in G^{K} with objective ϕ. ■

Lemma 3 and Lemma 4 yield Theorem 2.

Theorem 2 (Sure-winning reduction) *Player* 1 *has a deterministic observation-based sure-winning strategy in a game structure G of imperfect information for an objective ϕ if and only if Player* 1 *has a deterministic sure-winning strategy in the game structure G^{K} of perfect information for ϕ.*

3.2 Two interpretations of the μ-calculus

From the results of Section 3.1, we can solve a game G of imperfect information with objective ϕ by constructing the knowledge-based subset construction G^{K} and solving the resulting game of perfect information for the objective ϕ using standard methods. For the important class of ω-regular objectives, there exists a fixed-point theory —the μ-calculus— for this purpose [8]. When run on G^{K}, these fixed-point algorithms compute sets of sets of states of the game G. An important property of those sets is that they are *downward closed* with respect to set inclusion: if Player 1 has a deterministic strategy to win the game G when her knowledge is a set s, then she also has a deterministic strategy to win the game when her knowledge is s' with $s' \subseteq s$. And thus, if s is a sure-winning state of G^{K}, then so is s'. Based on this property, we devise a new algorithm for solving games of perfect information.

An *antichain* of nonempty sets of states is a set $q \subseteq 2^L \setminus \{\emptyset\}$ such that for all $s, s' \in q$, we have $s \not\subset s'$. Let $\mathcal{A}$ be the set of antichains of nonempty subsets of L, and consider the following partial order on $\mathcal{A}$: for all $q, q' \in \mathcal{A}$, let $q \sqsubseteq q'$ iff $\forall s \in q \cdot \exists s' \in q' : s \subseteq s'$. For $q \subseteq 2^L$, define the set of *maximal* elements of q by $\lceil q \rceil = \{s \in q \mid s \neq \emptyset \text{ and } \forall s' \in q : s \not\subset s'\}$. Clearly, $\lceil q \rceil$ is an antichain. The least upper bound of $q, q' \in \mathcal{A}$ is $q \sqcup q' = \lceil\{s \mid s \in q \text{ or } s \in q'\}\rceil$, and their greatest lower bound is $q \sqcap q' = \lceil\{s \cap s' \mid s \in q \text{ and } s' \in q'\}\rceil$. The definition of these two operators extends naturally to sets of antichains, and the greatest element of $\mathcal{A}$ is $\top = \{L\}$ and the least element is $\bot = \emptyset$. The partially ordered set $\langle \mathcal{A}, \sqsubseteq, \sqcup, \sqcap, \top, \bot\rangle$ forms a complete lattice. We view antichains of state sets as a symbolic representation of $\subseteq$-downward-closed sets of state sets.

A *game lattice* is a complete lattice V together with a *predecessor operator* $\mathsf{CPre} : V \to V$. Given a game structure $G = \langle L, l_0, \Sigma, \Delta, \mathcal{O}, \gamma\rangle$ of imperfect information, and its knowledge-based subset construction $G^{\mathsf{K}} = \langle \mathcal{L}, \{l_0\}, \Sigma, \Delta^{\mathsf{K}}\rangle$, we consider two game lattices: the *lattice of subsets* $\langle \mathcal{S}, \subseteq, \cup, \cap, \mathcal{L}, \emptyset\rangle$, where $\mathcal{S} = 2^{\mathcal{L}}$ and $\mathsf{CPre} : \mathcal{S} \to \mathcal{S}$ is defined by $\mathsf{CPre}(q) = \{s \in \mathcal{L} \mid \exists \sigma \in \Sigma \cdot \forall s' \in \mathcal{L} :$ if $(s, \sigma, s') \in \Delta^{\mathsf{K}}$, then $s' \in q\}$; and the *lattice of antichains* $\langle \mathcal{A}, \sqsubseteq, \sqcup, \sqcap, \{L\}, \emptyset\rangle$, with the operator $\lceil\mathsf{CPre}\rceil : \mathcal{A} \to \mathcal{A}$ defined by $\lceil\mathsf{CPre}\rceil(q) = \lceil\{s \in \mathcal{L} \mid \exists \sigma \in \Sigma \cdot \forall o \in \mathcal{O} \cdot \exists s' \in q : \mathsf{Post}_\sigma(s) \cap \gamma(o) \subseteq s'\}\rceil$.

The *μ-calculus formulas* are generated by the grammar

$$\varphi ::= o \mid x \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \mathsf{pre}(\varphi) \mid \mu x.\varphi \mid \nu x.\varphi$$

for atomic propositions $o \in \mathcal{O}$ and variables x. We can define $\neg o$ as a shortcut for $\bigvee_{o' \in \mathcal{O}\setminus\{o\}} o'$. A variable is *free* in a formula φ if it is not in the scope of a quantifier μx or νx. A formula φ is *closed*

Lattice of subsets
$[\![o]\!]^{\mathcal{S}}_{\mathcal{E}}= \{s \in \mathcal{L} \mid s \subseteq \gamma(o)\}$
$[\![x]\!]^{\mathcal{S}}_{\mathcal{E}}= \mathcal{E}(x)$
$[\![\varphi_1 \left\{ \begin{smallmatrix} \vee \\ \wedge \end{smallmatrix} \right\} \varphi_2]\!]^{\mathcal{S}}_{\mathcal{E}}=[\![\varphi_1]\!]^{\mathcal{S}}_{\mathcal{E}} \left\{ \begin{smallmatrix} \cup \\ \cap \end{smallmatrix} \right\} [\![\varphi_2]\!]^{\mathcal{S}}_{\mathcal{E}}$
$[\![\mathsf{pre}(\varphi)]\!]^{\mathcal{S}}_{\mathcal{E}}= \mathsf{CPre}([\![\varphi]\!]^{\mathcal{S}}_{\mathcal{E}})$
$[\![\left\{ \begin{smallmatrix} \mu \\ \nu \end{smallmatrix} \right\} x.\varphi]\!]^{\mathcal{S}}_{\mathcal{E}}= \left\{ \begin{smallmatrix} \bigcap \\ \bigcup \end{smallmatrix} \right\} \{q \mid q =[\![\varphi]\!]^{\mathcal{S}}_{\mathcal{E}[x\mapsto q]}\}$

Lattice of antichains
$[\![o]\!]^{\mathcal{A}}_{\mathcal{E}}= \{\gamma(o)\}$
$[\![x]\!]^{\mathcal{A}}_{\mathcal{E}}= \mathcal{E}(x)$
$[\![\varphi_1 \left\{ \begin{smallmatrix} \vee \\ \wedge \end{smallmatrix} \right\} \varphi_2]\!]^{\mathcal{A}}_{\mathcal{E}}=[\![\varphi_1]\!]^{\mathcal{A}}_{\mathcal{E}} \left\{ \begin{smallmatrix} \sqcup \\ \sqcap \end{smallmatrix} \right\} [\![\varphi_2]\!]^{\mathcal{A}}_{\mathcal{E}}$
$[\![\mathsf{pre}(\varphi)]\!]^{\mathcal{A}}_{\mathcal{E}}= \lceil \mathsf{CPre} \rceil ([\![\varphi]\!]^{\mathcal{A}}_{\mathcal{E}})$
$[\![\left\{ \begin{smallmatrix} \mu \\ \nu \end{smallmatrix} \right\} x.\varphi]\!]^{\mathcal{A}}_{\mathcal{E}}= \left\{ \begin{smallmatrix} \sqcap \\ \sqcup \end{smallmatrix} \right\} \{q \mid q =[\![\varphi]\!]^{\mathcal{A}}_{\mathcal{E}[x\mapsto q]}\}$

if it contains no free variable. Given a game lattice V, a *valuation* $\mathcal{E}$ for the variables is a function that maps every variable x to an element in V. For $q \in V$, we write $\mathcal{E}[x \mapsto q]$ for the valuation that agrees with $\mathcal{E}$ on all variables, except that x is mapped to q. Given a game lattice V and a valuation $\mathcal{E}$, each μ-calculus formula φ specifies an element $[\![\varphi]\!]^V_{\mathcal{E}}$ of V, which is defined inductively by the equations shown in the two tables below. If φ is a closed formula, then $[\![\varphi]\!]^V=[\![\varphi]\!]^V_{\mathcal{E}}$ for any valuation $\mathcal{E}$. The following theorem recalls that perfect-information games can be solved by evaluating fixed-point formulas in the lattice of subsets.

Theorem 3 (Symbolic solution of perfect-information games) [8] *For every ω-regular objective ϕ, there exists a closed μ-calculus formula $\mu\mathsf{Form}(\phi)$, called the* characteristic formula *of ϕ, such that for all game structures G of perfect information, the set of sure-winning states of G for ϕ is $[\![\mu\mathsf{Form}(\phi)]\!]^{\mathcal{S}}$.*

Downward closure. Given a set $q \in \mathcal{S}$, the *downward closure* of q is the set $q\!\downarrow = \{s \in \mathcal{L} \mid \exists s' \in q : s \subseteq s'\}$. Observe that in particular, for all $q \in \mathcal{S}$, we have $\emptyset \notin q\!\downarrow$ and $\lceil q \rceil\!\downarrow = q\!\downarrow$. The sets $q\!\downarrow$, for $q \in \mathcal{S}$, are the *downward-closed* sets. A valuation $\mathcal{E}$ for the variables in the lattice $\mathcal{S}$ of subsets is *downward closed* if every variable x is mapped to a downward-closed set, that is, $\mathcal{E}(x) = \mathcal{E}(x)\!\downarrow$.

Lemma 5 *For all downward-closed sets $q, q' \in \mathcal{S}$, $\lceil q \cap q' \rceil = \lceil q \rceil \sqcap \lceil q' \rceil$ and $\lceil q \cup q' \rceil = \lceil q \rceil \sqcup \lceil q' \rceil$.*

Lemma 6 *For all μ-calculus formulas φ and all downward-closed valuations $\mathcal{E}$ in the lattice of subsets, the set $[\![\varphi]\!]^{\mathcal{S}}_{\mathcal{E}}$ is downward closed.*

Proof. We prove this lemma by induction on the structure of φ.

- if $\varphi \equiv o$ for $o \in \mathcal{O}$. It is immediate to show that $[\![\varphi]\!]^{\mathcal{S}}_{\mathcal{E}}= [\![\varphi]\!]^{\mathcal{S}}_{\mathcal{E}}\!\downarrow$.
- if $\varphi \equiv x$ for a variable x. We have $[\![\varphi]\!]^{\mathcal{S}}_{\mathcal{E}}= \mathcal{E}(x)$ which is downward closed by hypothesis.
- if $\varphi \equiv \varphi_1 \left\{ \begin{smallmatrix} \vee \\ \wedge \end{smallmatrix} \right\} \varphi_2$ and both $[\![\varphi_1]\!]^{\mathcal{S}}_{\mathcal{E}}$ and $[\![\varphi_2]\!]^{\mathcal{S}}_{\mathcal{E}}$ are downward closed. Then we have $[\![\varphi]\!]^{\mathcal{S}}_{\mathcal{E}}$ $=[\![\varphi_1]\!]^{\mathcal{S}}_{\mathcal{E}} \left\{ \begin{smallmatrix} \cup \\ \cap \end{smallmatrix} \right\} [\![\varphi_2]\!]^{\mathcal{S}}_{\mathcal{E}}$ and the result follows from the fact that union and intersection of downward closed sets are downward closed.
- if $\varphi \equiv pre(\varphi_1)$ and $[\![\varphi_1]\!]^{\mathcal{S}}_{\mathcal{E}}$ is downward closed. We show that $[\![\varphi]\!]^{\mathcal{S}}_{\mathcal{E}}$ is downward closed. Let $s_1 \in[\![\varphi]\!]^{\mathcal{S}}_{\mathcal{E}}$ and let $s_2 \in \mathcal{L}$ such that $s_2 \subseteq s_1$. Let us show that $s_2 \in[\![\varphi]\!]^{\mathcal{S}}_{\mathcal{E}}$. By definition of

CPre, since $s_1 \in [\![pre(\varphi_1)]\!]^{\mathcal{S}}_{\mathcal{E}}$, there exists $\sigma \in \Sigma$ such that for any s_1', if $(s_1, \sigma, s_1') \in \Delta^{\mathsf{K}}$ then $s_1' \in [\![\varphi_1]\!]^{\mathcal{S}}_{\mathcal{E}}$. Consider any s_2' such that $(s_2, \sigma, s_2') \in \Delta^{\mathsf{K}}$. According to the definition of G^{K}, we have $s_2' = \mathsf{Post}_\sigma(s_2) \cap \gamma(o) \neq \emptyset$ for some $o \in \mathcal{O}$. Now, let $s_1' = \mathsf{Post}_\sigma(s_1) \cap \gamma(o)$. Since $s_2 \subseteq s_1$, we have $s_2' \subseteq s_1'$ and thus $s_1' \neq \emptyset$. Therefore $(s_1, \sigma, s_1') \in \Delta^{\mathsf{K}}$ and so $s_1' \in [\![\varphi_1]\!]^{\mathcal{S}}_{\mathcal{E}}$. As the latter set is downward closed, we also have $s_2' \in [\![\varphi_1]\!]^{\mathcal{S}}_{\mathcal{E}}$ and thus $s_2 \in [\![pre(\varphi_1)]\!]^{\mathcal{S}}_{\mathcal{E}}$.

- if $\varphi \equiv \nu x.\varphi_1$ and $[\![\varphi_1]\!]^{\mathcal{S}}_{\mathcal{E}'}$ is downward closed for any downward closed valuation $\mathcal{E}'$. By Tarski's theorem, $[\![\varphi]\!]^{\mathcal{S}}_{\mathcal{E}}$ is one of the set in the infinite sequence $q_0, q_1, \dots$ defined by $q_0 = \mathcal{L}$ and for every $i \geq 1$, $q_i = [\![\varphi]\!]^{\mathcal{S}}_{\mathcal{E}[x \mapsto q_{i-1}]}$. Since q_0 is downward closed, every q_i $(i \geq 1)$ is also downward closed by the induction hypothesis.

- if $\varphi \equiv \mu x.\varphi_1$ and $[\![\varphi_1]\!]^{\mathcal{S}}_{\mathcal{E}'}$ is downward closed for any downward closed valuation $\mathcal{E}'$. The proof is similar to the previous case.

■

Lemma 7 *For all μ-calculus formulas φ, and all downward-closed valuations $\mathcal{E}$ in the lattice of subsets, we have $\lceil [\![\varphi]\!]^{\mathcal{S}}_{\mathcal{E}} \rceil = [\![\varphi]\!]^{\mathcal{A}}_{\lceil \mathcal{E} \rceil}$, where $\lceil \mathcal{E} \rceil$ is a valuation in the lattice of antichains defined by $\lceil \mathcal{E} \rceil(x) = \lceil \mathcal{E}(x) \rceil$ for all variables x.*

Proof. We prove this by induction on the structure of φ.

- if $\varphi \equiv o$ for $o \in \mathcal{O}$. The claim is immediate.

- if $\varphi \equiv x$ for a variable x. We have $\lceil [\![\varphi]\!]^{\mathcal{S}}_{\mathcal{E}} \rceil = \lceil \mathcal{E}(x) \rceil = [\![\varphi]\!]^{\mathcal{A}}_{\lceil \mathcal{E} \rceil}$.

- if $\varphi \equiv \varphi_1 \left\{ \begin{smallmatrix} \vee \\ \wedge \end{smallmatrix} \right\} \varphi_2$ and both $\lceil [\![\varphi_1]\!]^{\mathcal{S}}_{\mathcal{E}} \rceil = [\![\varphi_1]\!]^{\mathcal{A}}_{\lceil \mathcal{E} \rceil}$ and $\lceil [\![\varphi_2]\!]^{\mathcal{S}}_{\mathcal{E}} \rceil = [\![\varphi_2]\!]^{\mathcal{A}}_{\lceil \mathcal{E} \rceil}$. Using Lemma 5 and Lemma 6, we have successively:

  $$\lceil [\![\varphi]\!]^{\mathcal{S}}_{\mathcal{E}} \rceil = \left\lceil [\![\varphi_1]\!]^{\mathcal{S}}_{\mathcal{E}} \left\{ \begin{smallmatrix} \cup \\ \cap \end{smallmatrix} \right\} [\![\varphi_2]\!]^{\mathcal{S}}_{\mathcal{E}} \right\rceil =$$
  $$\lceil [\![\varphi_1]\!]^{\mathcal{S}}_{\mathcal{E}} \rceil \left\{ \begin{smallmatrix} \sqcup \\ \sqcap \end{smallmatrix} \right\} \lceil [\![\varphi_2]\!]^{\mathcal{S}}_{\mathcal{E}} \rceil = [\![\varphi_1]\!]^{\mathcal{A}}_{\lceil \mathcal{E} \rceil} \left\{ \begin{smallmatrix} \sqcup \\ \sqcap \end{smallmatrix} \right\} [\![\varphi_2]\!]^{\mathcal{A}}_{\lceil \mathcal{E} \rceil}.$$

- if $\varphi \equiv pre(\varphi_1)$ and $\lceil [\![\varphi_1]\!]^{\mathcal{S}}_{\mathcal{E}} \rceil = [\![\varphi_1]\!]^{\mathcal{A}}_{\lceil \mathcal{E} \rceil}$.

  1. We prove the inclusion $[\![\varphi]\!]^{\mathcal{A}}_{\lceil \mathcal{E} \rceil} \subseteq \lceil [\![\varphi]\!]^{\mathcal{S}}_{\mathcal{E}} \rceil$. First, let $s \in \lceil \mathsf{CPre} \rceil([\![\varphi_1]\!]^{\mathcal{A}}_{\lceil \mathcal{E} \rceil})$. We know that there exists $\sigma \in \Sigma$ such that $\forall o \in \mathcal{O} \cdot \exists s'' \in [\![\varphi_1]\!]^{\mathcal{A}}_{\lceil \mathcal{E} \rceil}$: $\mathsf{Post}_\sigma(s) \cap \gamma(o) \subseteq s''$. Since $\lceil [\![\varphi_1]\!]^{\mathcal{S}}_{\mathcal{E}} \rceil = [\![\varphi_1]\!]^{\mathcal{A}}_{\lceil \mathcal{E} \rceil}$ (induction hypothesis), it is clear that for such σ, if $(s, \sigma, s') \in \Delta^{\mathsf{K}}$, then there exists $s'' \in \lceil [\![\varphi_1]\!]^{\mathcal{S}}_{\mathcal{E}} \rceil$ such that $s' \subseteq s''$. And since $[\![\varphi_1]\!]^{\mathcal{S}}_{\mathcal{E}}$ is downward closed (by Lemma 6) we have $s' \in [\![\varphi_1]\!]^{\mathcal{S}}_{\mathcal{E}}$, so that $s \in \mathsf{CPre}([\![\varphi_1]\!]^{\mathcal{S}}_{\mathcal{E}})$ (and thus $s \in [\![\varphi]\!]^{\mathcal{S}}_{\mathcal{E}}$).
     Second, we show that s is maximal in $[\![\varphi]\!]^{\mathcal{S}}_{\mathcal{E}}$. By contradiction, assume that there exists $s_1 \in [\![\varphi]\!]^{\mathcal{S}}_{\mathcal{E}}$ with $s \subset s_1$. Then, by the same argument as in the first part of the proof of the inclusion $\lceil [\![\varphi]\!]^{\mathcal{S}}_{\mathcal{E}} \rceil \subseteq [\![\varphi]\!]^{\mathcal{A}}_{\lceil \mathcal{E} \rceil}$ below, we have that s_1 satisfies the definition of $\lceil \mathsf{CPre} \rceil([\![\varphi_1]\!]^{\mathcal{A}}_{\lceil \mathcal{E} \rceil})$ up to the operator $\lceil \cdot \rceil$. This means that s is not maximal in $\lceil \mathsf{CPre} \rceil([\![\varphi_1]\!]^{\mathcal{A}}_{\lceil \mathcal{E} \rceil})$, a contradiction.

2. We prove the inclusion $\lceil \llbracket\varphi\rrbracket^{\mathcal{S}}_{\mathcal{E}} \rceil \subseteq \llbracket\varphi\rrbracket^{\mathcal{A}}_{\lceil\mathcal{E}\rceil}$. This is trivial if $\lceil \llbracket\varphi\rrbracket^{\mathcal{S}}_{\mathcal{E}} \rceil = \emptyset$. Otherwise, let us first show that $\llbracket\varphi_1\rrbracket^{\mathcal{A}}_{\lceil\mathcal{E}\rceil} \neq \emptyset$. Let $s \in \lceil \llbracket\varphi\rrbracket^{\mathcal{S}}_{\mathcal{E}} \rceil$. Then, there exists $\sigma \in \Sigma$ such that for any s', if $(s, \sigma, s') \in \Delta^{\mathsf{K}}$ then $s' \in \llbracket\varphi_1\rrbracket^{\mathcal{S}}_{\mathcal{E}}$. Since the transition relation of G is total and the observations partition the state space, we have $\mathsf{Post}_\sigma(s) \cap \gamma(o) \neq \emptyset$ for some $o \in \mathcal{O}$. Therefore, $\llbracket\varphi_1\rrbracket^{\mathcal{S}}_{\mathcal{E}}$ is nonempty and so is $\llbracket\varphi_1\rrbracket^{\mathcal{A}}_{\lceil\mathcal{E}\rceil}$.

   Now, we proceed with the proof of inclusion. Let $s \in \lceil \llbracket\varphi\rrbracket^{\mathcal{S}}_{\mathcal{E}} \rceil$, and let $\sigma \in \Sigma$ such that such that for any s', if $(s, \sigma, s') \in \Delta^{\mathsf{K}}$ then $s' \in \llbracket\varphi_1\rrbracket^{\mathcal{S}}_{\mathcal{E}}$. Let us show that $s \in \lceil\mathsf{CPre}\rceil(\llbracket\varphi_1\rrbracket^{\mathcal{A}}_{\lceil\mathcal{E}\rceil})$. First, consider an arbitrary observation $o \in \mathcal{O}$ and let $s' = \mathsf{Post}_\sigma(s) \cap \gamma(o)$. We must show that there exists $s'' \in \llbracket\varphi_1\rrbracket^{\mathcal{A}}_{\lceil\mathcal{E}\rceil}$ such that $s' \subseteq s''$. This is obvious if $s' = \emptyset$ since $\llbracket\varphi_1\rrbracket^{\mathcal{A}}_{\lceil\mathcal{E}\rceil}$ is nonempty. Otherwise, by the definition of G^{K}, we have $(s, \sigma, s') \in \Delta^{\mathsf{K}}$ and therefore $s' \in \llbracket\varphi_1\rrbracket^{\mathcal{S}}_{\mathcal{E}}$. Since $\llbracket\varphi_1\rrbracket^{\mathcal{A}}_{\lceil\mathcal{E}\rceil} = \lceil \llbracket\varphi_1\rrbracket^{\mathcal{S}}_{\mathcal{E}} \rceil$ (induction hypothesis), there exists $s'' \in \llbracket\varphi_1\rrbracket^{\mathcal{A}}_{\lceil\mathcal{E}\rceil}$ such that $s' \subseteq s''$, and thus s' satisfies the definition of $\lceil\mathsf{CPre}\rceil(\llbracket\varphi_1\rrbracket^{\mathcal{A}}_{\lceil\mathcal{E}\rceil})$ up to the operator $\lceil\cdot\rceil$.

   Second, let us show that s is maximal in $\lceil\mathsf{CPre}\rceil(\llbracket\varphi_1\rrbracket^{\mathcal{A}}_{\lceil\mathcal{E}\rceil})$. By contradiction, assume that there exists $s_1 \in \lceil\mathsf{CPre}\rceil(\llbracket\varphi_1\rrbracket^{\mathcal{A}}_{\lceil\mathcal{E}\rceil})$ with $s \subset s_1$. Then, by the same argument as in the first part of the proof of the inclusion $\llbracket\varphi\rrbracket^{\mathcal{A}}_{\lceil\mathcal{E}\rceil} \subseteq \lceil \llbracket\varphi\rrbracket^{\mathcal{S}}_{\mathcal{E}} \rceil$, we have $s_1 \in \llbracket\varphi\rrbracket^{\mathcal{S}}_{\mathcal{E}}$. This implies that $s \notin \lceil \llbracket\varphi\rrbracket^{\mathcal{S}}_{\mathcal{E}} \rceil$, a contradiction.

- if $\varphi \equiv \nu x.\varphi_1$ and $\lceil \llbracket\varphi_1\rrbracket^{\mathcal{S}}_{\mathcal{E}'} \rceil = \llbracket\varphi_1\rrbracket^{\mathcal{A}}_{\lceil\mathcal{E}'\rceil}$ for any downward closed valuation $\mathcal{E}'$. By Tarski's theorem, $\llbracket\varphi\rrbracket^{\mathcal{S}}_{\mathcal{E}}$ is one of the set in the infinite sequence $q_0, q_1, \dots$ defined by $q_0 = \mathcal{L}$ and for every $i \geq 1$, $q_i = \llbracket\varphi\rrbracket^{\mathcal{S}}_{\mathcal{E}[x \mapsto q_{i-1}]}$; and similarly, $\llbracket\varphi\rrbracket^{\mathcal{A}}_{\lceil\mathcal{E}\rceil}$ is one of the set in the infinite sequence $q'_0, q'_1, \dots$ defined by $q'_0 = \{L\}$ and for all $i \geq 1$, $q_i = \llbracket\varphi\rrbracket^{\mathcal{A}}_{\lceil\mathcal{E}\rceil[x \mapsto q'_{i-1}]}$. Observe that $q'_0 = \lceil q_0 \rceil$. By induction, assume that $q'_{i-1} = \lceil q_{i-1} \rceil$ for some $i \geq 1$. Then $q'_i = \lceil q_i \rceil$ as $\lceil\mathcal{E}\rceil[x \mapsto q'_{i-1}] = \lceil \mathcal{E}[x \mapsto q_{i-1}] \rceil$.
- if $\varphi \equiv \mu x.\varphi_1$ and $\lceil \llbracket\varphi_1\rrbracket^{\mathcal{S}}_{\mathcal{E}'} \rceil = \llbracket\varphi_1\rrbracket^{\mathcal{A}}_{\lceil\mathcal{E}'\rceil}$ for any downward closed valuation $\mathcal{E}'$. The proof is similar to the previous case.

■

Consider a game structure G of imperfect information and a parity objective ϕ. From Theorems 2 and 3 and Lemma 7, we can decide the existence of a deterministic observation-based sure-winning strategy for Player 1 in G for ϕ without explicitly constructing the knowledge-based subset construction G^{K}, by instead evaluating a fixed-point formula in the lattice of antichains.

Theorem 4 (Symbolic solution of imperfect-information games) *Let G be a game structure of imperfect information with initial state l_0. For every ω-regular objective ϕ, Player 1 has a deterministic observation-based strategy in G for ϕ if and only if $\{l_0\} \sqsubseteq \llbracket\mu\mathsf{Form}(\phi)\rrbracket^{\mathcal{A}}$.*

Corollary 1 *Let G be a game structure of imperfect information, let p be a priority function, and let ℓ be a state of G. Whether ℓ is a sure-winning state in G for the parity objective* $\mathsf{Parity}(p)$ *can be decided in* Exptime.

Corollary 1 is proved as follows: for a parity objective ϕ, an equivalent μ-calculus formula φ can be obtained, where the size and the fixed-point quantifier alternations of φ is polynomial in ϕ. Thus given G and ϕ, we can evaluate φ in G^{K} in Exptime.

4 Almost Winning

Given a game structure G of imperfect information, we first construct a game structure H in which the knowledge of Player 1 is made explicit. However, the construction is different from the one used for sure winning. Then, we establish certain equivalences between randomized strategies in G and H. Finally, we show how the reduction can be used to obtain a symbolic EXPTIME algorithm for computing almost-winning states in G for Büchi objectives. An EXPTIME algorithm for almost winning for coBüchi objectives under imperfect information remains unknown.

4.1 Subset construction for almost winning

Given a game structure of imperfect information $G = \langle L, l_0, \Sigma, \Delta, \mathcal{O}, \gamma \rangle$, we construct the game structure $H = \mathsf{Knw}(G) = \langle Q, q_0, \Sigma, \Delta_H \rangle$ as follows: $Q = \{ (s, \ell) \mid \exists o \in \mathcal{O} : s \subseteq \gamma(o) \text{ and } \ell \in s \}$; the initial state is $q_0 = (\{l_0\}, l_0)$; the transition relation $\Delta_H \subseteq Q \times \Sigma \times Q$ is defined by $((s, \ell), \sigma, (s', \ell')) \in \Delta_H$ iff there is an observation $o \in \mathcal{O}$ such that $s' = \mathsf{Post}^G_\sigma(s) \cap \gamma(o)$ and $(\ell, \sigma, \ell') \in \Delta$. Intuitively, when H is in state (s, ℓ), it corresponds to G being in state ℓ and the knowledge of Player 1 being s. Two states $q = (s, \ell)$ and $q' = (s', \ell')$ of H are *equivalent*, written $q \approx q'$, if $s = s'$, that is when the knowledge of Player 1 is the same in the two states. Two prefixes $\rho = q_0\sigma_0q_1 \ldots \sigma_{n-1}q_n$ and $\rho' = q'_0\sigma'_0q'_1 \ldots \sigma'_{n-1}q'_n$ of H are *equivalent*, written $\rho \approx \rho$, if for all $0 \le i \le n$, we have $q_i \approx q'_i$, and for all $0 \le i \le n-1$, we have $\sigma_i = \sigma'_i$. Two plays $\pi, \pi' \in \mathsf{Plays}(H)$ are *equivalent*, written $\pi \approx \pi'$, if for all $i \ge 0$, we have $\pi(i) \approx \pi'(i)$. For a state $q \in Q$, we denote by $[q]_\approx = \{ q' \in Q \mid q \approx q' \}$ the $\approx$-equivalence class of q. We define equivalence classes for prefixes and plays similarly. We cannot reuse the results of Section 3 to compute almost-winning states of G, as the randomized strategies in H should not distinguish equivalent states.

Equivalence-preserving strategies and objectives. A strategy α for Player 1 in H is *positional* if it is independent of the prefix of plays and depends only on the last state, that is, for all $\rho, \rho' \in \mathsf{Prefs}(H)$ with $\mathsf{Last}(\rho) = \mathsf{Last}(\rho')$, we have $\alpha(\rho) = \alpha(\rho')$. A positional strategy α can be viewed as a function $\alpha : Q \to \mathcal{D}(\Sigma)$. A strategy α for Player 1 in H is *equivalence-preserving* if for all $\rho, \rho' \in \mathsf{Prefs}(H)$ with $\rho \approx \rho'$, we have $\alpha(\rho) = \alpha(\rho')$. We denote by $\mathcal{A}_H$, $\mathcal{A}^P_H$, and $\mathcal{A}^\approx_H$ the set of all Player-1 strategies, the set of all positional Player-1 strategies, and the set of all equivalence-preserving Player-1 strategies in H, respectively. We write $\mathcal{A}^{\approx(P)}_H = \mathcal{A}^\approx_H \cap \mathcal{A}^P_H$ for the set of equivalence-preserving positional strategies.

An objective ϕ for H is a subset of $(Q \times \Sigma)^\omega$, that is, the objective ϕ is a set of plays. The objective ϕ is *equivalence-preserving* if for all plays $\pi \in \phi$, we have $[\pi]_\approx \subseteq \phi$.

Relating prefixes and plays. We define a mapping $h : \mathsf{Prefs}(G) \to \mathsf{Prefs}(H)$ that maps prefixes in G to prefixes in H as follows: given $\rho = \ell_0\sigma_0\ell_1\sigma_1 \ldots \sigma_{n-1}\ell_n$, let $h(\rho) = q_0\sigma_0q_1\sigma_1 \ldots \sigma_{n-1}q_n$, where for all $0 \le i \le n$, we have $q_i = (s_i, \ell_i)$, and for all $0 \le i \le n-1$, we have $s_i = \mathsf{K}(\gamma^{-1}(\rho(i)))$. The following properties hold: (i) for all $\rho, \rho' \in \mathsf{Prefs}(G)$, if $\gamma^{-1}(\rho) = \gamma^{-1}(\rho')$, then $h(\rho) \approx h(\rho')$; and (ii) for all $\rho, \rho' \in \mathsf{Prefs}(H)$, if $\rho \approx \rho'$, then $\gamma^{-1}(h^{-1}(\rho)) = \gamma^{-1}(h^{-1}(\rho'))$. The mapping $h : \mathsf{Plays}(G) \to \mathsf{Plays}(H)$ for plays is defined similarly, and has similar properties.

Relating strategies for Player 1. We define two strategy mappings $\overline{g} : \mathcal{A}_H \to \mathcal{A}_G$ and $\overline{h} : \mathcal{A}_G \to \mathcal{A}_H$. Given a Player-1 strategy α_H in H, we construct a Player-1 strategy $\alpha_G = \overline{g}(\alpha_H)$ in G as follows: for all $\rho \in \mathsf{Prefs}(G)$, let $\alpha_G(\rho) = \alpha_H(h(\rho))$. Similarly, given a Player-1 strategy α_G in G, we construct a Player-1 strategy $\alpha_H = \overline{h}(\alpha_G)$ in H as follows: for all $\rho \in \mathsf{Prefs}(H)$, let $\alpha_H(\rho) = \alpha_G(h^{-1}(\rho))$. The following properties hold: (i) for all strategies $\alpha_H \in \mathcal{A}_H$, if α_H is equivalence-preserving, then

$\overline{g}(\alpha_H)$ is observation-based; and (*ii*) for all strategies $\alpha_G \in \mathcal{A}_G$, if α_G is observation-based, then $\overline{h}(\alpha_G)$ is equivalence-preserving.

Relating strategies for Player 2. Observe that for all $q \in Q$, all $\sigma \in \Sigma$, and all $\ell' \in L$, we have $|\{\, s' \mid (q, \sigma, q') \in \Delta_H \wedge q' = (s', \ell') \,\}| \leq 1$. Given a Player-2 strategy β_H in H, we construct a Player-2 strategy $\beta_G = \overline{g}(\beta_H)$ as follows: for all $\rho \in \mathsf{Prefs}(G)$, $\sigma \in \Sigma$, and $\ell' \in L$, let $\beta_G(\rho, \sigma)(\ell') = \beta_H(h(\rho), \sigma)(s', \ell')$ if for $s = \mathsf{K}(\gamma^{-1}(\rho))$ and $\ell = \mathsf{Last}(\rho)$, we have $((s, \ell), \sigma, (s', \ell')) \in \Delta_H$ for some (and then unique) s', and $\beta_G(\rho, \sigma)(\ell') = 0$ otherwise. Similarly, given a Player-2 strategy β_G in G, we construct a Player-2 strategy $\beta_H = \overline{h}(\beta_G)$ in H as follows: for all $\rho \in \mathsf{Prefs}(H)$, all $\sigma \in \Sigma$, and all (s, ℓ), let $\beta_H(\rho, \sigma)((s, \ell)) = \beta_G(h^{-1}(\rho), \sigma)(\ell)$.

Lemma 8 *The following assertions hold.*

1. *For all $\rho_H \in \mathsf{Prefs}(H)$, for every equivalence preserving strategy α_H, for every strategy β_H we have*
$$\mathrm{Pr}_{q_0}^{\alpha_H, \beta_H}(\mathsf{Cone}(\rho_H)) = \mathrm{Pr}_{l_0}^{\overline{g}(\alpha_H), \overline{g}(\beta_H)}(h^{-1}(\mathsf{Cone}(\rho_H))).$$
2. *For all $\rho_G \in \mathsf{Prefs}(G)$, for every observation-based strategy α_G, for every strategy β_G we have*
$$\mathrm{Pr}_{l_0}^{\alpha_G, \beta_G}(\mathsf{Cone}(\rho_G)) = \mathrm{Pr}_{q_0}^{\overline{h}(\alpha_G), \overline{h}(\beta_G)}(h(\mathsf{Cone}(\rho_G))).$$

Proof. The following properties follow from the construction of strategies in Section 4.1.

1. For all $\rho_H \in \mathsf{Prefs}(H)$, for every equivalence preserving strategy α_H, for every strategy β_H and for all ρ'_H such that $|\rho'_H| = |\rho_H| + 1$, we have $\mathrm{Pr}_{q_0}^{\alpha_H, \beta_H}\big(\mathsf{Cone}(\rho'_H) \mid \mathsf{Cone}(\rho_H)\big) = \mathrm{Pr}_{l_0}^{\overline{g}(\alpha_H), \overline{g}(\beta_H)}\big(h^{-1}(\mathsf{Cone}(\rho'_H)) \mid h^{-1}(\mathsf{Cone}(\rho_H))\big)$.
2. For all $\rho_G \in \mathsf{Prefs}(G)$, for every observation-based strategy α_G, for every strategy β_G and for all ρ'_G such that $|\rho'_G| = |\rho_G| + 1$, we have $\mathrm{Pr}_{l_0}^{\alpha_G, \beta_G}\big(\mathsf{Cone}(\rho'_G) \mid \mathsf{Cone}(\rho_G)\big) = \mathrm{Pr}_{q_0}^{\overline{h}(\alpha_G), \overline{h}(\beta_G)}\big(h(\mathsf{Cone}(\rho'_G)) \mid h(\mathsf{Cone}(\rho_G))\big)$.

The proof for the first part is as follows: observe that

$$\begin{aligned}
\mathrm{Pr}_{q_0}^{\alpha_H, \beta_H}(\mathsf{Cone}(q_0)) &= \mathrm{Pr}_{l_0}^{\overline{g}(\alpha_H), \overline{g}(\beta_H)}(h^{-1}(\mathsf{Cone}(q_0))) \\
&= \mathrm{Pr}_{l_0}^{\overline{g}(\alpha_H), \overline{g}(\beta_H)}(\mathsf{Cone}(l_0)) = 1.
\end{aligned}$$

The result follows from the above property and induction. The second part follows analogously. ■

Theorem 5 (Almost-winning reduction) *Let G be a game structure of imperfect information, and let $H = \mathsf{Knw}(G)$. For all Borel objectives ϕ for G, all observation-based Player-1 strategies α_G in G, and all Player-2 strategies β_G in G, we have $\mathrm{Pr}_{l_0}^{\alpha_G, \beta_G}(\phi) = \mathrm{Pr}_{q_0}^{\overline{h}(\alpha_G), \overline{h}(\beta_G)}(h(\phi))$. Dually, for all equivalence-preserving Borel objectives ϕ for H, all equivalence-preserving Player-1 strategies α_H in H, and all Player-2 strategies β_H in H, we have $\mathrm{Pr}_{q_0}^{\alpha_H, \beta_H}(\phi) = \mathrm{Pr}_{l_0}^{\overline{g}(\alpha_H), \overline{g}(\beta_H)}(h^{-1}(\phi))$.*

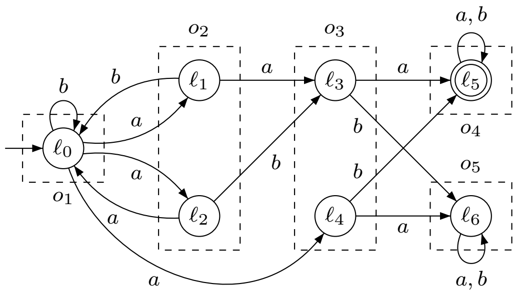

Figure 2: Memory and randomization are necessary to almost win the objective $\mathsf{Buchi}(\{o_4\})$.

Proof. By the Carathe̒odary unique-extension theorem, a probability measure defined on cones has a unique extension to all Borel objectives. The theorem then follows from Lemma 8. ∎

Corollary 2 follows from Theorem 5.

Corollary 2 *For every Borel objective Φ_G for G, we have*

$$\sup_{\alpha_G \in \mathcal{A}_G^O} \inf_{\beta_G \in \mathcal{B}_G} \mathrm{Pr}_{l_0}^{\alpha_G,\beta_G}(\Phi_G) = \sup_{\alpha_H \in \mathcal{A}_H^{\approx}} \inf_{\beta_H \in \mathcal{B}_H} \mathrm{Pr}_{q_0}^{\alpha_H,\beta_H}(h(\Phi_G));$$

$$\exists \alpha_G \in \mathcal{A}_G^O.\ \forall \beta_G \in \mathcal{B}_G:\ \mathrm{Pr}_{l_0}^{\alpha_G,\beta_G}(\Phi_G) = 1$$
$$\textit{iff}\ \exists \alpha_H \in \mathcal{A}_H^{\approx}.\ \forall \beta_H \in \mathcal{B}_H:\ \mathrm{Pr}_{q_0}^{\alpha_H,\beta_H}(h(\Phi_G)) = 1.$$

4.2 Almost winning for Büchi objectives

We first illustrate the need of memory and randomization for almost-winning in imperfect information games with Büchi objectives.

Example 2 (Memory is needed to almost-win) *Consider the example of Figure 2. The objective of Player 1 is to reach a state with observation o_4.*

We show that Player 1 has no observation-based sure-winning strategy in this game. This is because when we fix an observation-based strategy for Player 1, Player 2 has a spoiling strategy to maintain the game into the states $\{\ell_0, \ell_1, \ell_2\}$. Indeed, at ℓ_0, the only reasonable choice for Player 1 is to play a. Then Player 2 can choose to go either in ℓ_1 or ℓ_2. In both cases, the observation will be the same for Player 1. After seeing $o_1 a o_2$, if the strategy of Player 1 is to play a then Player 2 chooses ℓ_2, otherwise, if Player 1 strategy is to play b then Player 2 chooses ℓ_1. This can be repeated and so Player 2 has a spoiling strategy against any observation-based strategy of Player 1.

We now show that almost-winning strategies exist for Player 1. Consider that Player 1 plays an observation-based randomized strategy α as follows: after a sequence of observations τ,

- *if* $\mathsf{Last}(\tau) = o_1$, *then* $\alpha(\tau)(a) = 1$ *and* $\alpha(\tau)(b) = 0$,
- *if* $\mathsf{Last}(\tau) = o_2$, *then* $\alpha(\tau)(a) = 0.5$ *and* $\alpha(\tau)(b) = 0.5$,
- *if* $\tau = \tau' \cdot o_1 \cdot \sigma \cdot o_3$, *then* $\alpha(\tau)(a) = 0$ *and* $\alpha(\tau)(b) = 1$,

- *if $\tau = \tau' \cdot o_2 \cdot \sigma \cdot o_3$, then $\alpha(\tau)(a) = 1$ and $\alpha(\tau)(b) = 0$,*
- *otherwise take, arbitrarily, $\alpha(\tau)(a) = 1$ and $\alpha(\tau)(b) = 0$.*

The strategy α is almost-winning against any randomized strategy of Player 2. Note that the strategy α uses memory and this is necessary because when receiving observation o_3, Player 1 has to play a if the previous state satisfied observation o_1 and b if the previous state satisfied o_2.

Given a game structure G of imperfect information, let $H = \mathsf{Knw}(G)$. Given a set $\mathcal{T} \subseteq \mathcal{O}$ of target observations, let $B_{\mathcal{T}} = \{ (s, \ell) \in Q \mid \exists o \in \mathcal{T} : s \subseteq \gamma(o) \}$. Then $h(\mathsf{Buchi}(\mathcal{T})) = \mathsf{Buchi}(B_{\mathcal{T}}) = \{ \pi_H \in \mathsf{Plays}(H) \mid \mathsf{Inf}(\pi_H) \cap B_{\mathcal{T}} \neq \emptyset \}$. We first show that almost winning in H for the Büchi objective $\mathsf{Buchi}(B_{\mathcal{T}})$ with respect to equivalence-preserving strategies is equivalent to almost winning with respect to equivalence-preserving positional strategies. Formally, for $B_{\mathcal{T}} \subseteq Q$, let $Q^{\approx}_{\mathsf{AS}} = \{ q \in Q \mid \exists \alpha_H \in \mathcal{A}^{\approx}_H \cdot \forall \beta_H \in \mathcal{B}_H \cdot \forall q' \in [q]_\approx : \Pr_{q'}^{\alpha_H,\beta_H}(\mathsf{Buchi}(B_{\mathcal{T}})) = 1 \}$, and $Q^{\approx(P)}_{\mathsf{AS}} = \{ q \in Q \mid \exists \alpha_H \in \mathcal{A}^{\approx(P)}_H \cdot \forall \beta_H \in \mathcal{B}_H \cdot \forall q' \in [q]_\approx : \Pr_{q'}^{\alpha_H,\beta_H}(\mathsf{Buchi}(B_{\mathcal{T}})) = 1 \}$. We will prove that $Q^{\approx}_{\mathsf{AS}} = Q^{\approx(P)}_{\mathsf{AS}}$. Lemma 9 follows from the construction of H from G, and yields Lemma 10.

Lemma 9 *For all $q_1 \in Q$, and all $\sigma \in \Sigma$, if $(q_1, \sigma, q_1') \in \Delta_H$, then for all $q_2' \in [q_1']_\approx$, there exists $q_2 \in [q_1]_\approx$ such that $(q_2, \sigma, q_2') \in \Delta_H$.*

Lemma 10 *Given an equivalence-preserving Player-1 strategy $\alpha_H \in \mathcal{A}_H$, a prefix $\rho \in \mathsf{Prefs}(H)$, and a state $q \in Q$, if there exists a Player-2 strategy $\beta_H \in \mathcal{B}_H$ such that $\Pr_q^{\alpha_H,\beta_H}(\mathsf{Cone}(\rho)) > 0$, then for every prefix $\rho' \in \mathsf{Prefs}(H)$ with $\rho \approx \rho'$, there exist a Player-2 strategy $\beta'_H \in \mathcal{B}_H$ and a state $q' \in [q]_\approx$ such that $\Pr_{q'}^{\alpha_H,\beta'_H}(\mathsf{Cone}(\rho')) > 0$.*

Observe that $Q \setminus Q^{\approx}_{\mathsf{AS}} = \{ q \in Q \mid \forall \alpha_H \in \mathcal{A}^{\approx}_H \cdot \exists \beta_H \in \mathcal{B}_H \cdot \exists q' \in [q]_\approx : \Pr_{q'}^{\alpha_H,\beta_H}(\mathsf{Buchi}(B_{\mathcal{T}})) < 1 \}$. It follows from Lemma 10 that if a play starts in $Q^{\approx}_{\mathsf{AS}}$ and reaches $Q \setminus Q^{\approx}_{\mathsf{AS}}$ with positive probability, then for all equivalence-preserving strategies for Player 1, there is a Player 2 strategy that ensures that the Büchi objective $\mathsf{Buchi}(B_{\mathcal{T}})$ is satisfied with probability strictly lower than 1.

Notation. For a state $q \in Q$ and $Y \subseteq Q$, let $\mathsf{Allow}(q, Y) = \{ \sigma \in \Sigma \mid \mathsf{Post}^H_\sigma(q) \subseteq Y \}$. For a state $q \in Q$ and $Y \subseteq Q$, let $\mathsf{Allow}([q]_\approx, Y) = \bigcap_{q' \in [q]_\approx} \mathsf{Allow}(q', Y)$.

Lemma 11 *For all $q \in Q^{\approx}_{\mathsf{AS}}$, we have $\mathsf{Allow}([q]_\approx, Q^{\approx}_{\mathsf{AS}}) \neq \emptyset$.*

Proof. Assume towards contradiction that there exists $q \in Q^{\approx}_{\mathsf{AS}}$ such that $\mathsf{Allow}([q]_\approx, Q^{\approx}_{\mathsf{AS}}) = \emptyset$. Then for all $\sigma \in \Sigma$ there exists $q' \in [q]_\approx$ such that $\mathsf{Post}^H_\sigma(q') \cap (Q \setminus Q^{\approx}_{\mathsf{AS}}) \neq \emptyset$. Hence for every equivalence preserving strategy α_H there exists $q' \in [q]_\approx$ such that $\alpha_H(q')(\sigma) > 0$ and $\mathsf{Post}^H_\sigma(q') \cap (Q \setminus Q^{\approx}_{\mathsf{AS}}) \neq \emptyset$. Hence for every equivalence strategy α_H there is a state $q' \in [q]_\approx$ and a strategy β_H for Player 2 such that $Q \setminus Q^{\approx}_{\mathsf{AS}}$ is reached with positive probability. This contradicts that $[q]_\approx \subseteq Q^{\approx}_{\mathsf{AS}}$. ∎

Lemma 12 *Given a state $q \in Q^{\approx}_{\mathsf{AS}}$, let $\alpha_H \in \mathcal{A}_H$ be an equivalence-preserving Player-1 strategy such that for all Player-2 strategies $\beta_H \in \mathcal{B}_H$ and all states $q' \in [q]_\approx$, we have $\Pr_{q'}^{\alpha_H,\beta_H}(\mathsf{Buchi}(B_{\mathcal{T}})) = 1$. Let $\rho = q_0\sigma_0q_1 \ldots \sigma_{n-1}q_n$ be a prefix in $\mathsf{Prefs}(H)$ such that for all $0 \leq i \leq n$, we have $q_i \in Q^{\approx}_{\mathsf{AS}}$. If there is a Player-2 strategy $\beta_H \in \mathcal{B}_H$ and a state $q' \in [q]_\approx$ such that $\Pr_{q'}^{\alpha_H,\beta_H}(\mathsf{Cone}(\rho)) > 0$, then $\mathsf{Supp}(\alpha_H(\rho)) \subseteq \mathsf{Allow}([q_n]_\approx, Q^{\approx}_{\mathsf{AS}})$.*

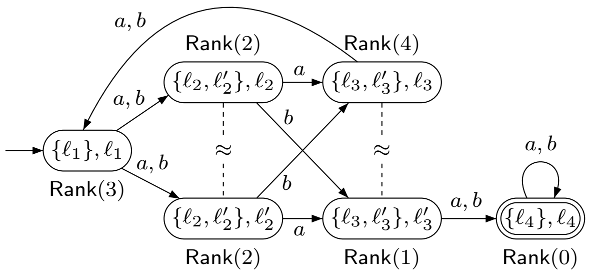

Figure 3: Game structure $H = \mathsf{Knw}(G)$ (for G of Figure 1).

Proof. Fix an almost-winning strategy α_H. Assume towards contradiction for a history ρ_H satisfying the conditions of the lemma that there exists $\sigma \in \mathsf{Supp}(\alpha_H(\rho_H)) \setminus \mathsf{Allow}([q_n], Q^{\approx}_{\mathsf{AS}})$. Then there exists $q_n' \in [q_n]_\approx$ such that $\mathsf{Post}^H_\sigma(q_n') \cap (Q \setminus Q^{\approx}_{\mathsf{AS}}) \neq \emptyset$. Then there exists ρ_H' such that $\rho_H \approx \rho_H'$ and $\mathsf{Last}(\rho_H') = q_n'$. Then by Lemma 10 there exists a strategy β_H' and $q' \in [q]_\approx$ such that $\Pr^{\alpha_H, \beta_H'}_{q'}(\mathsf{Cone}(\rho_H')) > 0$. Then given ρ_H' and the strategy α_H there exists a Player 2 strategy such that $Q \setminus Q^{\approx}_{\mathsf{AS}}$ is reached with positive probability. This contradicts that α_H is an almost-winning strategy. ■

Notation. We inductively define the *ranks* of states in $Q^{\approx}_{\mathsf{AS}}$ as follows: let $\mathsf{Rank}(0) = B_{\mathcal{T}} \cap Q^{\approx}_{\mathsf{AS}}$, and for all $j \geq 0$, let $\mathsf{Rank}(j+1) = \mathsf{Rank}(j) \cup \{ q \in Q^{\approx}_{\mathsf{AS}} \mid \exists \sigma \in \mathsf{Allow}([q]_\approx, Q^{\approx}_{\mathsf{AS}}) : \mathsf{Post}^H_\sigma(q) \subseteq \mathsf{Rank}(j) \}$. Let $j^* = \min\{ j \geq 0 \mid \mathsf{Rank}(j) = \mathsf{Rank}(j+1) \}$, and let $Q^* = \mathsf{Rank}(j^*)$. We say that the set $\mathsf{Rank}(j+1) \setminus \mathsf{Rank}(j)$ contains the *states of rank* $j+1$, for all $j \geq 0$.

Example 3 *Given the game structure G of imperfect information from Figure 1, the game structure $H = \mathsf{Knw}(G)$ is shown in Figure 3. All states are almost winning for the Büchi objective $\mathsf{Buchi}(\{ (\{\ell_4\}, \ell_4) \})$. The ranks of the states are shown next to the states. The positional strategy that plays both a and b with equal probability is almost winning at all states. For the states q with ranks 1, 3, and 4, if the rank of q is j, then $\mathsf{Post}^H_a(q) \subseteq \mathsf{Rank}(j-1)$ and $\mathsf{Post}^H_b(q) \subseteq \mathsf{Rank}(j-1)$. For the states with rank 2, if $q = (\{ \ell_2, \ell_2' \}, \ell_2)$, then $\mathsf{Post}^H_b(q) \subseteq \mathsf{Rank}(1)$; and if $q = (\{ \ell_2, \ell_2' \}, \ell_2')$, then $\mathsf{Post}^H_a(q) \subseteq \mathsf{Rank}(1)$.*

Lemma 13 $Q^* = Q^{\approx}_{\mathsf{AS}}$.

Proof. By definition, $Q^* \subseteq Q^{\approx}_{\mathsf{AS}}$. We now prove that $Q^{\approx}_{\mathsf{AS}} \subseteq Q^*$. Assume towards a contradiction that $X = Q^{\approx}_{\mathsf{AS}} \setminus Q^* \neq \emptyset$. For all states $q \in X$ and all $\sigma \in \mathsf{Allow}([q]_\approx, Q^{\approx}_{\mathsf{AS}})$, we have $\mathsf{Post}^H_\sigma(q) \cap X \neq \emptyset$, because otherwise q would have been in Q^*. Hence, for all $q \in X$ and all $\sigma \in \mathsf{Allow}([q]_\approx, Q^{\approx}_{\mathsf{AS}})$, there exists a $q' \in X$ such that $(q, \sigma, q') \in \Delta_H$. Fix a strategy β_H for Player 2 as follows: for a state $q \in X$ and the input letter $\sigma \in \mathsf{Allow}([q]_\approx, Q^{\approx}_{\mathsf{AS}})$, choose a successor $q' \in X$ such that $(q, \sigma, q') \in \Delta_H$. Consider a state $q \in X$ and an equivalence-preserving almost-winning strategy α_H for Player 1 from q for the objective $\mathsf{Buchi}(B_{\mathcal{T}})$. By Lemma 12, for every prefix ρ satisfying the condition of Lemma 12, we have $\mathsf{Supp}(\alpha_H(\rho)) \subseteq \mathsf{Allow}([\mathsf{Last}(\rho)]_\approx, Q^{\approx}_{\mathsf{AS}})$. It follows that $\Pr^{\alpha_H, \beta_H}_q(\mathsf{Safe}(X)) = 1$. Since $B_{\mathcal{T}} \cap Q^{\approx}_{\mathsf{AS}} \subseteq Q^*$, it follows that $B_{\mathcal{T}} \cap X = \emptyset$. Hence $\Pr^{\alpha_H, \beta_H}_q(\mathsf{Reach}(B_{\mathcal{T}})) = 0$, and therefore $\Pr^{\alpha_H, \beta_H}_q(\mathsf{Buchi}(B_{\mathcal{T}})) = 0$. This contradicts the fact that α_H is an almost-winning strategy. ■

Equivalence-preserving positional strategy. Consider the equivalence-preserving positional strategy α^p_H for Player 1 in H, which is defined as follows: for a state $q \in Q^{\approx}_{\mathsf{AS}}$, choose all moves in $\mathsf{Allow}([q]_\approx, Q^{\approx}_{\mathsf{AS}})$ uniformly at random.

Lemma 14 *For all states $q \in Q^{\approx}_{\mathsf{AS}}$ and all Player-2 strategies β_H in H, we have $\Pr_q^{\alpha^p_H,\beta_H}(\mathsf{Safe}(Q^{\approx}_{\mathsf{AS}})) = 1$ and $\Pr_q^{\alpha^p_H,\beta_H}(\mathsf{Reach}(B_{\mathcal{T}} \cap Q^{\approx}_{\mathsf{AS}})) = 1$.*

Proof. By Lemma 13, we have $Q^* = Q^{\approx}_{\mathsf{AS}}$. Let $z = |Q^*|$.

- For a state $q \in Q^{\approx}_{\mathsf{AS}}$, we have $\mathsf{Post}^H_\sigma(q) \subseteq Q^{\approx}_{\mathsf{AS}}$ for all $\sigma \in \mathsf{Allow}([q]_\approx, Q^{\approx}_{\mathsf{AS}})$. It follows for all states $q \in Q^{\approx}_{\mathsf{AS}}$ and all strategies β_H for Player 2, we have $\Pr_q^{\alpha^p_H,\beta_H}(\mathsf{Safe}(Q^{\approx}_{\mathsf{AS}})) = 1$.
- For a state $q \in (\mathsf{Rank}(j+1) \setminus \mathsf{Rank}(j))$, there exists $\sigma \in \mathsf{Allow}([q]_\approx, Q^{\approx}_{\mathsf{AS}})$ such that $\mathsf{Post}^H_\sigma(q) \subseteq \mathsf{Rank}(j)$. For a set $Y \subseteq Q$, let $\Diamond^j(Y)$ denote the set of prefixes that reach Y after at most j steps. It follows that for all states $q \in \mathsf{Rank}(j+1)$ and all strategies β_H for Player 2, we have

$$\Pr_q^{\alpha^p_H,\beta_H}(\Diamond^1(\mathsf{Rank}(j))) \geq \frac{1}{|\Sigma|}.$$

Let $B = B_{\mathcal{T}} \cap Q^{\approx}_{\mathsf{AS}}$. By induction on the ranks it follows that for all states $q \in Q^*$ and all strategies β for Player 2:

$$\Pr_q^{\alpha^p_H,\beta_H}(\Diamond^z(\mathsf{Rank}(0))) = \Pr_q^{\alpha^p_H,\beta_H}(\Diamond^z(B)) \geq \left(\frac{1}{|\Sigma|}\right)^z = r > 0.$$

For $m > 0$, we have $\Pr_q^{\alpha^p_H,\beta_H}(\Diamond^{m\cdot z}(B)) \geq 1 - (1-r)^m$. Thus:

$$\Pr_q^{\alpha^p_H,\beta_H}(\mathsf{Reach}(B)) = \lim_{m\to\infty} \Pr_q^{\alpha^p_H,\beta_H}(\Diamond^{m\cdot z}(B)) \geq \lim_{m\to\infty} 1 - (1-r)^m = 1.$$

The lemma follows. ∎

Lemma 14 implies that, given the Player-1 strategy α^p_H, the set $Q^{\approx}_{\mathsf{AS}}$ is never left, and the states in $B_{\mathcal{T}} \cap Q^{\approx}_{\mathsf{AS}}$ are reached with probability 1. Since this happens for every state in $Q^{\approx}_{\mathsf{AS}}$, it follows that the set $B_{\mathcal{T}} \cap Q^{\approx}_{\mathsf{AS}}$ is visited infinitely often with probability 1, that is, the Büchi objective $\mathsf{Buchi}(B_{\mathcal{T}})$ is satisfied with probability 1. This analysis, together with the fact that $[q_0]_\approx$ is a singleton and Corollary 2, proves that $Q^{\approx}_{\mathsf{AS}} = Q^{\approx(P)}_{\mathsf{AS}}$. Theorem 6 follows.

Theorem 6 (Positional almost winning for Büchi objectives under imperfect information) *Let G be a game structure of imperfect information, and let $H = \mathsf{Knw}(G)$. For all sets $\mathcal{T}$ of observations, there exists an observation-based almost-winning strategy for Player 1 in G for the objective $\mathsf{Buchi}(\mathcal{T})$ iff there exists an equivalence-preserving positional almost-winning strategy for Player 1 in H for the objective $\mathsf{Buchi}(B_{\mathcal{T}})$.*

Symbolic algorithm. We present a symbolic quadratic-time (in the size of H) algorithm to compute the set $Q^{\approx}_{\mathsf{AS}}$. For $Y \subseteq Q$ and $X \subseteq Y$, let $\mathsf{Apre}(Y,X) = \{\, q \in Y \mid \exists \sigma \in \mathsf{Allow}([q]_\approx, Y) : \mathsf{Post}^H_\sigma(q) \subseteq X \,\}$; and $\mathsf{Spre}(Y) = \{\, q \in Y \mid \mathsf{Allow}([q]_\approx, Y) \neq \emptyset \,\}$. Note that $\mathsf{Spre}(Y) = \mathsf{Apre}(Y,Y)$. Let

$$\phi = \nu Y.\mu X.\big(\mathsf{Apre}(Y,X) \vee (B_{\mathcal{T}} \wedge \mathsf{Spre}(Y))\big)$$

and let $Z = [\![\phi]\!]$.

Lemma 15 *$Z = Q^{\approx}_{\mathsf{AS}}$.*

Proof of Lemma 15. We prove $Z = Q^{\approx}_{\mathsf{AS}}$ by proving inclusion in both directions. We have $Z = [\![\phi]\!]$ and $\phi = \nu Y.\mu X.\big(\mathsf{Apre}(Y,X) \vee (B_{\mathcal{T}} \wedge \mathsf{Spre}(Y))\big)$

1. We first show that $Z \subseteq Q^{\approx}_{\mathsf{AS}}$. Since Z is a fixed-point of ϕ we have

$$Z = [\![\mu X.\big(\mathsf{Apre}(Z,X) \vee (B_{\mathcal{T}} \wedge \mathsf{Spre}(Z))\big)]\!] \,.$$

We analyze the evaluation of Z as the fixed-point as follows: let $X_0 = \emptyset$ and $X_{i+1} = \mathsf{Apre}(Z,X_i) \vee (B_{\mathcal{T}} \wedge \mathsf{Spre}(Z))$. Observe that since $X_0 = \emptyset$ we have $\mathsf{Apre}(Z,X_0) = \emptyset$ and hence $X_1 = B_{\mathcal{T}} \cap \mathsf{Spre}(Z) \subseteq B_{\mathcal{T}}$. Let $j^* = \min\{\, i \mid X_{i+1} = X_i \,\}$ and we have $Z = X_{j^*}$. Consider the equivalence preserving strategy α^p_H for Player 1 that at a state $q \in Z$ plays all moves in $\mathsf{Allow}([q]_{\approx}, Z)$ uniformly at random. For all $q \in Z$, for all $q' \in [q]_{\approx}$, and for all $\sigma \in \mathsf{Allow}([q]_{\approx}, Z)$, we have $\mathsf{Post}^H_\sigma(q') \subseteq Z$. It follows that for all strategies β_H for Player 2 and for all states $q \in Z$ we have $\Pr_q^{\alpha^p_H,\beta_H}(\mathsf{Safe}(Z)) = 1$. Also for a state $q \in (X_{i+1} \backslash X_i) \backslash B_{\mathcal{T}}$ we have there exists $\sigma \in \mathsf{Allow}([q]_{\approx}, Z)$ such that $\mathsf{Post}^H_\sigma(q) \subseteq X_i$, i.e., for a state $q \in (X_{i+1} \setminus X_i) \setminus B_{\mathcal{T}}$, given α^p_H against all strategies β_H the next state is in X_i with probability at least $\frac{1}{|\Sigma|}$. Arguments similar to Lemma 14 establishes that α^p_H is an almost-winning strategy for all states $q \in Z$. Hence we have $Z \subseteq Q^{\approx}_{\mathsf{AS}}$.

2. We now show that $Q^{\approx}_{\mathsf{AS}} \subseteq Z$. We first show that $Q^{\approx}_{\mathsf{AS}}$ satisfies that

$$Q^{\approx}_{\mathsf{AS}} = [\![\mu X.\big(\mathsf{Apre}(Q^{\approx}_{\mathsf{AS}},X) \vee (B_{\mathcal{T}} \wedge \mathsf{Spre}(Q^{\approx}_{\mathsf{AS}}))\big)]\!] \,.$$

Observe that $Q^{\approx}_{\mathsf{AS}} = \mathsf{Spre}(Q^{\approx}_{\mathsf{AS}})$. We now analyze the evaluation of $Q^{\approx}_{\mathsf{AS}}$ as the fixed-point Q^* as shown in Lemma 13. Let $X_0 = \emptyset$, then $\mathsf{Apre}(Q^{\approx}_{\mathsf{AS}}, X_0) = \emptyset$. Hence $X_1 = \mathsf{Apre}(Q^{\approx}_{\mathsf{AS}}) \vee (B_{\mathcal{T}} \wedge \mathsf{Spre}(Q^{\approx}_{\mathsf{AS}})) = B_{\mathcal{T}} \wedge Q^{\approx}_{\mathsf{AS}} = \mathsf{Rank}(0)$ (as defined before Lemma 13). By the definition of $\mathsf{Rank}(j+1)$ from $\mathsf{Rank}(j)$ and the definition of $\mathsf{Apre}(\cdot,\cdot)$ and $\mathsf{Spre}(\cdot)$ it follows that for all $i > 0$, given $\mathsf{Rank}(i-1) = X_i$, we have $\mathsf{Rank}(i) = X_{i+1} = \mathsf{Apre}(Q^{\approx}_{\mathsf{AS}}, X_i) \vee (B_{\mathcal{T}} \wedge \mathsf{Spre}(Q^{\approx}_{\mathsf{AS}}))$. By induction we have $Q^* = [\![\mu X.\big(\mathsf{Apre}(Q^{\approx}_{\mathsf{AS}},X) \vee (B_{\mathcal{T}} \wedge \mathsf{Spre}(Q^{\approx}_{\mathsf{AS}}))\big)]\!]$. Since $Q^* = Q^{\approx}_{\mathsf{AS}}$ (by Lemma 13) we have

$$Q^{\approx}_{\mathsf{AS}} = [\![\mu X.\big(\mathsf{Apre}(Q^{\approx}_{\mathsf{AS}},X) \vee (B_{\mathcal{T}} \wedge \mathsf{Spre}(Q^{\approx}_{\mathsf{AS}}))\big)]\!] \,.$$

Since Z is the greatest fixed-point we have $Q^{\approx}_{\mathsf{AS}} \subseteq Z$.

The result follows. ■

Theorem 7 (Complexity of almost winning for Büchi objectives under imperfect information) *Let G be a game structure of imperfect information, let $\mathcal{T}$ be a set of observations, and let ℓ be a state of G. Whether ℓ is an almost-winning state in G for the Büchi objective* $\mathsf{Buchi}(\mathcal{T})$ *can be decided in* EXPTIME.

The facts that $Z = Q^{\approx}_{\mathsf{AS}}$ and that H is exponential in the size of G yield Theorem 7. The arguments for the proofs of Theorem 6 and 7 do not directly extend to coBüchi or parity objectives. In fact, Theorem 6 does not hold for parity objectives in general, for the following reason: in concurrent games with parity objectives with more than two priorities, almost-winning strategies may require infinite memory; for an example, see [5]. Such concurrent games are reducible to semiperfect-information games [4], and semiperfect-information games are reducible to the imperfect-information games we study. Hence a reduction to finite game structures of perfect information in order to obtain randomized positional strategies is not possible with respect to almost winning for general parity objectives. Theorem 6 and Theorem 7 may hold for coBüchi objectives, but there does not seem to be a simple extension of our arguments for Büchi objectives to the coBüchi case. The results that correspond to Theorems 6 and 7 for coBüchi objectives are open.

Direct symbolic algorithm. As in Section 3.2, the subset structure H does not have to be constructed explicitly. Instead, we can evaluate a fixed-point formula on a well-chosen lattice. The fixed-point formula to compute the set $Q^{\approx}_{\mathsf{AS}}$ is evaluated on the lattice $\langle 2^Q, \subseteq, \cup, \cap, Q, \emptyset \rangle$. It is easy to show that the sets computed by the fixed-point algorithm are downward closed for the following order on Q: for $(s, \ell), (s', \ell') \in Q$, let $(s, \ell) \preceq (s', \ell')$ iff $\ell = \ell'$ and $s \subseteq s'$. Then, we can define an antichain over Q as a set of pairwise $\preceq$-incomparable elements of Q, and compute the almost-sure winning states in the lattice of antichains over Q, without explicitly constructing the exponential game structure H.

5 Lower Bounds

We show that deciding the existence of a deterministic (resp. randomized) observation-based sure-winning (resp. almost-winning) strategy for Player 1 in games of imperfect information is EXPTIME-hard already for reachability objectives. A first proof for sure-winning was given in [19]. We give all the details of the reduction used in the proof and show that it extends to almost winning as well.

Sure winning. To show the lower bound result, we use a reduction of the membership problem for polynomial space Alternating Turing Machine. An *alternating Turing machine* (ATM) is a tuple $M = \langle Q, q_0, g, \Sigma_i, \Sigma_t, \delta, F \rangle$ where:

- Q is a finite set of control states;
- $q_0 \in Q$ is the initial state;
- $g : Q \to \{\wedge, \vee\}$;
- $\Sigma_i = \{0, 1\}$ is the input alphabet;
- $\Sigma_t = \{0, 1, 2\}$ is the tape alphabet and 2 is the *blank* symbol;
- $\delta \subseteq Q \times \Sigma_t \times Q \times \Sigma_t \times \{-1, 1\}$ is a transition relation; and
- $F \subseteq Q$ is the set of accepting states.

We say that M is a *polynomial space* ATM if for some polynomial $p(\cdot)$, the space used by M on any input word w is bounded by $p(|w|)$.

Without loss of generality, we make the hypothesis that the initial control state of the machine is a $\vee$-state and that transitions link $\vee$-state to $\wedge$-state and vice versa. A word w is *accepted* by an ATM M if there exists a run tree of M on w whose all leaf nodes are accepting configurations (see [3] for details). The AND-OR graph of the polynomial space ATM (M, p) on the input word $w \in \Sigma^*$ is $G(M, p) = \langle S_\vee, S_\wedge, s_0, \Rightarrow, R \rangle$ where

- $S_\vee = \{(q, h, t) \mid q \in Q,\ g(q) = \vee, 1 \le h \le p(|w|) \text{ and } t \in \Sigma_t^{p(|w|)}\}$;
- $S_\wedge = \{(q, h, t) \mid q \in Q,\ g(q) = \wedge, 1 \le h \le p(|w|) \text{ and } t \in \Sigma_t^{p(|w|)}\}$;
- $s_0 = (q_0, 1, t)$ where $t = w.\Sigma_t^{p(|w|)-|w|}$;
- $((q_1, h_1, t_1), (q_2, h_2, t_2)) \in \Rightarrow$ iff there exists $(q_1, t_1(h_1), q, \gamma, d) \in \delta$ such that $q_2 = q$, $h_2 = h_1 + d$, $t_2(h_1) = \gamma$ and $t_2(i) = t_1(i)$ for all $i \neq h_1$;
- $R = \{(q, h, t) \in S_\vee \cup S_\wedge \mid q \in F\}$.

A word w is *accepted* by (M, p) iff R is reachable in $G(M, p)$. The *membership problem* is to decide if a given word w is accepted by a given polynomial space ATM (M, p). This problem is known to be ExpTime-hard [3].

Idea of the reduction. Given a polynomial space ATM M and a word w, we construct a game of size polynomial in the size of (M, w) to simulate the execution of M on w. Player 1 makes choices in $\vee$-states and Player 2 makes choices in $\wedge$-states. Furthermore, Player 1 is responsible for maintaining the symbol under the tape head. The objective is to reach an accepting configuration of the ATM.

Each turn proceeds as follows. In an $\vee$-state, by choosing a letter (t, a) in the alphabet of the game, Player 1 reveals (i) the transition t of the ATM that he has chosen (this way he also reveals the symbol that is currently under the tape head) and (ii) the symbol a under the next position of the tape head. If Player 1 lies about the current or the next symbol under the tape head, he should loose the game, otherwise the game proceeds. The machine is now in an $\wedge$-state and Player 1 has no choice: he announces a special symbol ϵ and Player 2, by resolving nondeterminism on ϵ, chooses a transition of the Turing machine which is compatible with the current symbol under the tape head revealed by Player 1 at the previous turn. The state of the ATM is updated and the game proceeds. The transition chosen by Player 2 is visible in the next state of the game and so Player 1 can update his knowledge about the configuration of the ATM. Player 1 wins whenever an accepting configuration of the ATM is reached, that is w is accepted.

The difficulty is to ensure that Player 1 looses when he announces a wrong content of the cell under the tape head. As the number of configurations of the polynomial ATM is exponential, we cannot directly encode the full configuration of the ATM in the states of the game. To overcome this difficulty, we use the power of imperfect information as follows. Initially, Player 2 chooses a position k, $1 \le k \le p(|w|)$, on the tape: this number as well as the symbol $\sigma \in \{0, 1, 2\}$ that lies in the tape cell number k is maintained all along the game in the non-observable portion of the game states. The pair (σ, k) is thus private to Player 2 and invisible to Player 1. Hence, at any point in the game, Player 2 can check whether Player 1 is lying when announcing the content of cell number k, and go to a sink state if Player 1 cheats (no other states can be reached from there). Since Player 1 does not know which cell is monitored by Player 2 (k is private), to avoid loosing, he should not lie about any of the tape cells and thus he should faithfully simulate the machine. Then, he wins the game if and only if the ATM accepts the words w.

Almost winning. To establish lower bound for almost-winning, we can use the same reduction. Randomization can not help Player I in this game. Indeed, at any point of the game, if Player I takes a chance in either not faithfully simulating the ATM or lying about the symbol under the tape head, the sink state is reached. In those case, the probability to reach the sink state is positive and so the probability to win the game is strictly less than one. We now present the details of the reduction of the hardness proof.

Reduction. Given a polynomial space ATM (M, p), with $M = \langle Q, q_0, g, \Sigma_i, \Sigma_t, \delta, F\rangle$ and a word w, we construct the following game structure $G_{M,p,w} = \langle L, l_0, \Sigma, \Delta, \mathcal{O}, \gamma\rangle$, where:

- The set of positions $L = \{\mathsf{init}\} \cup \{\mathsf{sink}\} \cup L_1 \cup L_2$ where: $L_1 = (\delta \cup \{-\}) \times Q \times \{1, \dots, p(|w|)\} \times \{1, \dots, p(|w|)\} \times \Sigma_t$. A state (t, q, h, k, σ) consists of a transition $t \in \delta$ of the ATM chosen by Player 2 at the previous round or $-$ if this is the first round where Player 1 plays, the current control state q of M, the position h of the tape head, the pair (k, σ) such that the k-th symbol of the tape is σ, this pair (k, σ) will be kept invisible for Player 1. $L_2 = Q \times \{1, \dots, p(|w|)\} \times \Sigma_t \times \{1, \dots, p(|w|)\} \times \Sigma_t$. A state $(q, h, \gamma, k, \sigma)$ consists of q, h, k, σ as in L_1 and γ is the symbol that Player 1 claims to be under the tape head. The objective for Player 1 will be to reach a state $\ell \in L$ associated with an accepting control state of M.
- $l_0 = \mathsf{init}$.
- $\Sigma = \{\epsilon\} \cup (\delta \times \Sigma_t)$.
- The transition relation Δ contains the following sets of transitions:
  - I_1 that contains transitions $(\mathsf{init}, \epsilon, (-, q_0, 1, k, \sigma))$ where (i) $1 \le k \le p(|w|)$ and (ii) $\sigma = w(k)$ if $1 \le k \le |w|$ and $\sigma = 2$ otherwise. I_2 that contains transitions $(\mathsf{init}, (t, \gamma), \mathsf{sink})$ where $(t, \gamma) \in \delta \times \Sigma_t$. So, at the initial state init, Player 1 has to play ϵ in order to avoid entering sink. By resolving nondeterminism on ϵ, Player 2 chooses a tape cell to monitor.
  - S that contains transitions $(\mathsf{sink}, \sigma, \mathsf{sink})$ for all $\sigma \in \Sigma$. When the sink state is entered, the game stays there forever.
  - $L_{1.1}$ that contains transitions $(\ell_1, \epsilon, \mathsf{sink})$ for all $\ell_1 \in L_1$; $L_{1.2}$ that contains transitions $((t, q, h, k, \sigma), ((q_1, \gamma_1, q_2, \gamma_2, d), \gamma_3), \mathsf{sink})$ where $q_1 \ne q$ or $\neg(1 \le h+d \le p(|w|))$; $L_{1.3}$ contains the transitions $((t, q, h, k, \sigma), ((q_1, \gamma_1, q_2, \gamma_2, d), \gamma_3), \mathsf{sink})$ where $h = k \wedge \gamma_1 \ne \sigma$ or $h + d = k \wedge \gamma_3 \ne \sigma$; $L_{1.4}$ contains the transitions $((t, q, h, k, \sigma), ((q_1, \gamma_1, q_2, \gamma_2, d), \gamma_3), (q_2, h + d, \gamma_3, k, \sigma'))$ such that $q = q_1$, $1 \le h + d \le p(|w|)$, $h = k \to (\gamma_1 = \sigma \wedge \sigma' = \gamma_2)$, and $h \ne k \to \sigma' = \sigma$. Those transitions are associated with states of the game where Player 1 chooses a transition of the ATM to execute (if he proposes ϵ, the game evolves to the sink state, see $L_{1.1}$). The transition proposed by Player 1 should be valid for the current control state of the ATM and the head should not exit the bounded tape after execution of the transition by the ATM, otherwise the game evolves to the sink state, see $L_{1.2}$. When choosing a letter, Player 1 also reveals the current letter under the tape head (given by the transition) as well as the letter under the next position of the tape head. If one of those positions is the one that is monitored by Player 2, the game evolves to the sink state in case Player 1 lies, see $L_{1.3}, L_{1.4}$.
  - $L_{2.1}$ contains the transitions $((q, h, \gamma_1, k, \sigma), \epsilon, ((q_1, \gamma_2, q_2, \gamma_3, d), q_3, h+d, k, \sigma'))$ such that $q = q_1$, $q_2 = q_3$, $\gamma_1 = \gamma_2$, $1 \le h + d \le p(|w|)$, $h = k \to \sigma' = \gamma_3$, and $h \ne k \to \sigma' = \sigma$; $L_{2.2}$ contains the transitions $((q, h, \gamma_1, k, \sigma), \epsilon, \mathsf{sink})$ such that there does not exist a transition $(q, \gamma_1, q_1, \gamma_2, d) \in \delta$ with $1 \le h + d \le p(|w|)$; $L_{2.3}$ contains the transitions $((q, h, \gamma_1, k, \sigma), (t, \gamma), \mathsf{sink})$ where $(t, \gamma) \in \Sigma \setminus \{\epsilon\}$. Those transitions are associated with states of the game where Player 2 chooses the next transition of the ATM to execute.

Player 1 should play ϵ otherwise the game goes to the sink state (see $L_{2.3}$). Also the game goes to the sink state if there is no valid transition to execute in the ATM (see $L_{2.2}$). In the other cases, when Player 1 proposes ϵ, Player 2 chooses a valid transition by resolving nondeterminism. The copy of the monitored cell is updated if necessary.

- $\mathcal{O} = \{\mathsf{init}, \mathsf{sink}\} \cup \mathcal{O}_1 \cup \mathcal{O}_2$ where $\mathcal{O}_1 = \{(t, q, h) \mid \exists(t, q, h, k, \sigma) \in L_1\}$ and $\mathcal{O}_1 = \{(q, h, \gamma) \mid \exists(q, h, \gamma, k, \sigma) \in L_2\}$.
- γ is defined as follows: $\gamma(\mathsf{init}) = \{\mathsf{init}\}$, $\gamma(\mathsf{sink}) = \{\mathsf{sink}\}$, for all $(t, q, h) \in \mathcal{O}_1$, $\gamma(t, q, h) = \{(t, q, h, k, \sigma) \in L_1\}$, for all $(q, h, \gamma) \in \mathcal{O}_2$, $\gamma(q, h, \gamma) = \{(q, h, \gamma, k, \sigma) \in L_2\}$.

Finally, the objective ϕ of this game for Player 1 is to reach a state where the associated control state of the ATM is accepting, i.e. $\phi = \{o_1 o_2 \dots o_n \dots \in \mathcal{O}^\omega \mid \exists i \geq 0 : (o_i = (t, q, h) \in \mathcal{O}_1 \vee o_i = (q, h, \gamma) \in \mathcal{O}_2) \wedge q \in F\}$.

It follows that Player 1 has an observation-based sure-winning (or almost-winning) strategy in the game $G_{M,p,w}$ for the objective ϕ iff the word w is accepted by the polynomial space ATM (M, p). This gives us Lemma 16 and Theorem 8 follows from the lemma.

Lemma 16 *Player 1 has a deterministic (resp. randomized) observation-based sure-winning (resp. almost-winning) strategy in the game $G_{M,p,w}$ for the objective ϕ iff the word w is accepted by the polynomial space ATM (M, p).*

Theorem 8 (Lower bounds) *Let G be a game structure of imperfect information, let $\mathcal{T}$ be a set of observations, and let ℓ be a state of G. Deciding whether ℓ is a sure-winning state in G for the reachability objective $\mathsf{Reach}(\mathcal{T})$ is EXPTIME-hard. Deciding whether ℓ is an almost-winning state in G for $\mathsf{Reach}(\mathcal{T})$ is also EXPTIME-hard.*

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