DOMAIN-COMPLETE AND LCS-COMPLETE SPACES

Matthew de Brecht

京都大学 KYOTO UNIVERSITY
Graduate School of Human and Environmental Studies
Faculty of Integrated Human Studies

科研費 KAKENHI

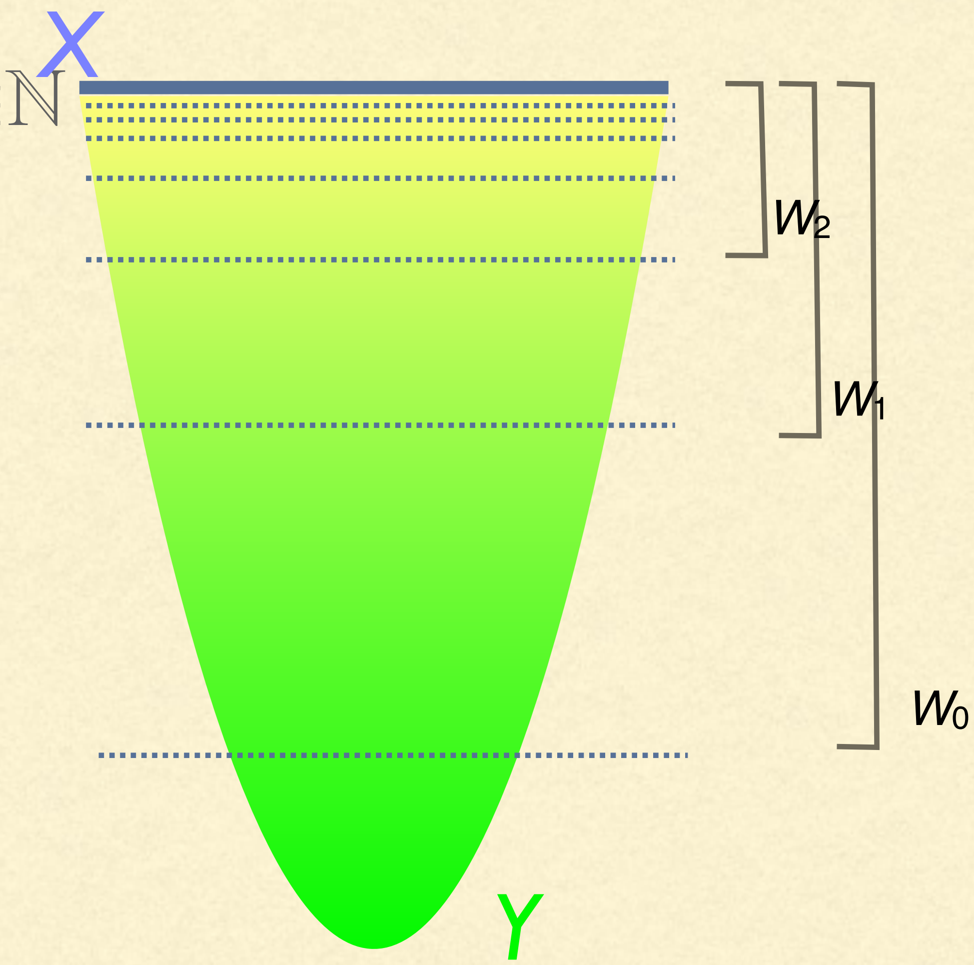
Jean Goubault-Larrecq
Xiaodong Jia
Zhenchao Lyu

école normale supérieure paris—saclay
cnrs
université PARIS-SACLAY
Labex DigiCosme

- Beyond domains and quasi-Polish spaces
- Motivating example: measure extension theorems
- Locating LCS-complete spaces
- If time permits: Stone duality, consonance, …

G-DELTA SUBSETS

- **G_δ** = countable intersection of opens W_n, $n \in \mathbb{N}$ (with the subspace topology)



X is G_δ in Y

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- **G_δ** = countable intersection of opens W_n, $n \in \mathbb{N}$
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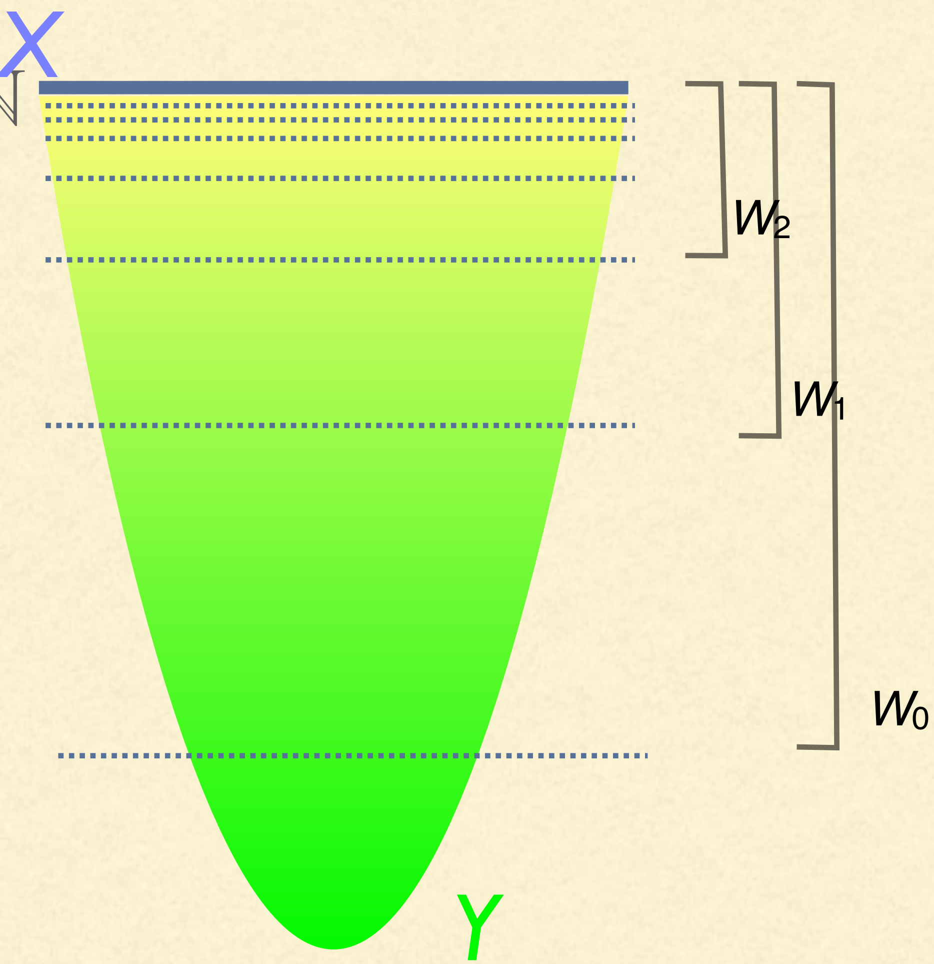
- Every **Polish space** X is G_δ in its space Y of formal balls
  and Y is an ω-continuous dcpo
  [Edalat,Heckmann98]



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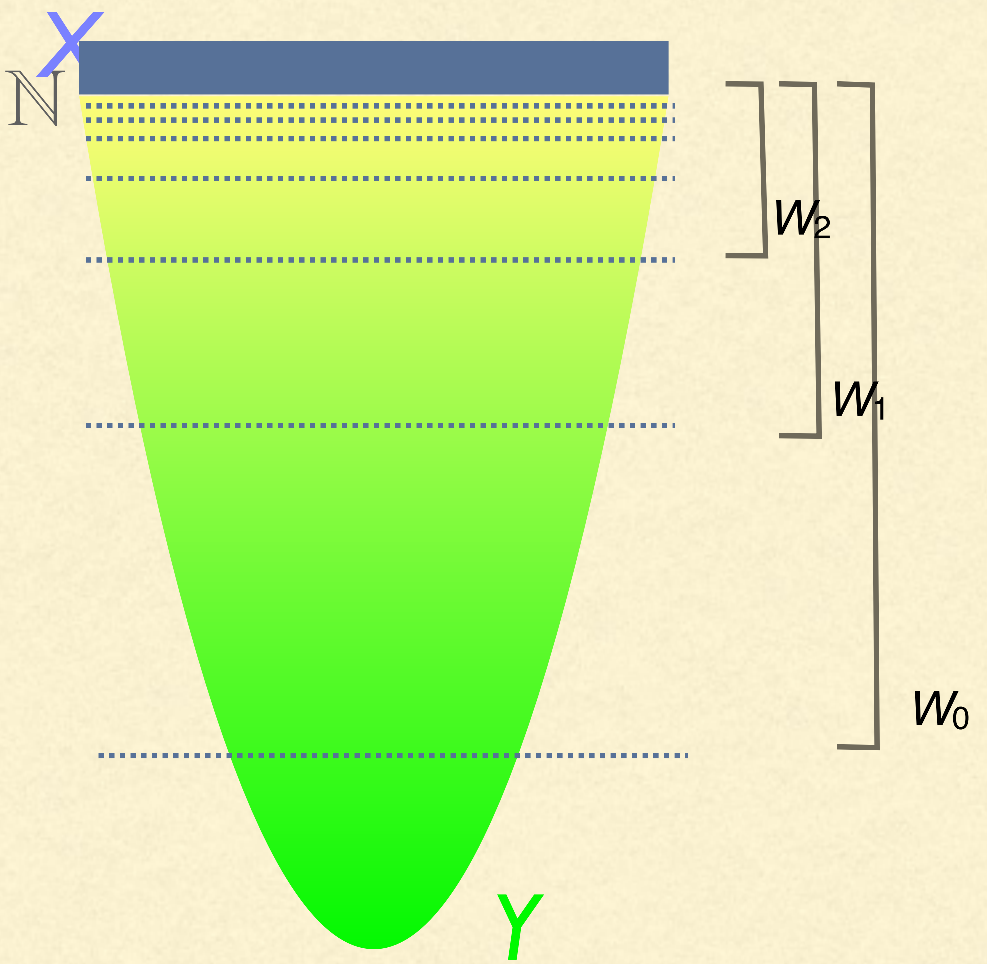
- Same for **quasi-Polish spaces** =
  topological space underlying separable
  Smyth-complete quasi-metric [deBrecht13]



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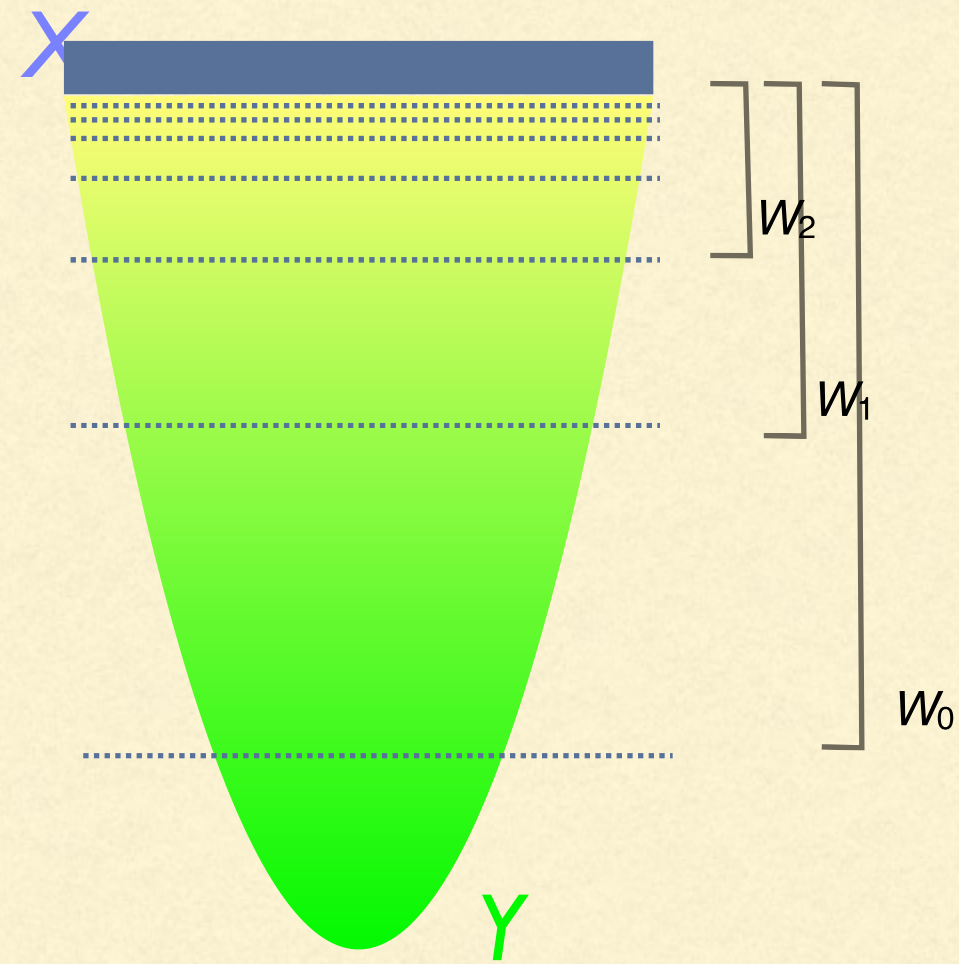
*-COMPLETE SPACES

- In fact:
  G_δ subsets of <u>ω-continuous dcpos</u>
  = **quasi-Polish spaces** [GL,Ng17]



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- In fact:
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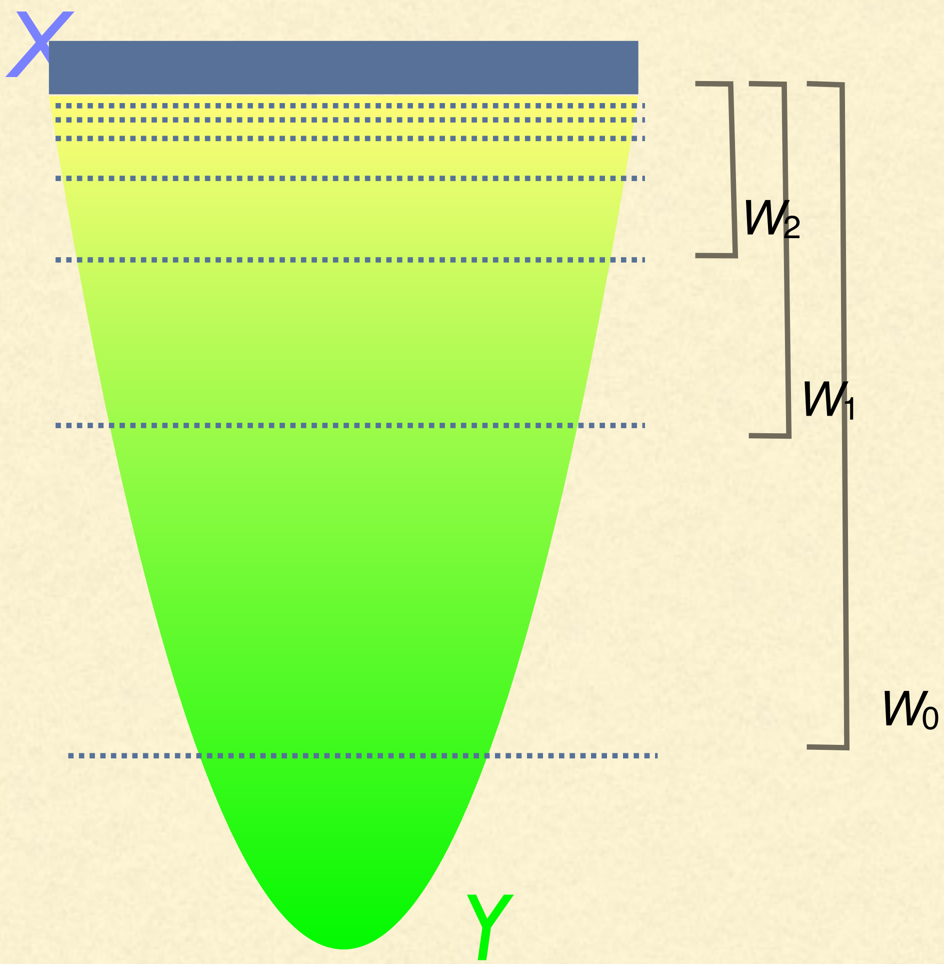
- **Defn.** *X* is **domain-complete** iff
  X is G_δ in a <u>continuous dcpo</u> *Y*



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- **Defn.** *X* is **domain-complete** iff
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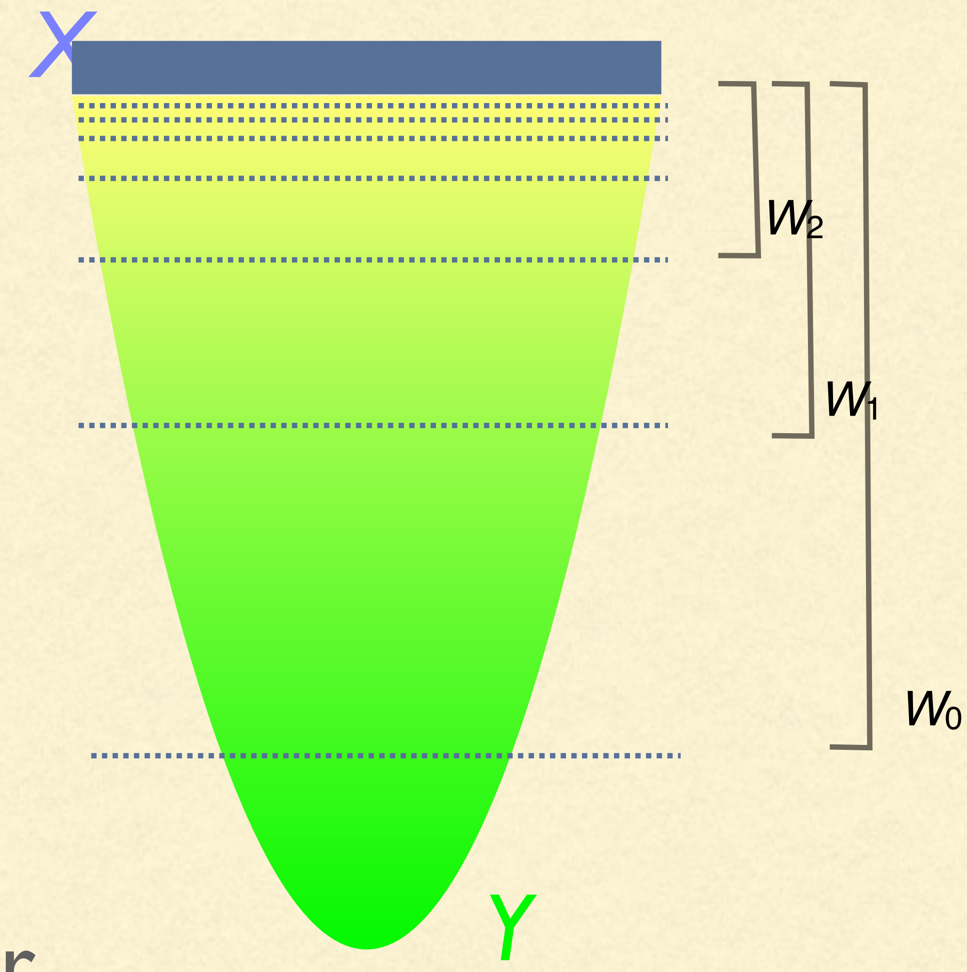
- **Defn.** *X* is **LCS-complete** iff
  X is G_δ in a locally compact sober space *Y*



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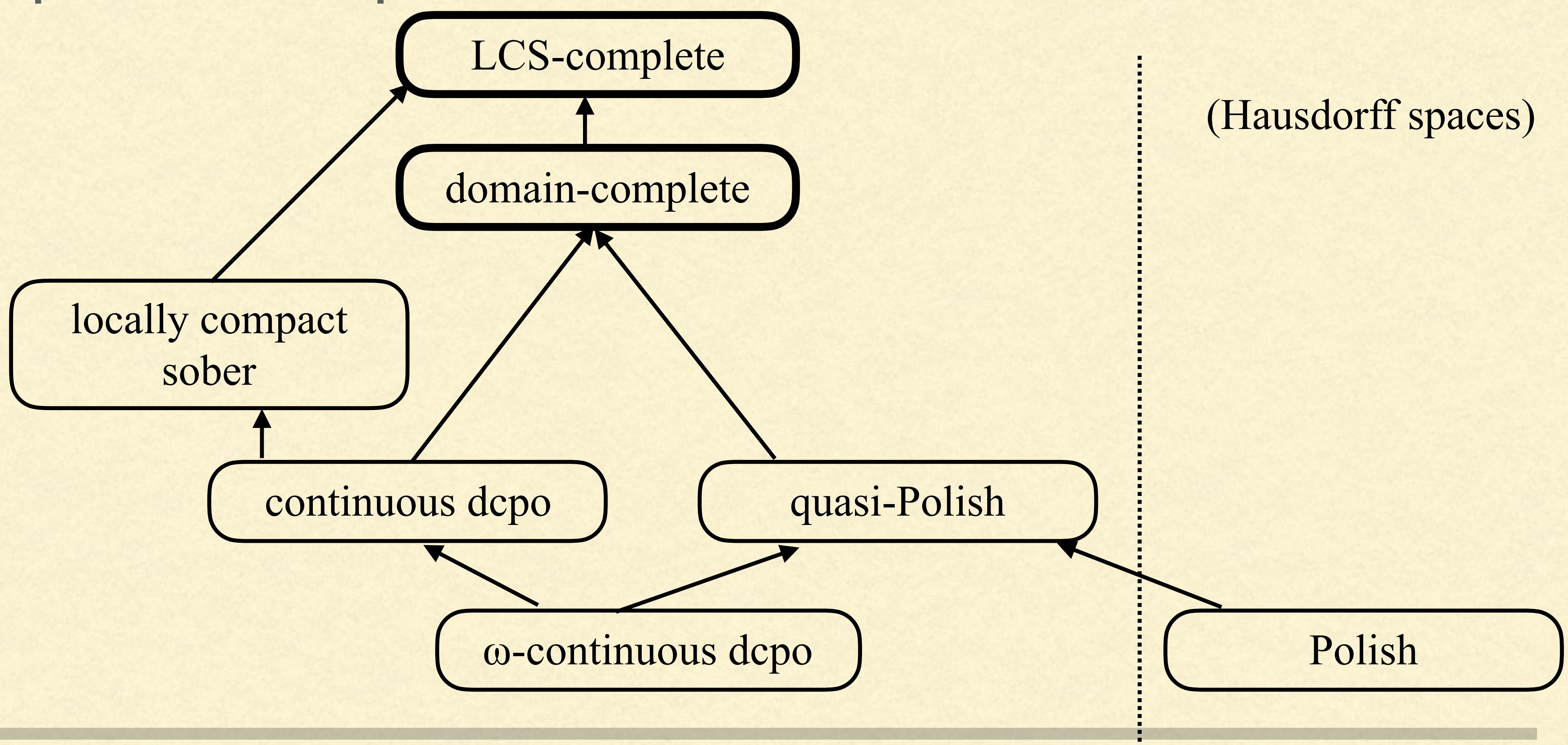
MOTIVATION

- LCS-complete spaces generalize
  — domains
  — Polish, quasi-Polish spaces



MOTIVATION

- LCS-complete spaces generalize
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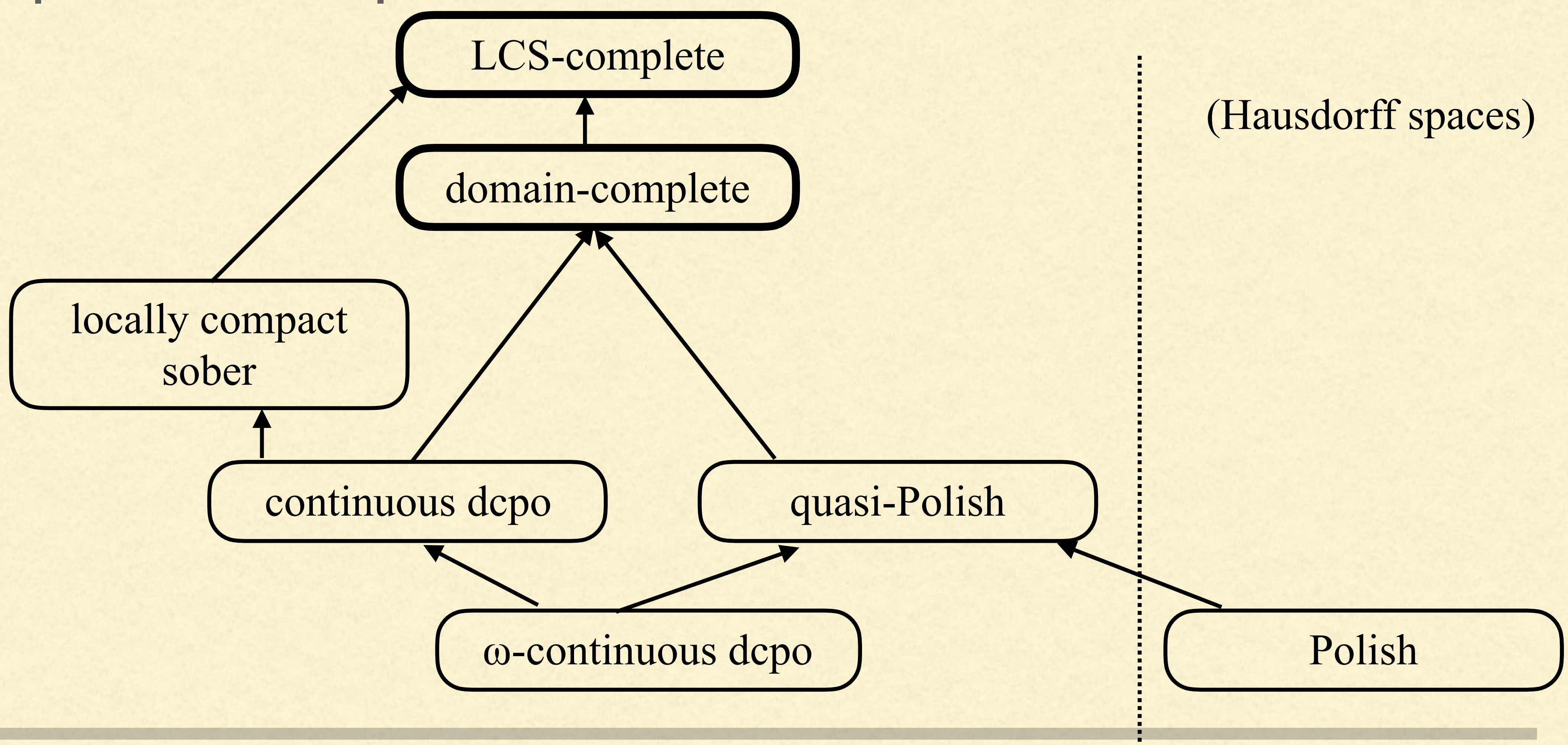
  — Polish, quasi-Polish spaces
- Useful theorems! (next)



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VALUATIONS AND MEASURES

- Continuous valuation ν:
  — $\nu : \mathbf{O}X \to \mathbb{R}_+ \cup \{\infty\}$
  measures **open** sets
  — Scott-continuous
  — $\nu(\varnothing)=0$
  — $\nu(U \cup V)+\nu(U \cap V)$
  $=\nu(U)+\nu(V)$

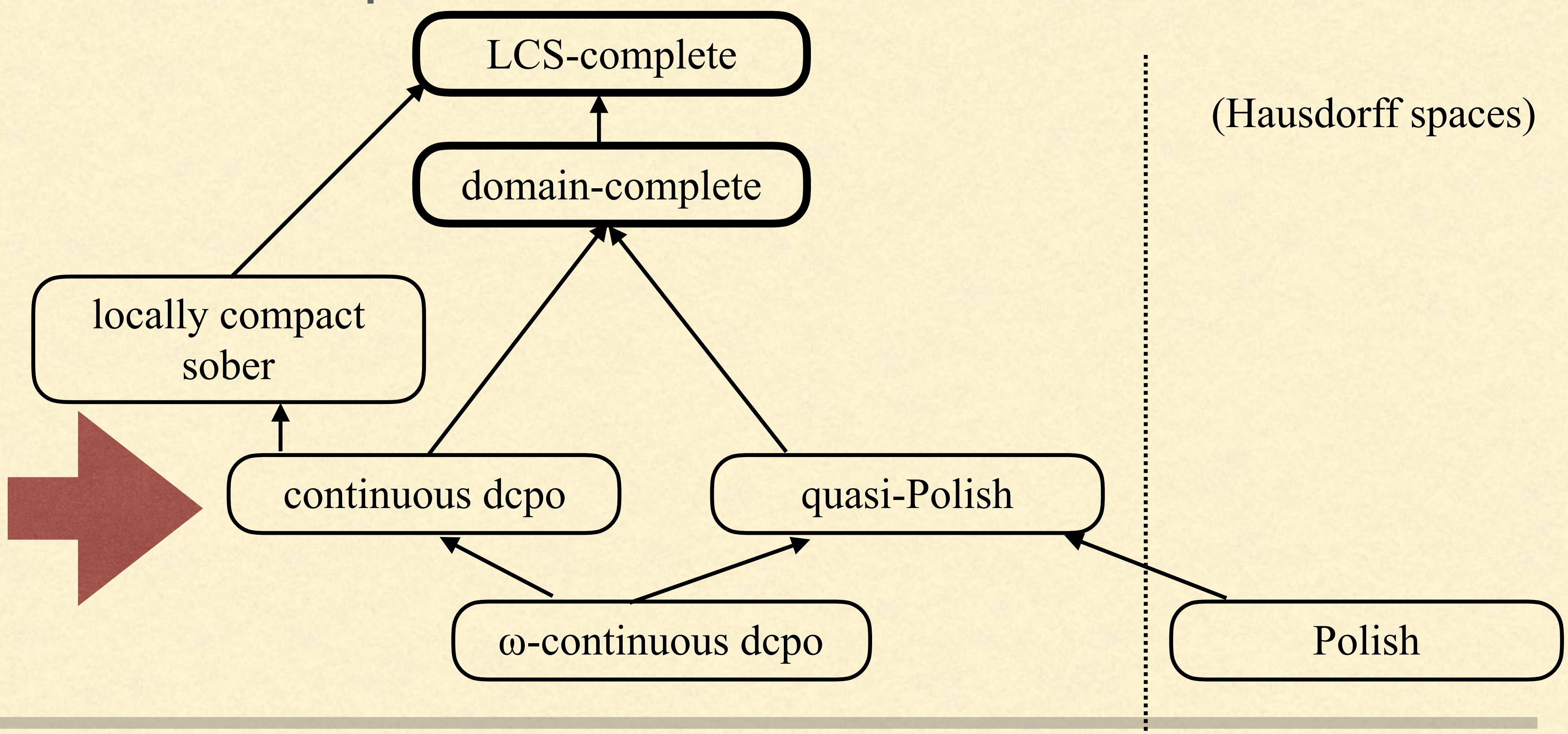
- Measure μ:
  — $\mu : \mathcal{B}(X) \to \mathbb{R}_+ \cup \{\infty\}$
  measures **Borel** sets
  — $\mu(\varnothing)=0$
  — $(E_n)_{n\in\mathbb{N}}$ pairwise disjoint
  $\Rightarrow \mu(\bigcup_n E_n)=\sum_n \mu(E_n)$

- **Fact.** Every measure on a countably-based space X **restricts** to a continuous valuation.

- Conversely…

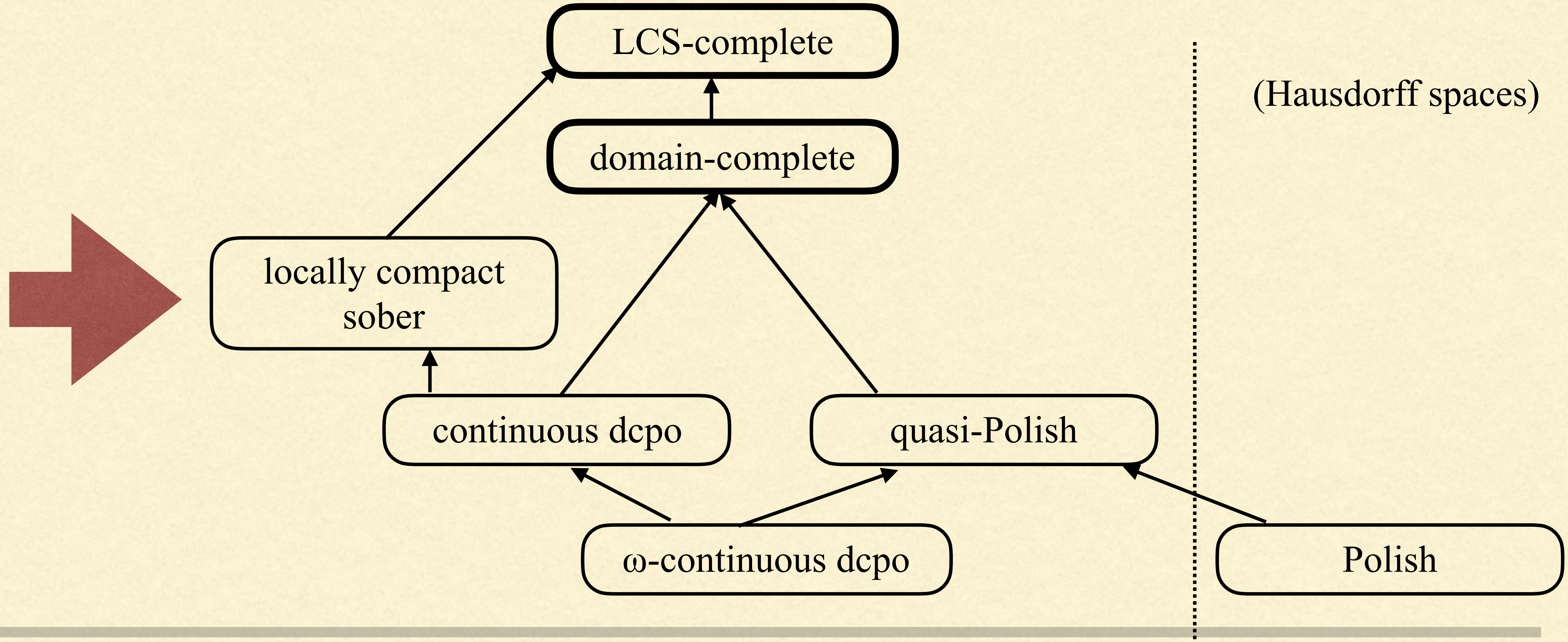
MEASURE EXTENSION THEOREMS

- **Thm** [Alvarez-Manilla,Edalat,Saheb-Djahromi00 + Jones90]
  Every (finite) continuous valuation **extends** to a measure
  — on a continuous dcpo.



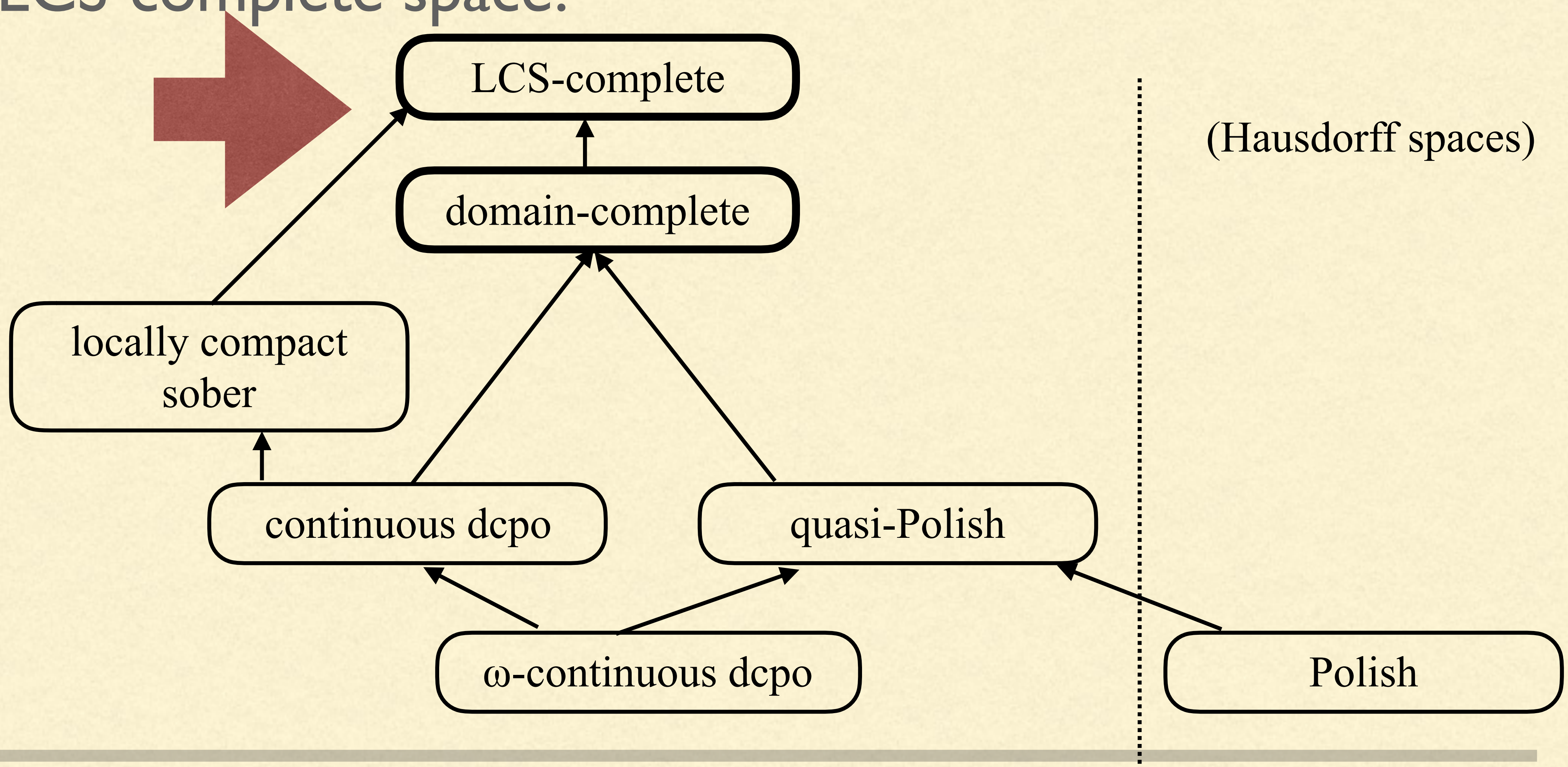
MEASURE EXTENSION THEOREMS

- **Thm** [Alvarez-Manilla00; Keimel,Lawson05]
  Every (loc. finite) continuous valuation **extends** to a measure — on a locally compact sober space.



MEASURE EXTENSION THEOREMS

- **Thm** [this paper]
  Every continuous valuation extends to a measure — on an LCS-complete space.



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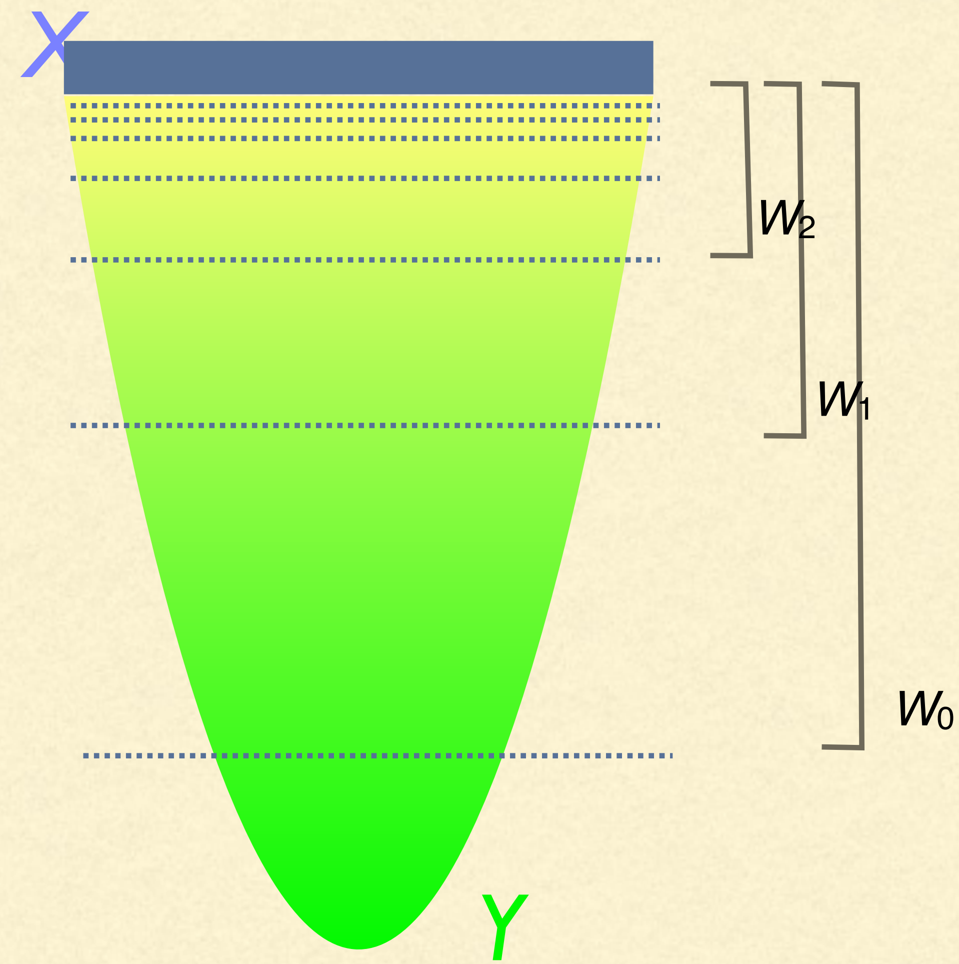

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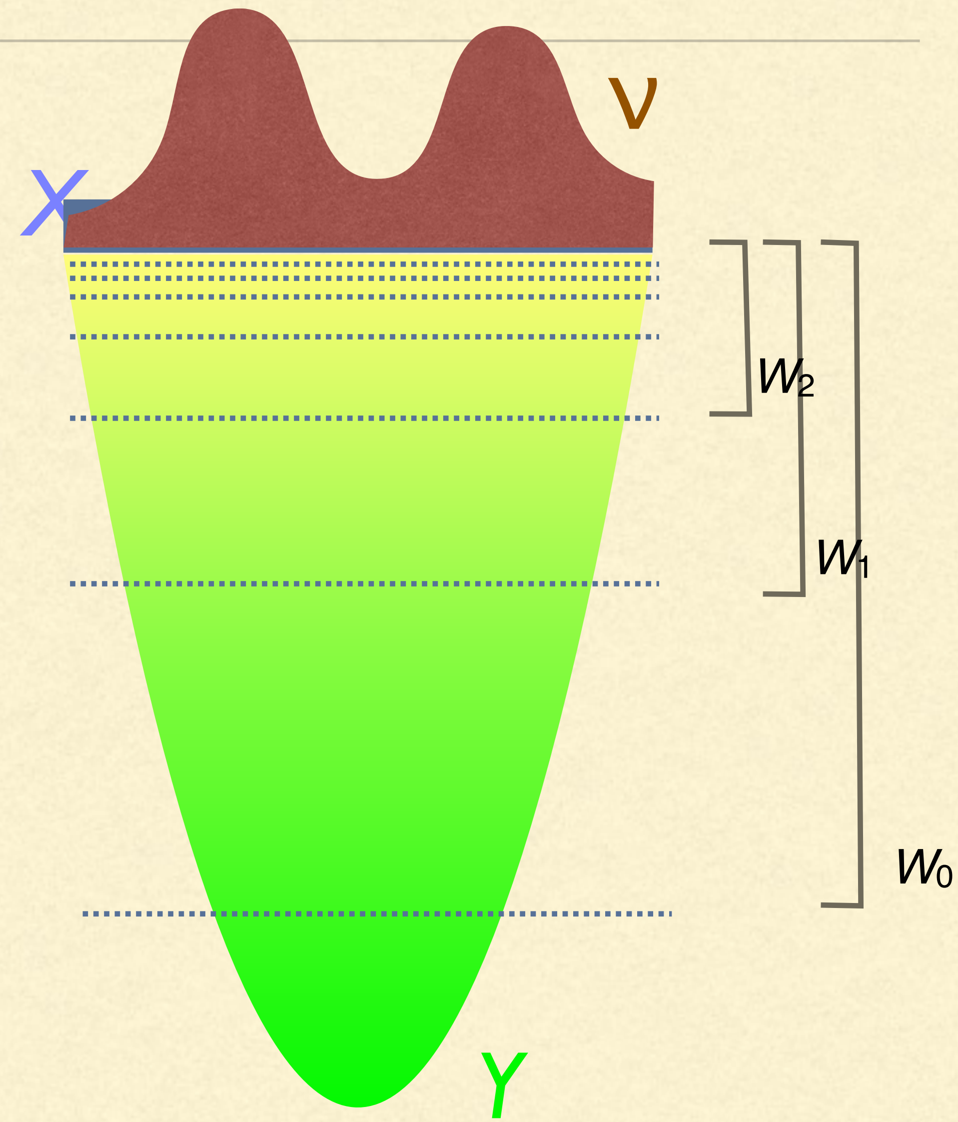

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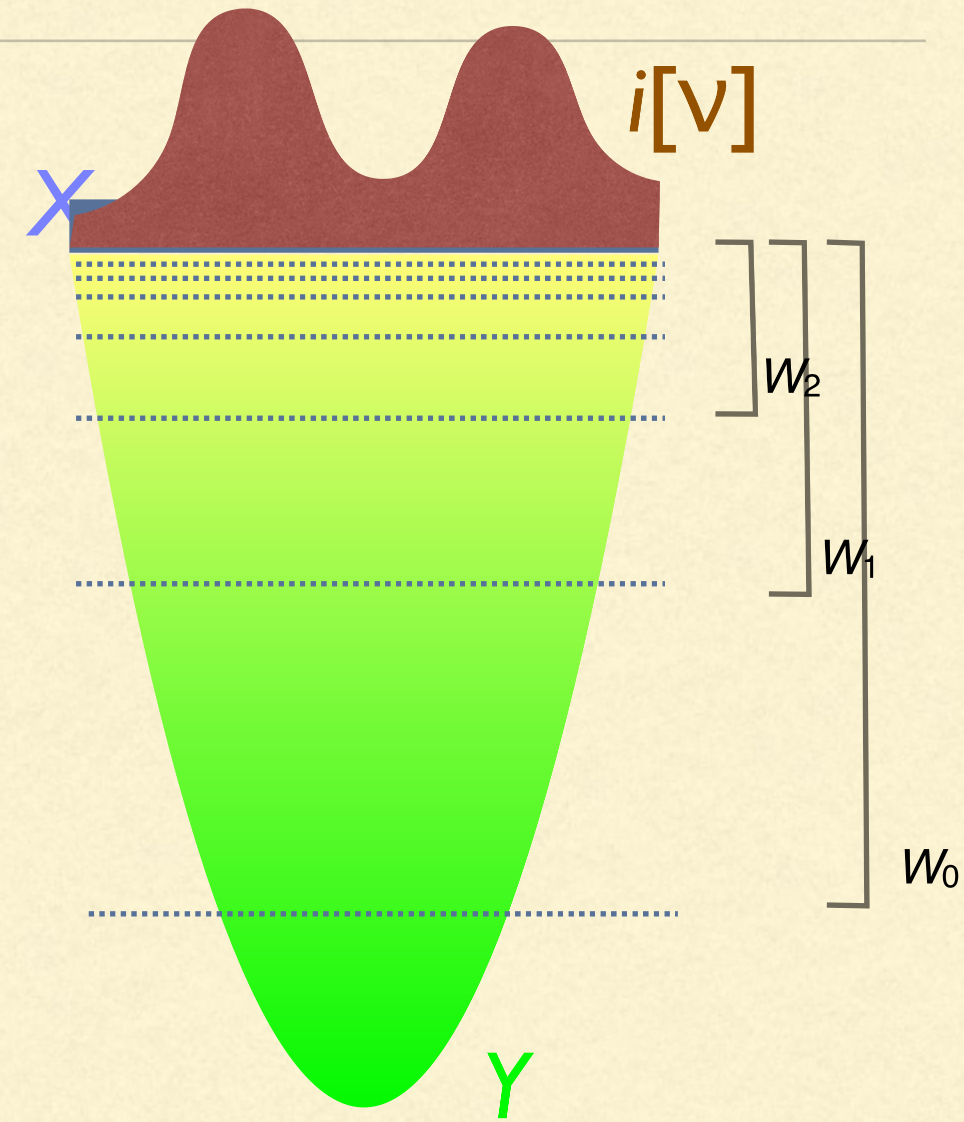
MEASURE EXTENSION THEOREMS

- **Thm.** Every continuous valuation ν extends to a measure — on an LCS-complete space *X*.

- **Proof.**
  Let $i: X \to Y$ = inclusion map
  $i[\nu]$ is a continuous valuation on Y
  $i[\nu](V) = \nu(V \cap X)$



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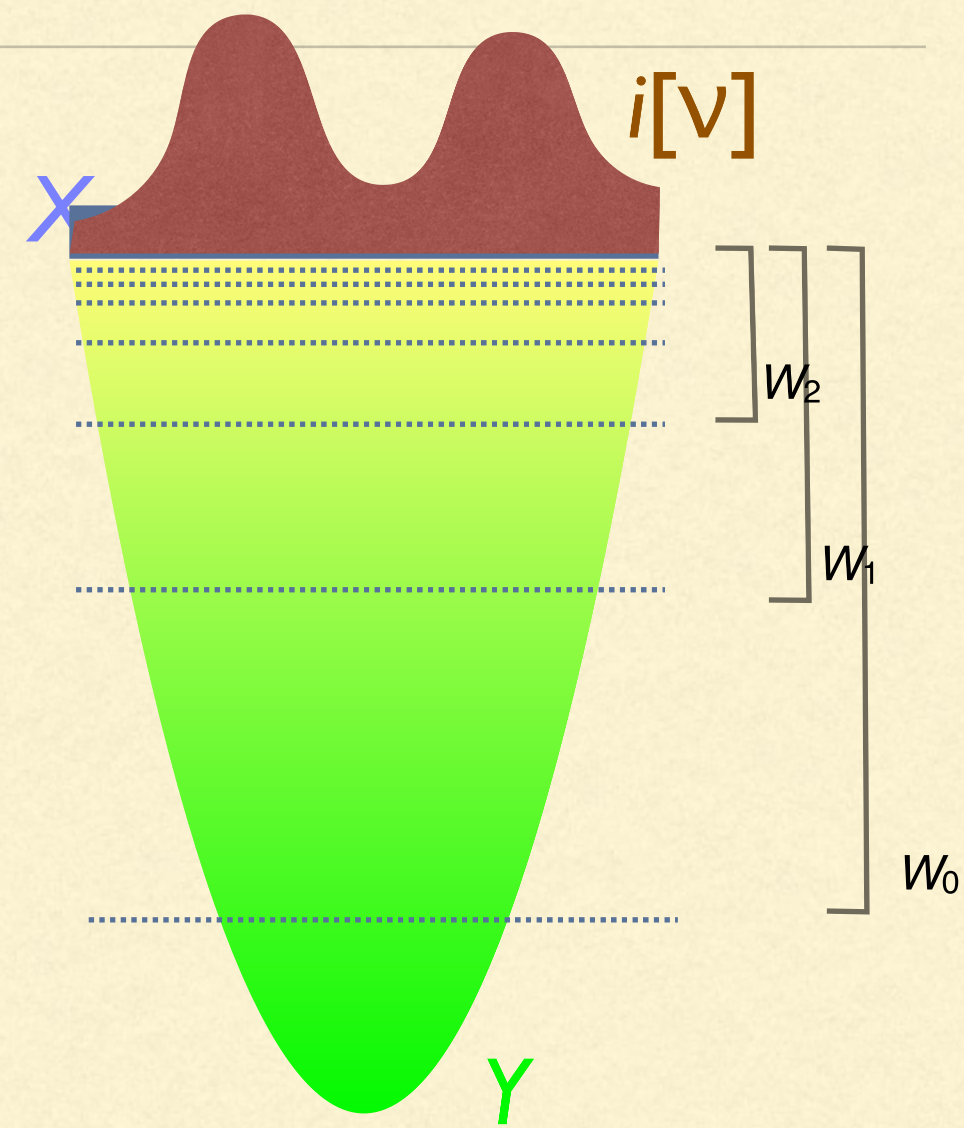
- **Thm.** Every continuous valuation ν extends to a measure — on an LCS-complete space X.
- **Proof.** $i[\nu](V) = \nu(V \cap X)$
- $i[\nu]$ extends to a measure μ on Y by [AM00,KL05]
- hence on X, which is Borel in Y



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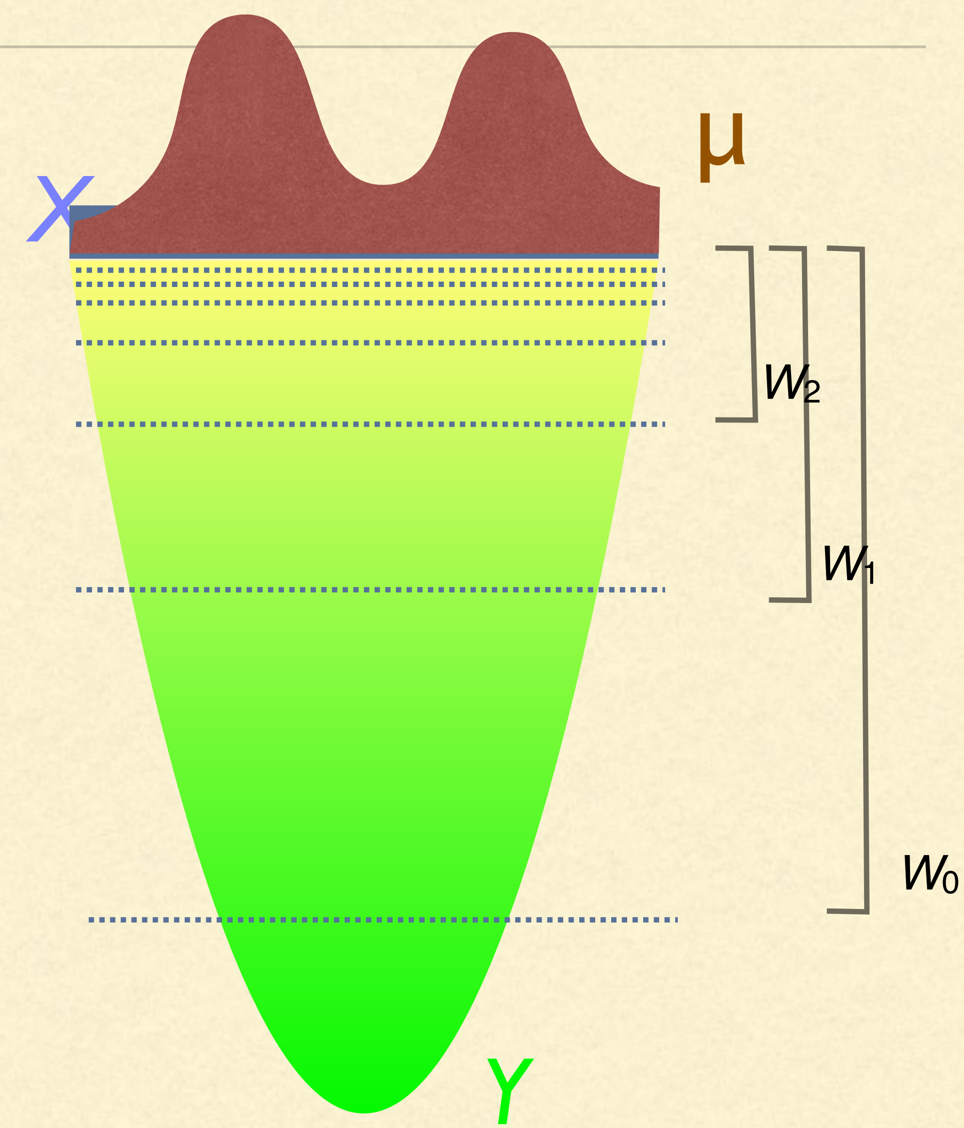

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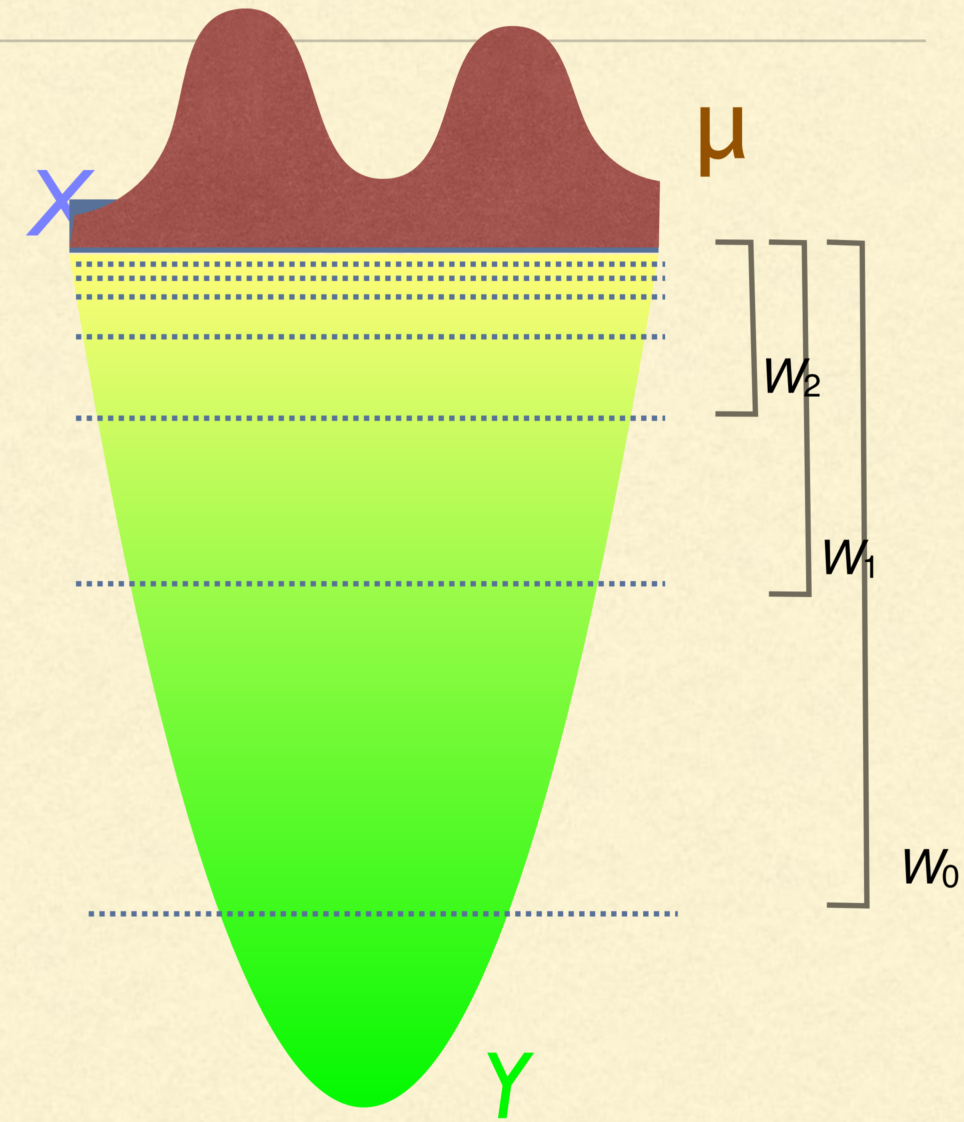
- for every open U of X,
  $U = V \cap X$ for some open V of Y
  $= \bigcap_n (V \cap W_n)$, so
  $\mu(U) = \inf_n \mu(V \cap W_n) = \inf_n \nu(U) = \nu(U)$. □



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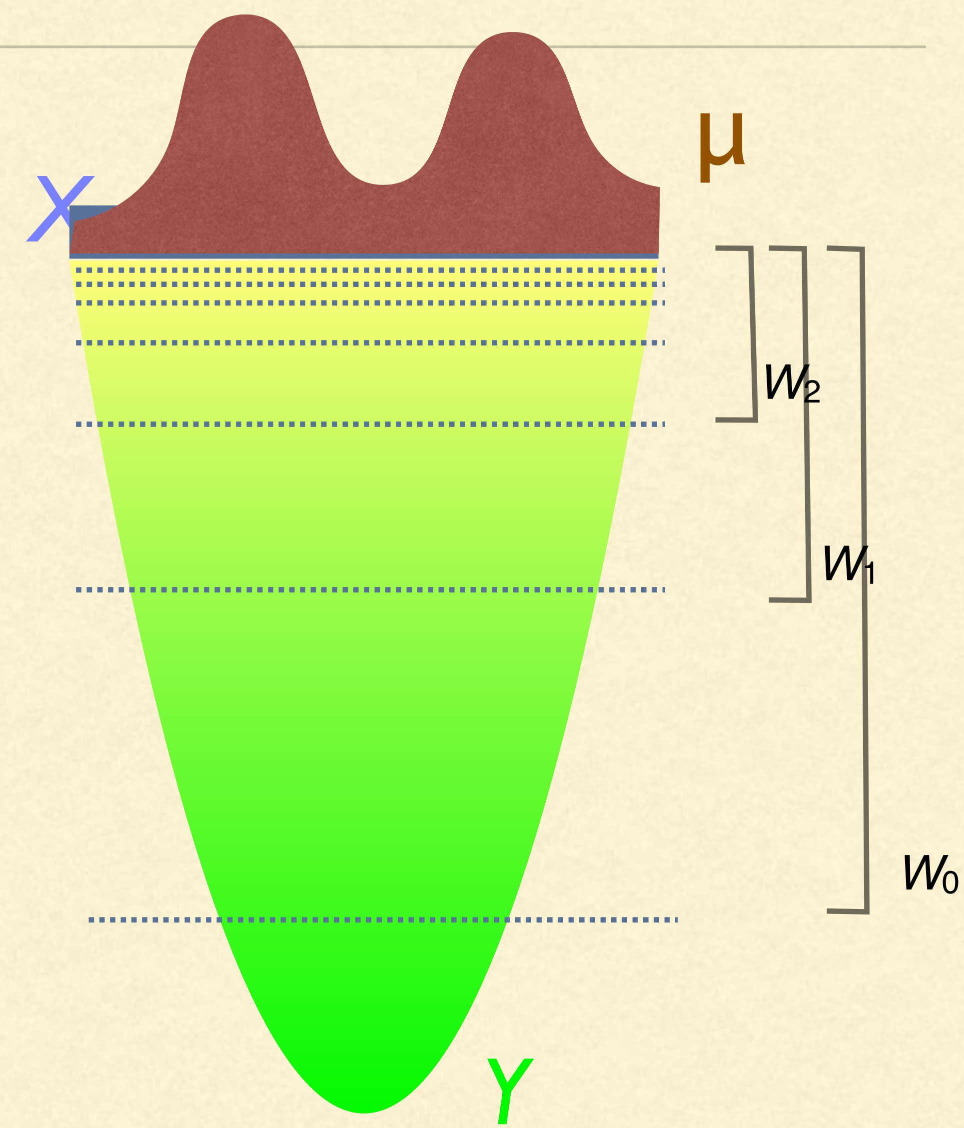
works only if ν (hence μ) bounded…



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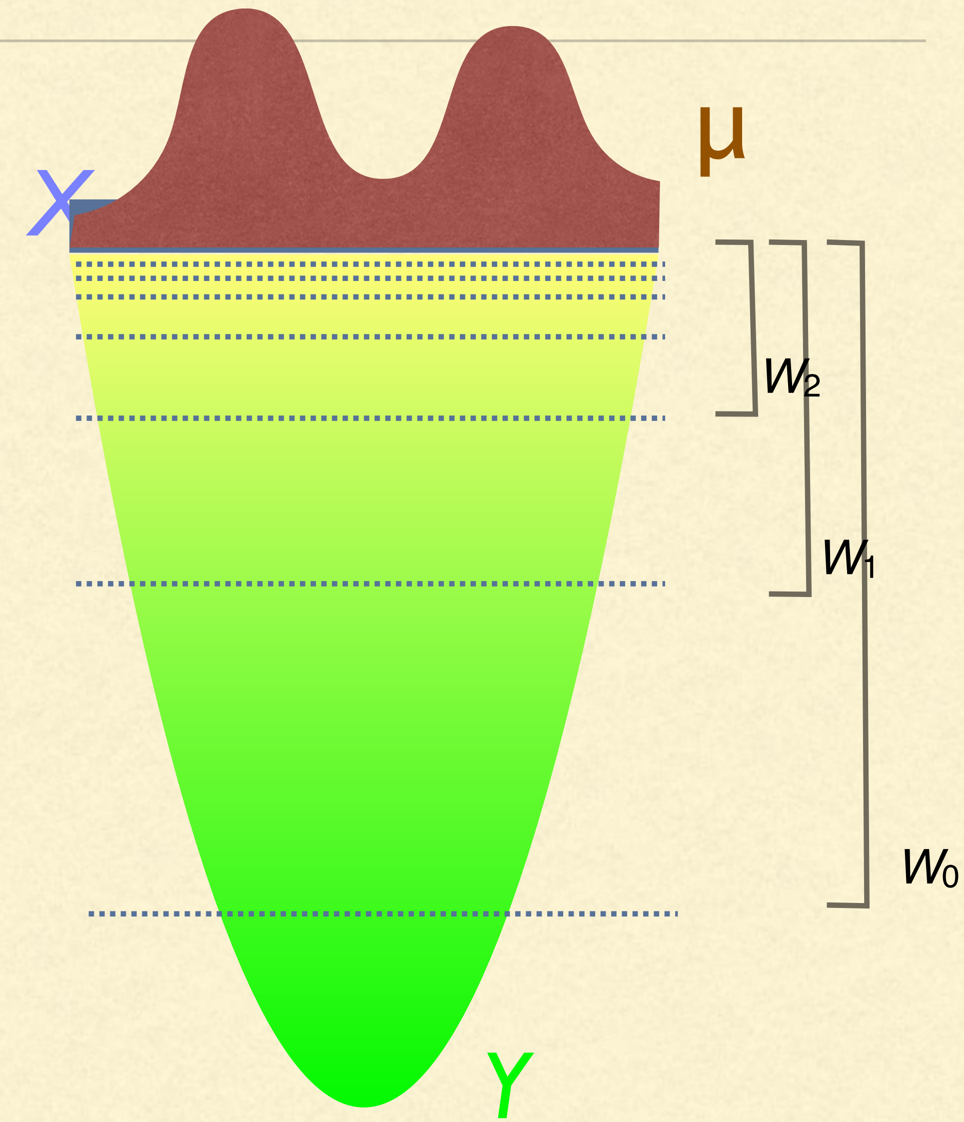

…otherwise use tricks introduced by Heckmann (1996)

MEASURE EXTENSION THEOREMS

- **Thm.** Every continuous valuation ν
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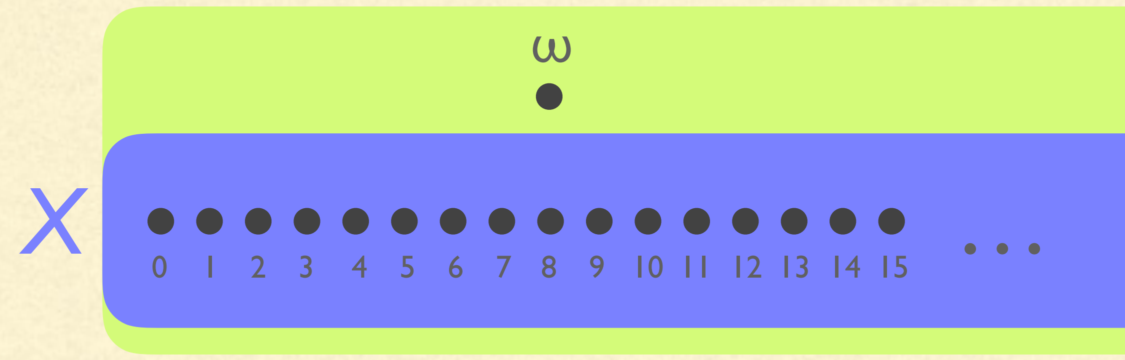
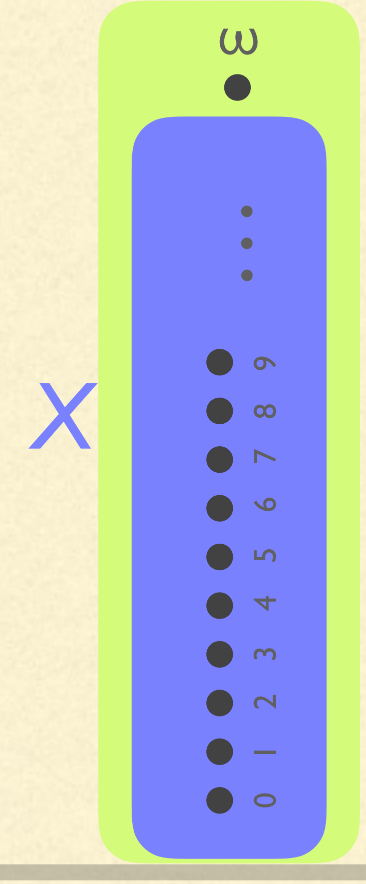
  This is **tight** [deBrecht95]



(cofinite topology)



(Scott topology)

- (Right) both X are F_σ in their sobrifications

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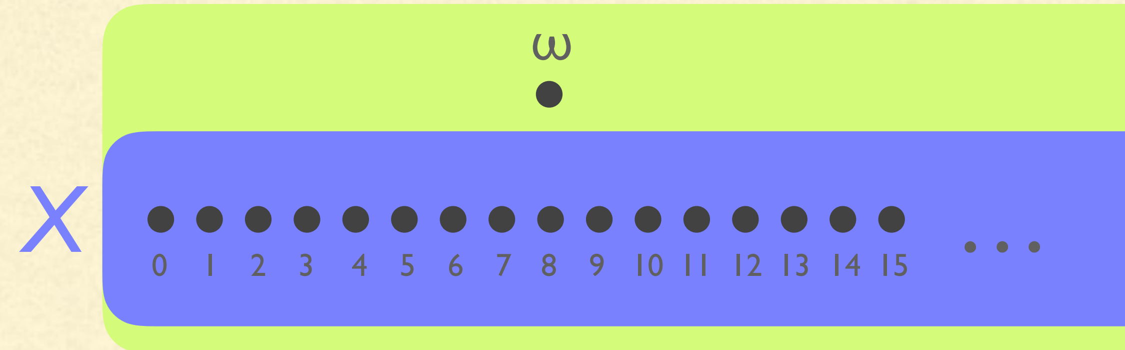
- Take ν / ν(*U*)=1 for every non-empty *U*



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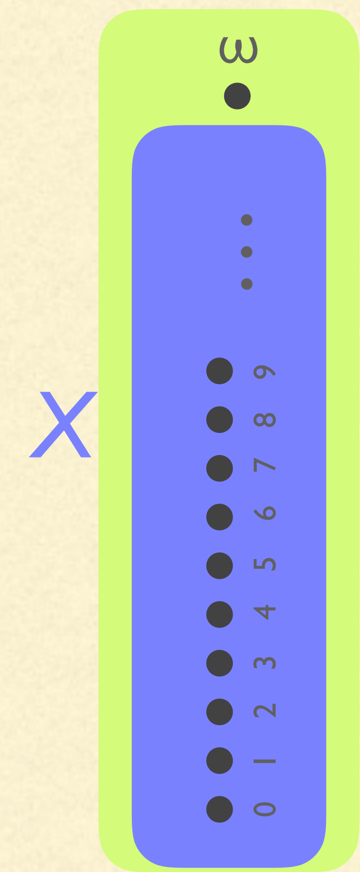

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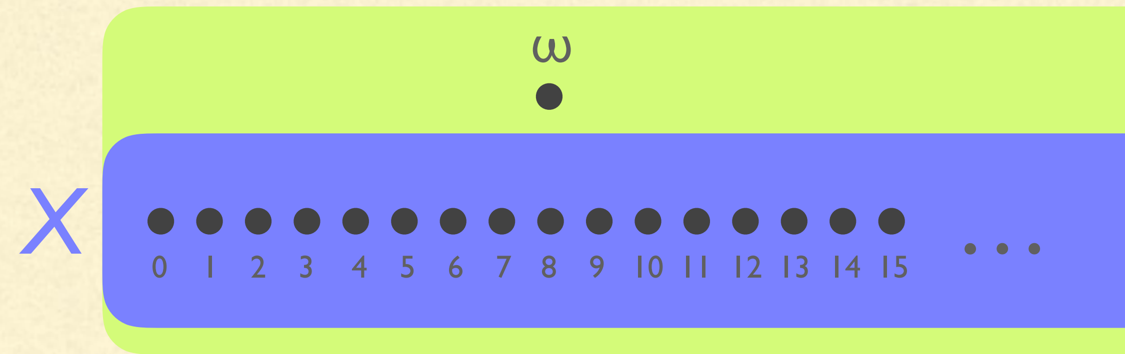

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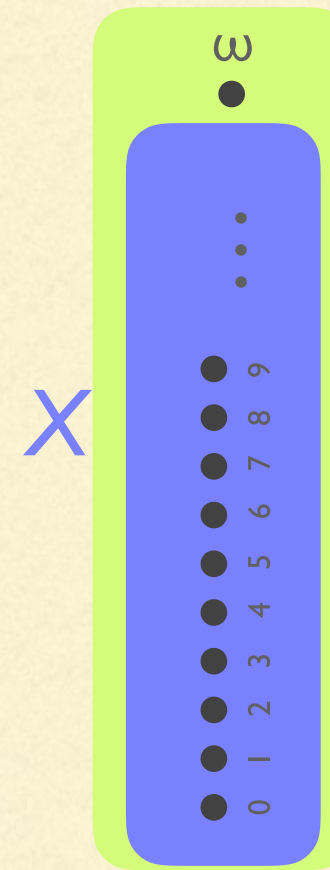

(Scott topology)

- (Right) both X are F_σ in their sobrifications
- Take ν / $\nu(U)=1$ for every non-empty U
- Any μ extending ν must satisfy $\mu(\{n\})=0$ hence $\mu=0$… which does not extend ν. □

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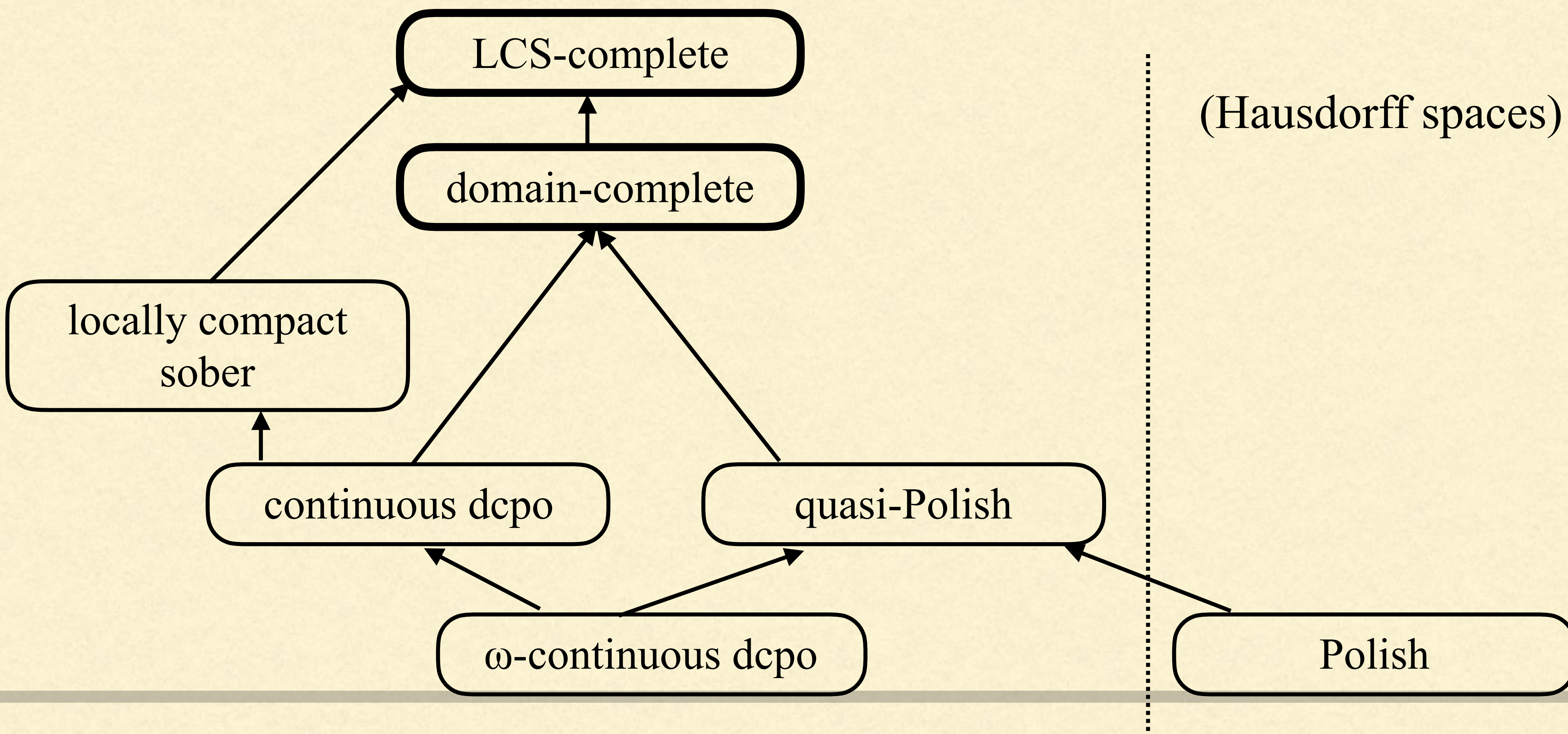
LOCATING *-COMPLETE SPACES

- Čech's **topologically complete** spaces [1937]= G_δ of compact T_2 spaces
  contain all completely metrizable spaces



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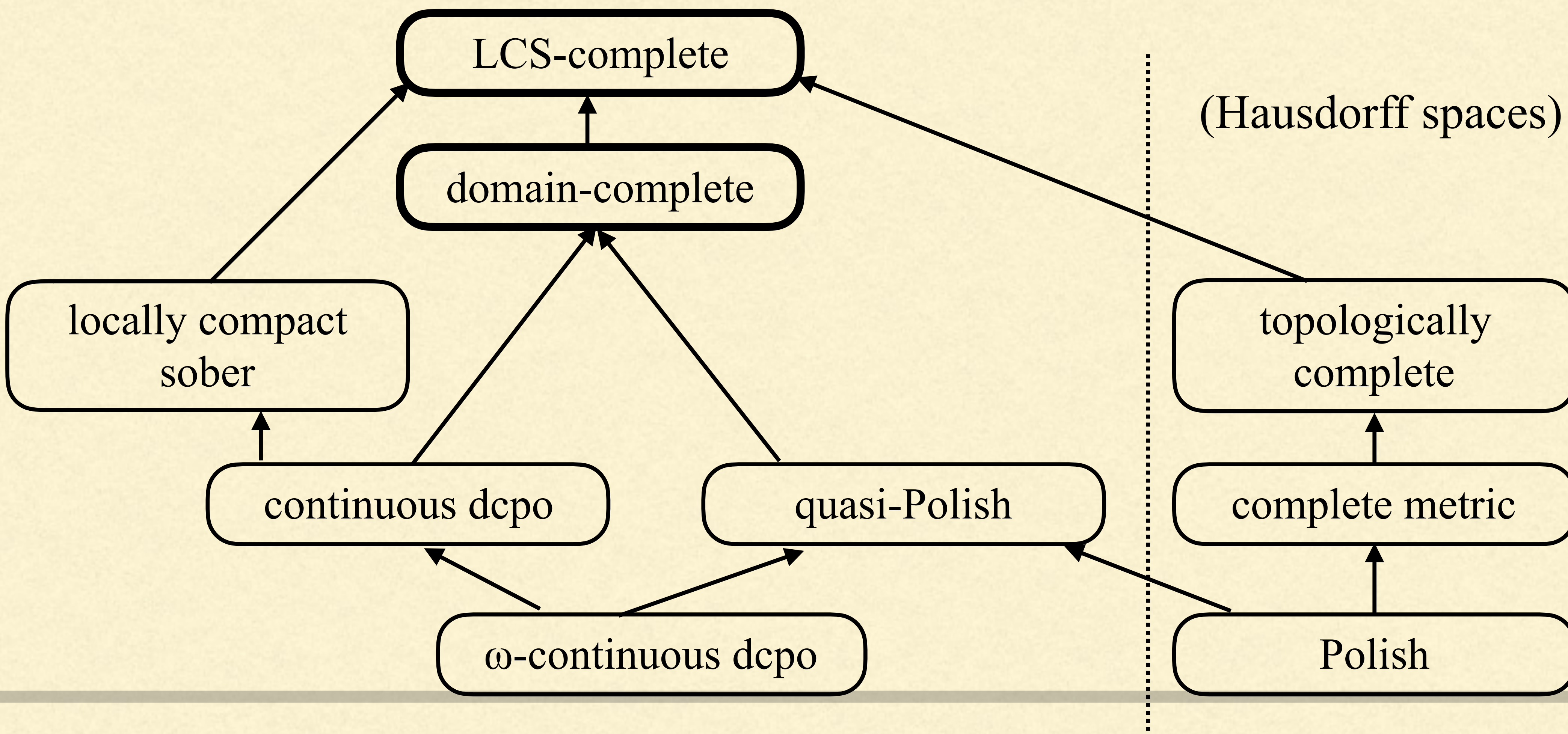
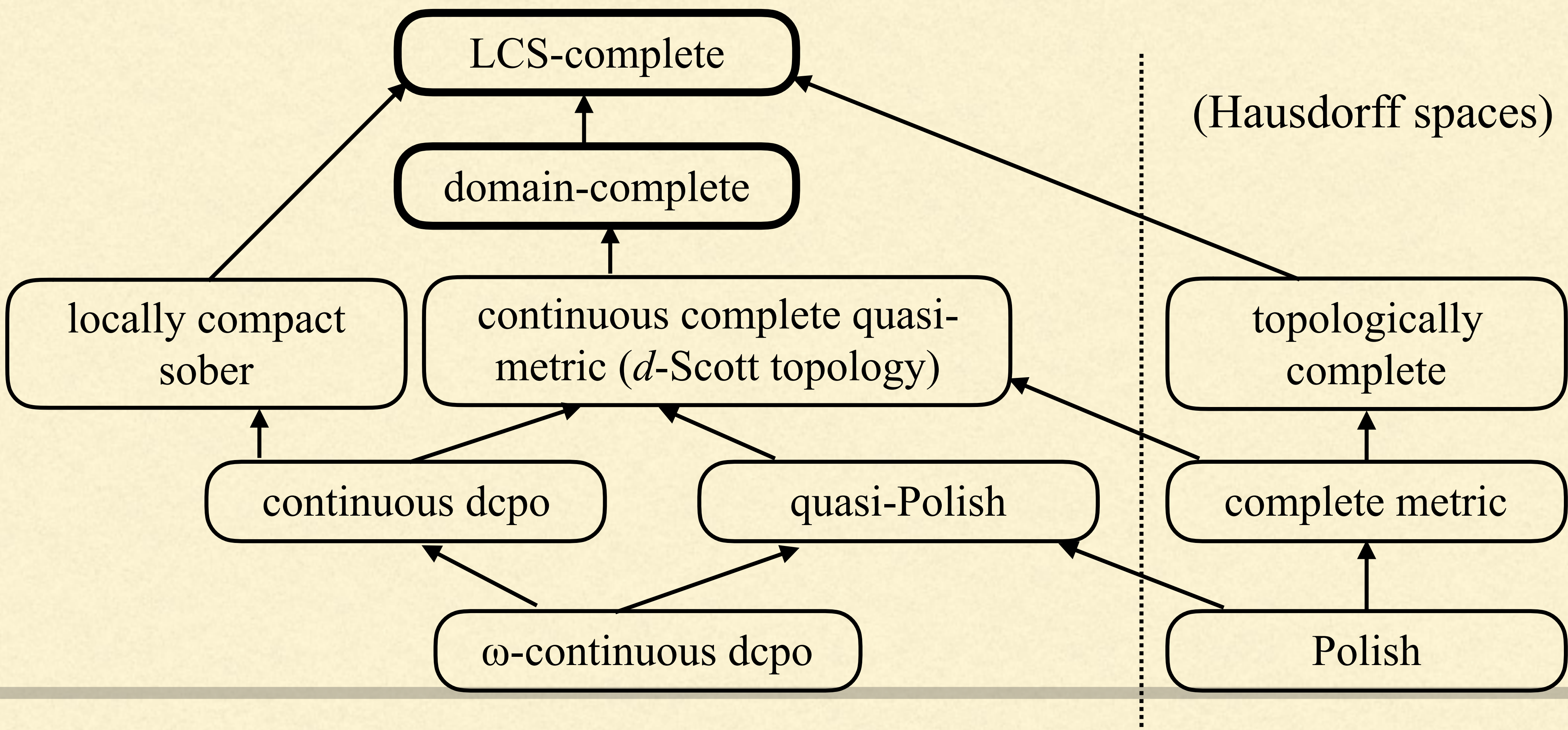
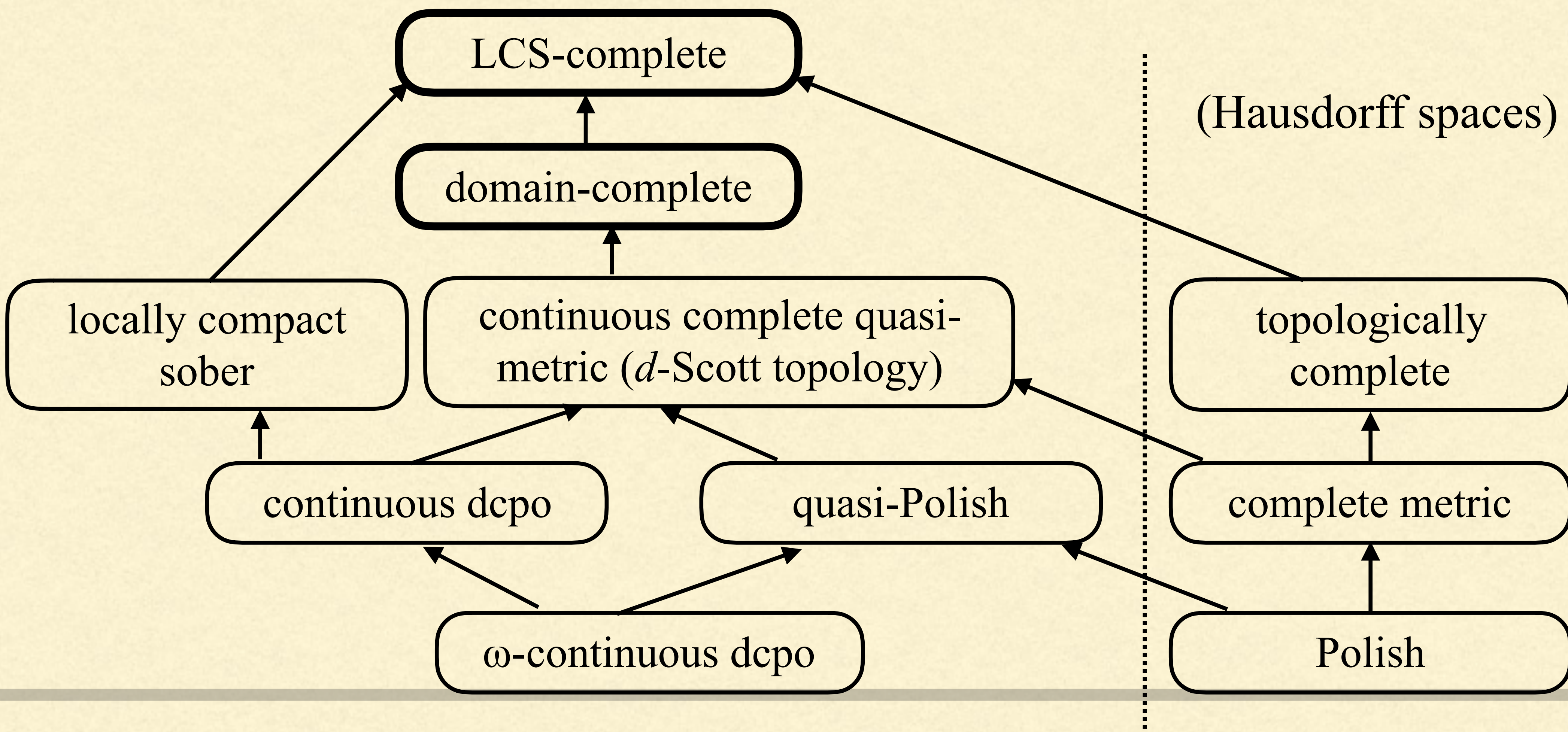


LOCATING *-COMPLETE SPACES

- **Continuous complete quasi-metric** spaces [Kostanek, Waszkiewicz10] embed as G_δ subsets of their poset of formal balls — a continuous dcpo.



PROPERTIES


LCS-complete
(Hausdorff spaces)
domain-complete
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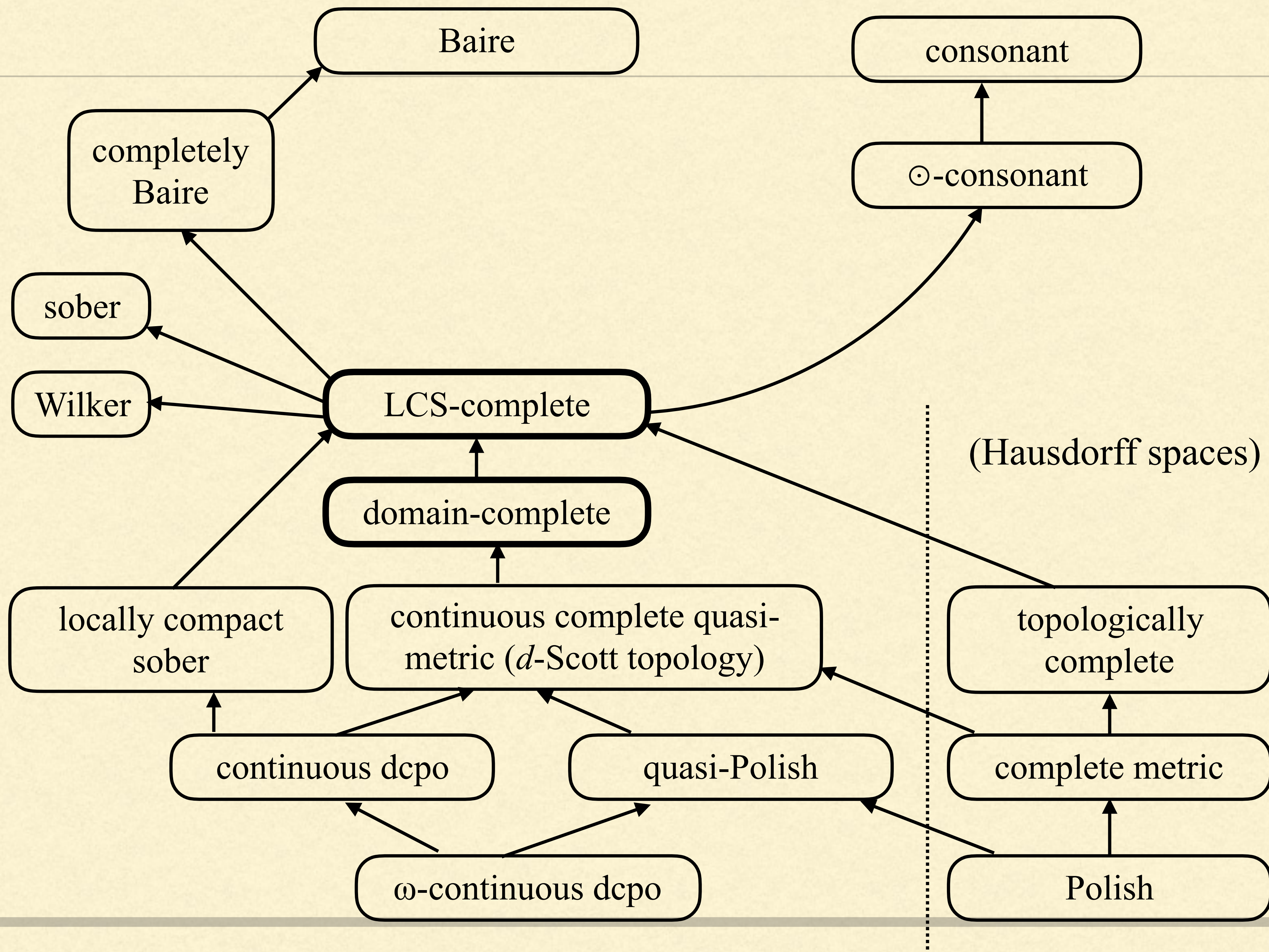
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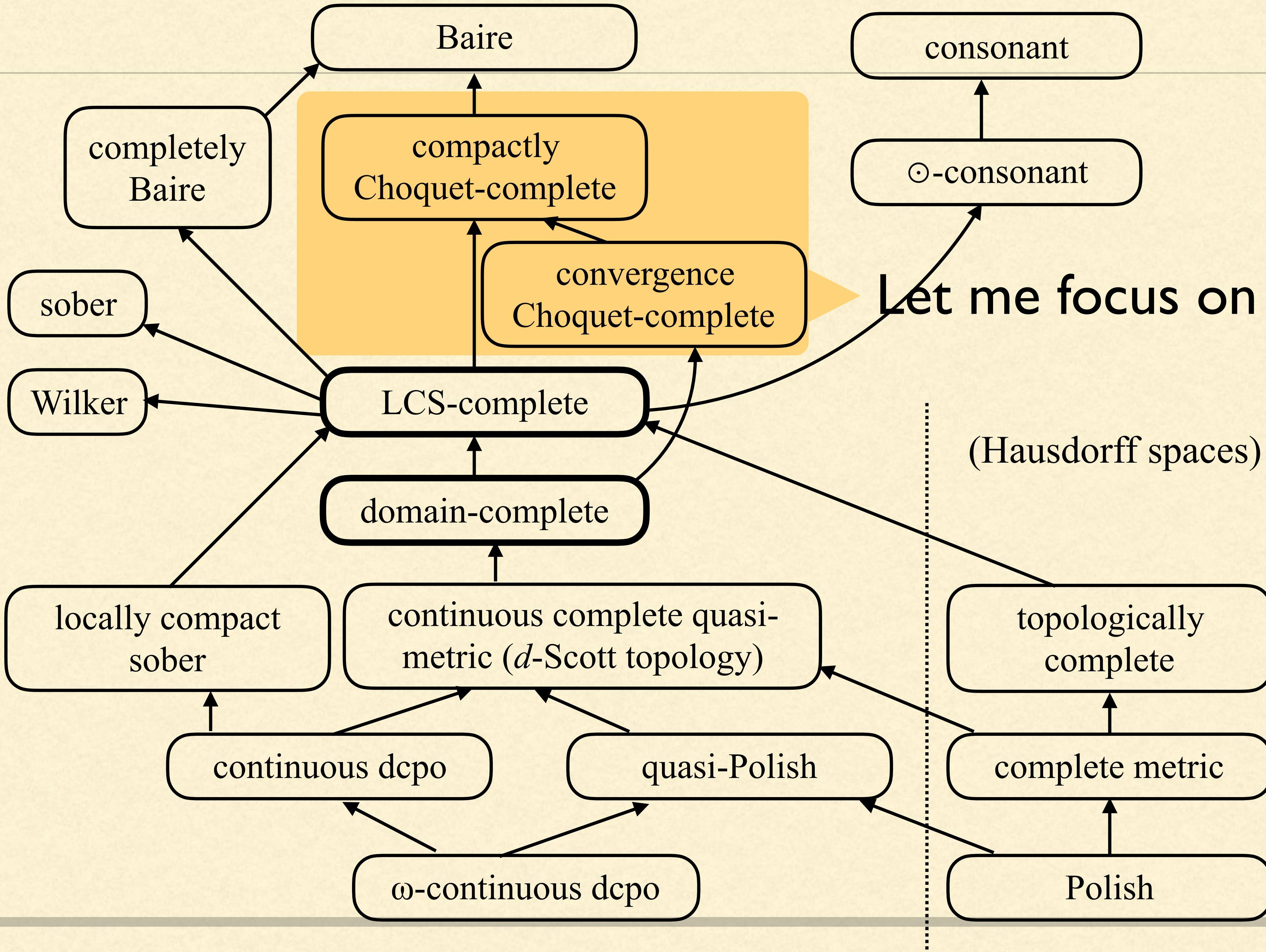

Let me focus on those

THE STRONG CHOQUET GAME

- Two players α, β

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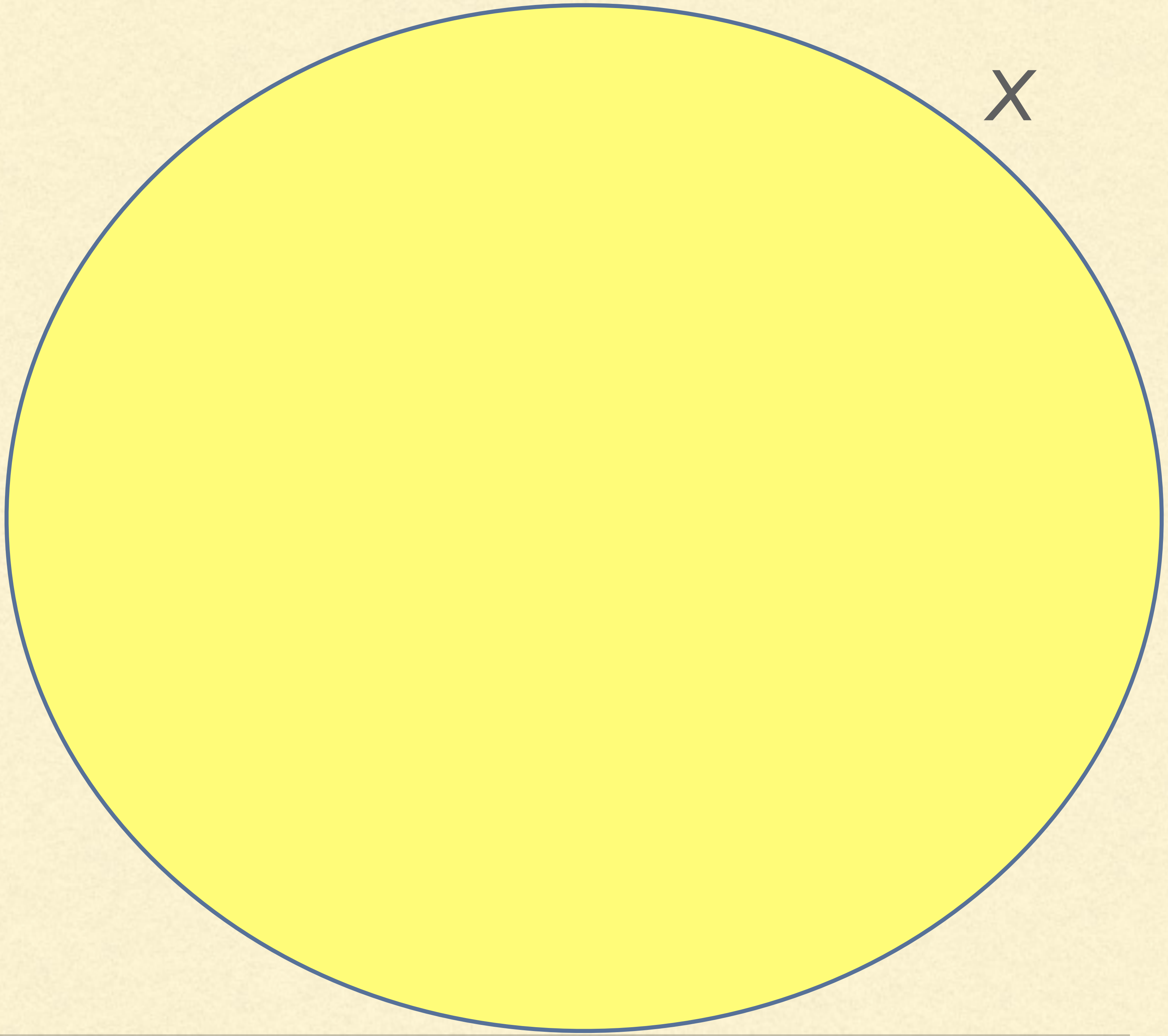
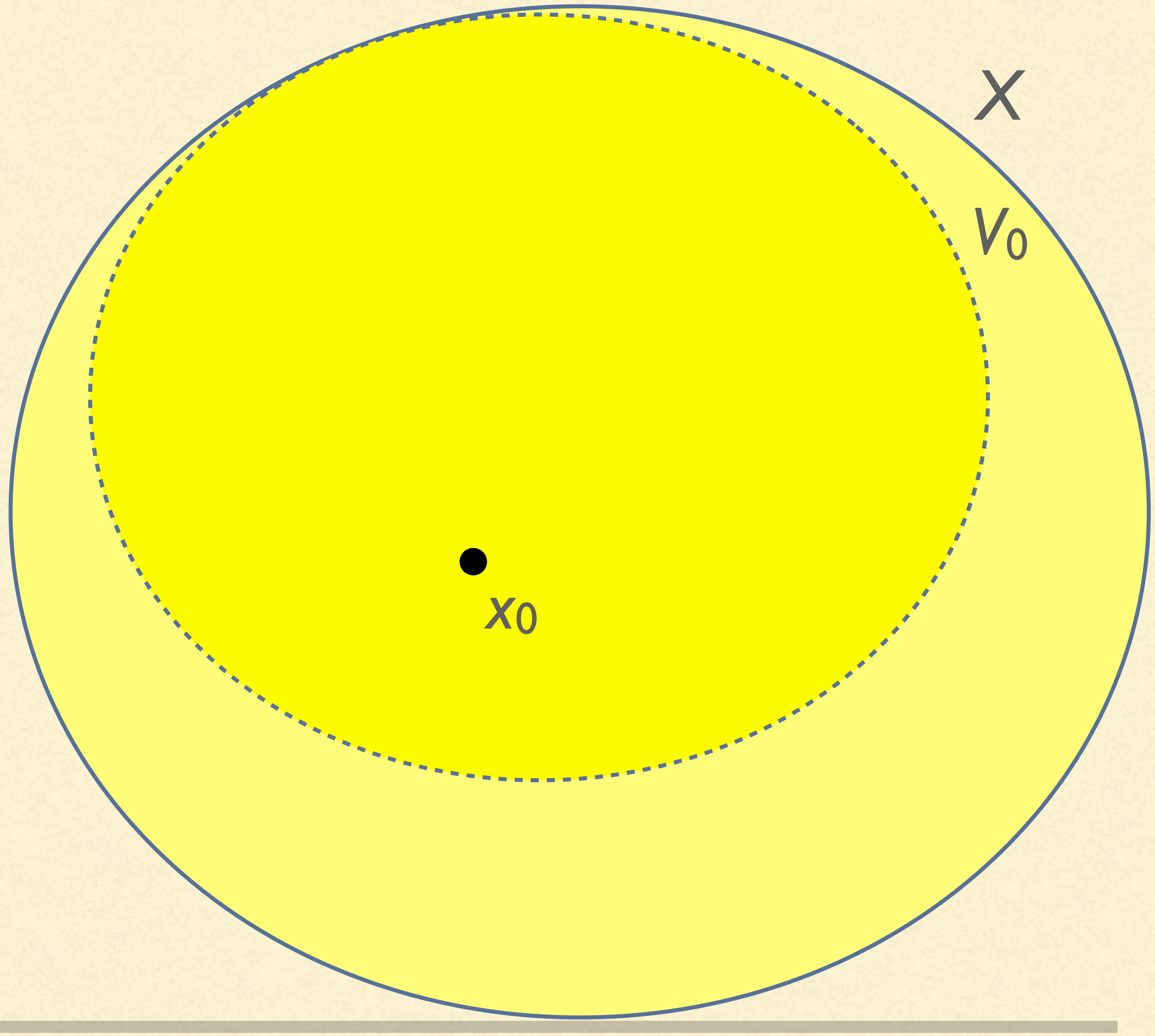


THE STRONG CHOQUET GAME

- Two players α, β
- β picks open V_0, $x_0 \in V_0$



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- Two players α, β
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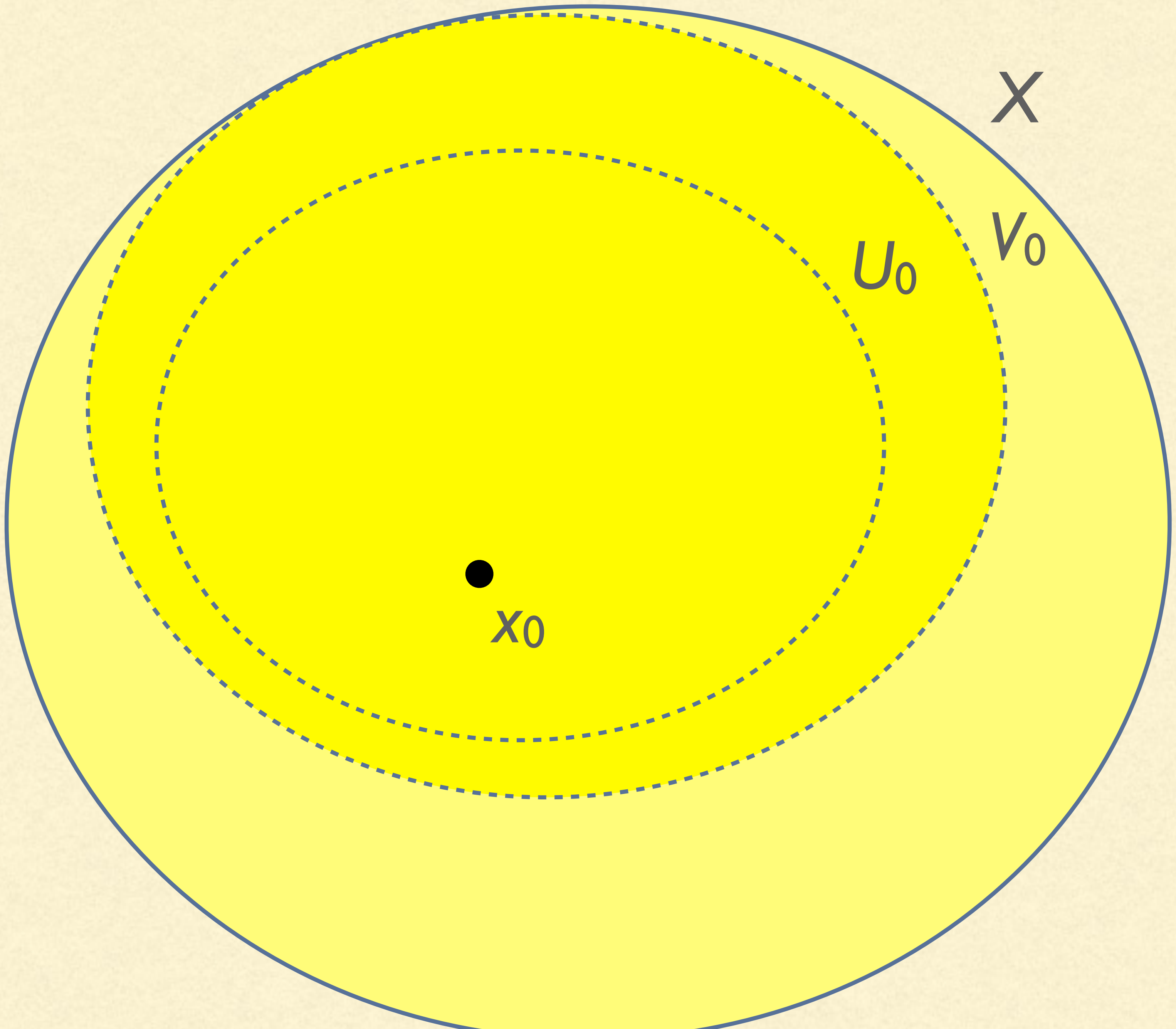
- α picks smaller open U_0 containing x_0



THE STRONG CHOQUET GAME

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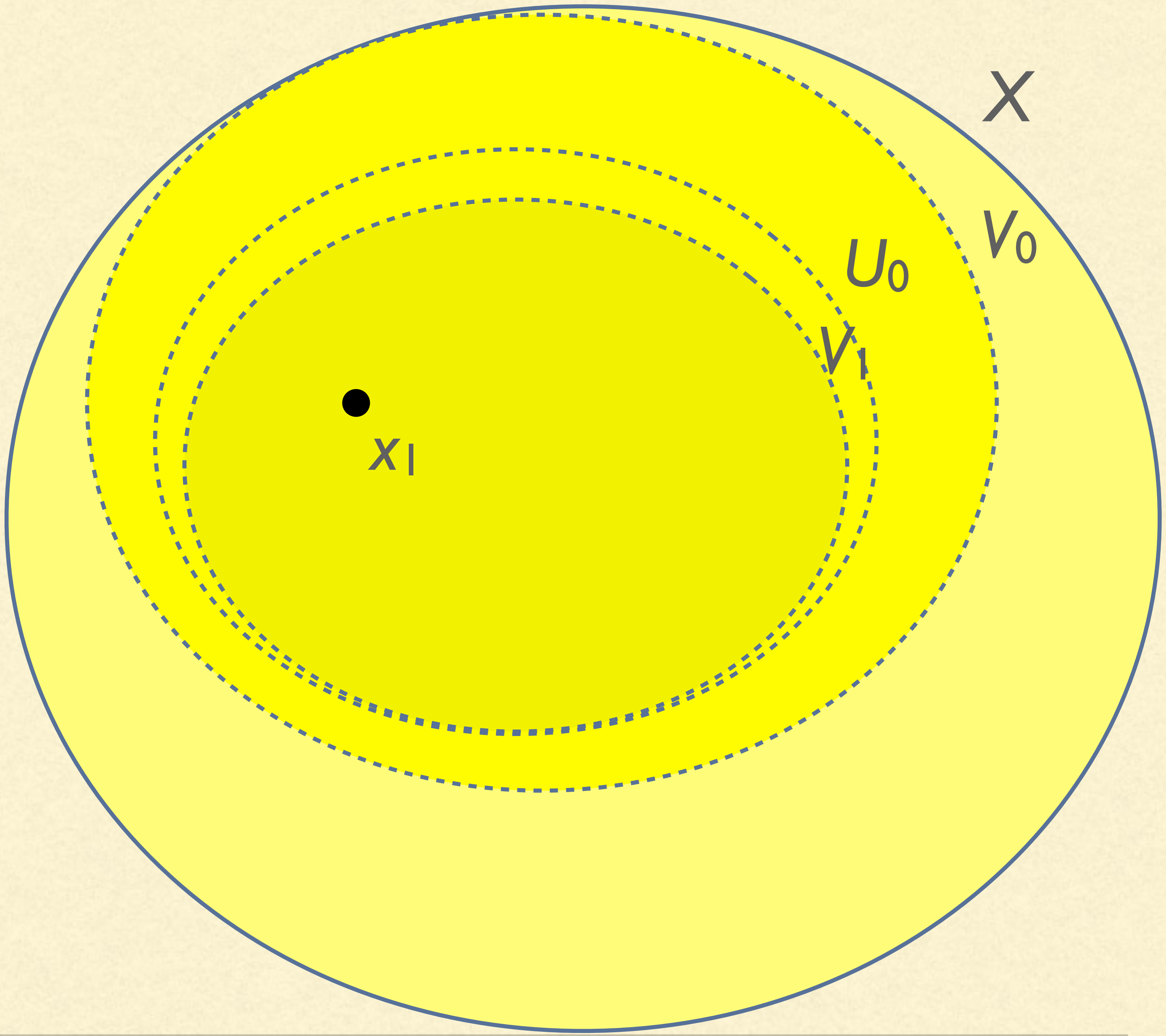
- β picks smaller open V_1, $x_1 \in V_1$



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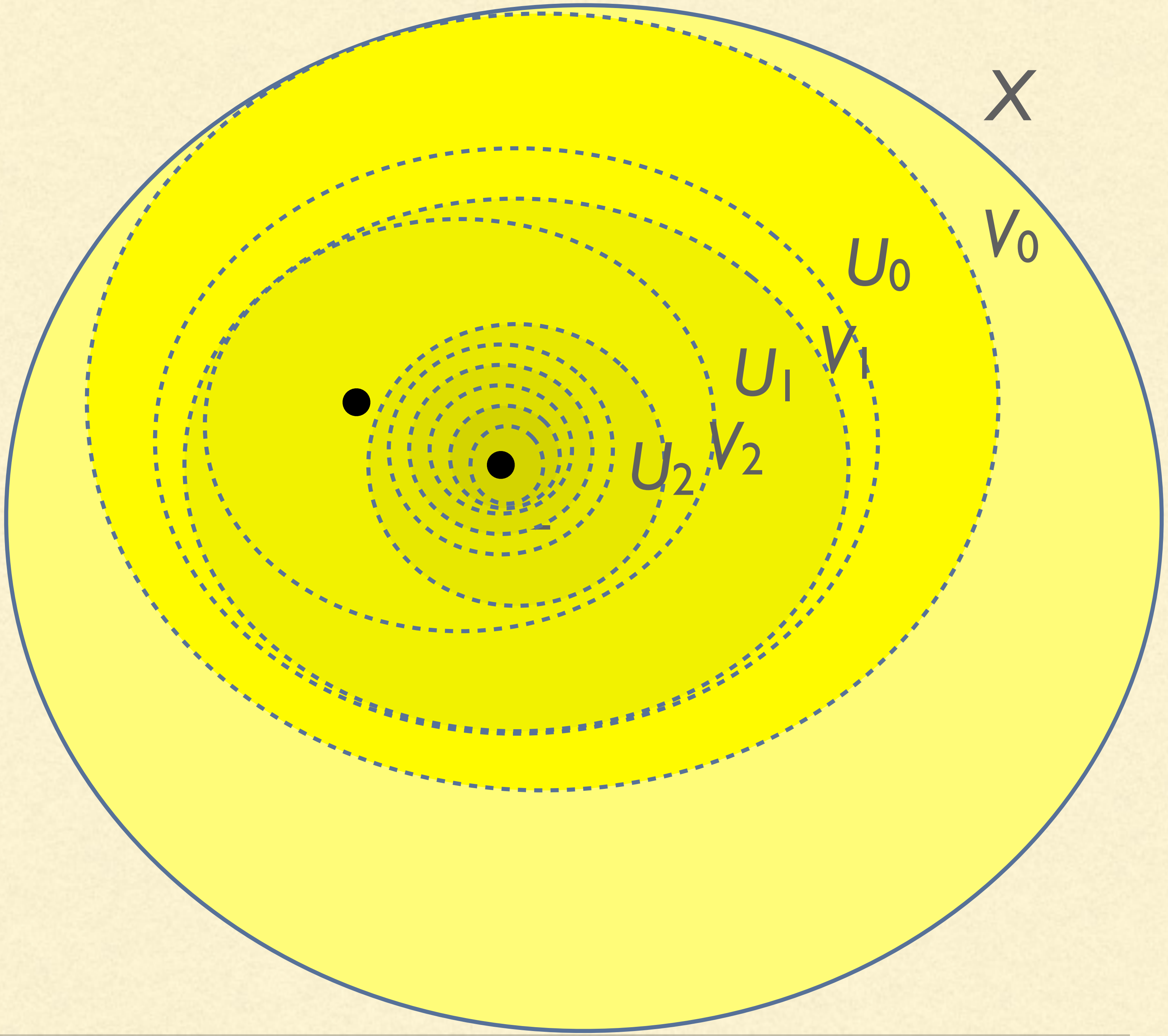

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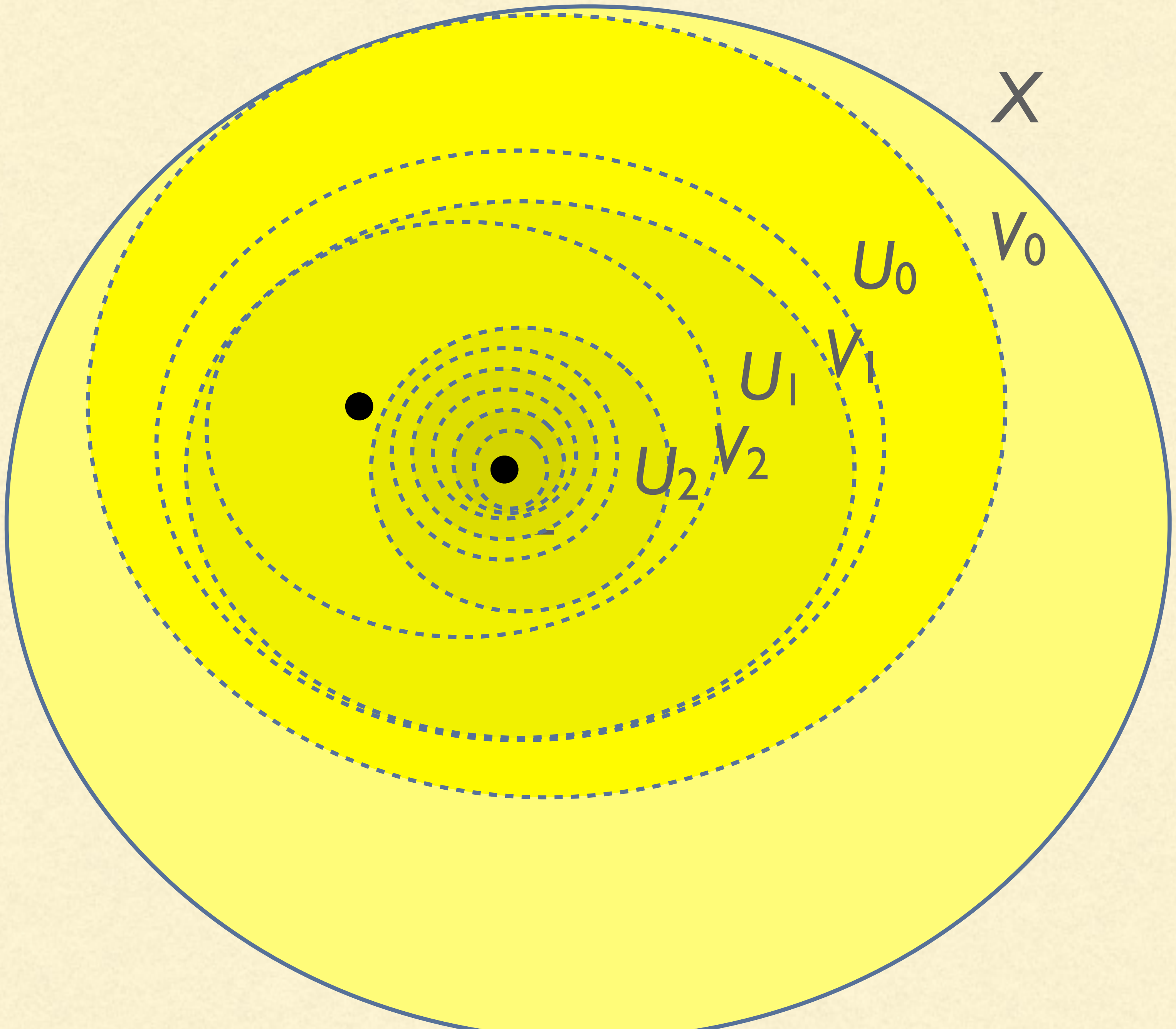
- X **Choquet-complete** iff whatever β's strategy, α can ensure $\bigcap_n U_n \neq \emptyset$



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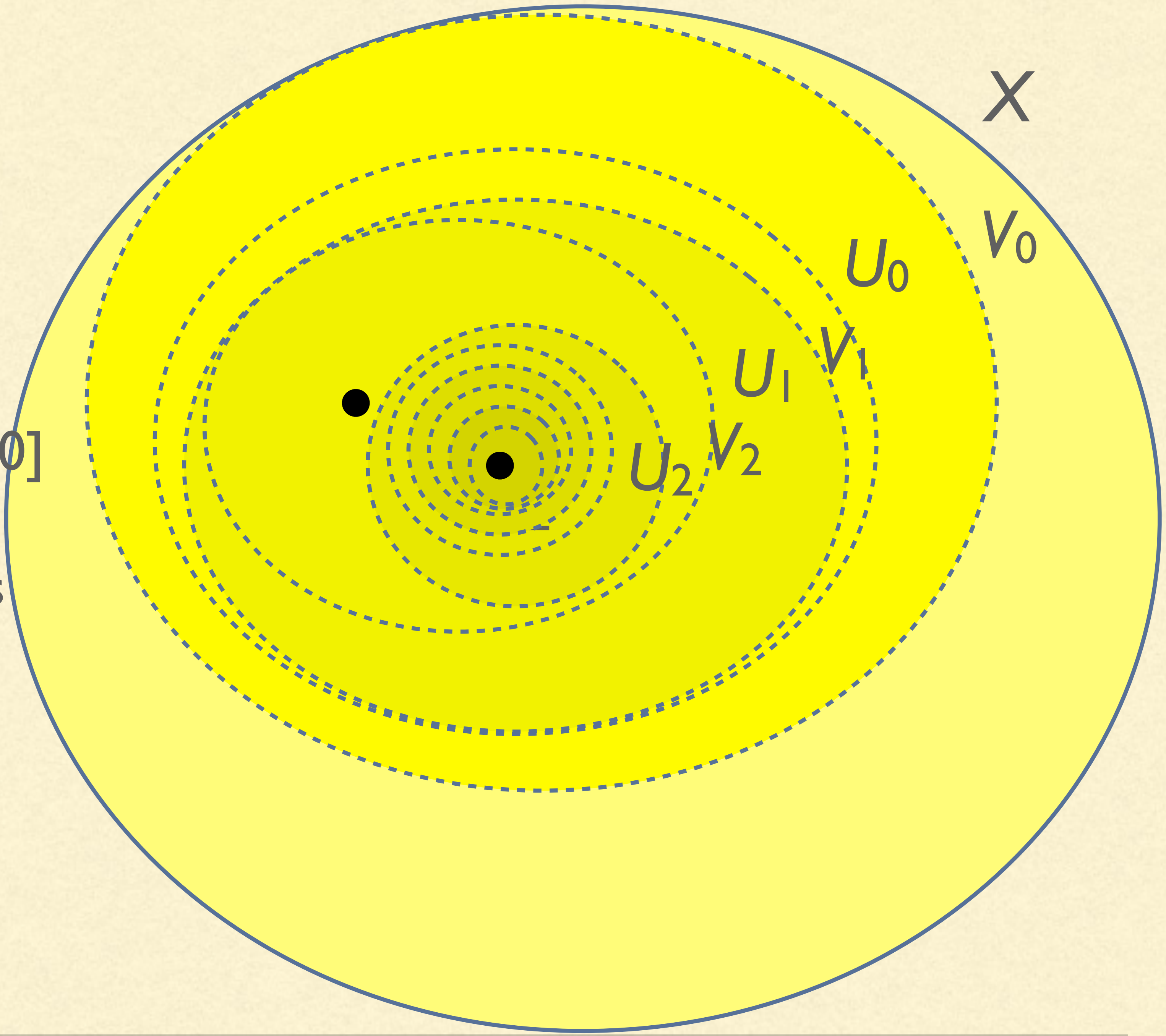
- X **convergence Choquet-complete** [Dorais,Mummert10] iff α can ensure that $(U_n)_n$ is a base of neighborhoods of some point.



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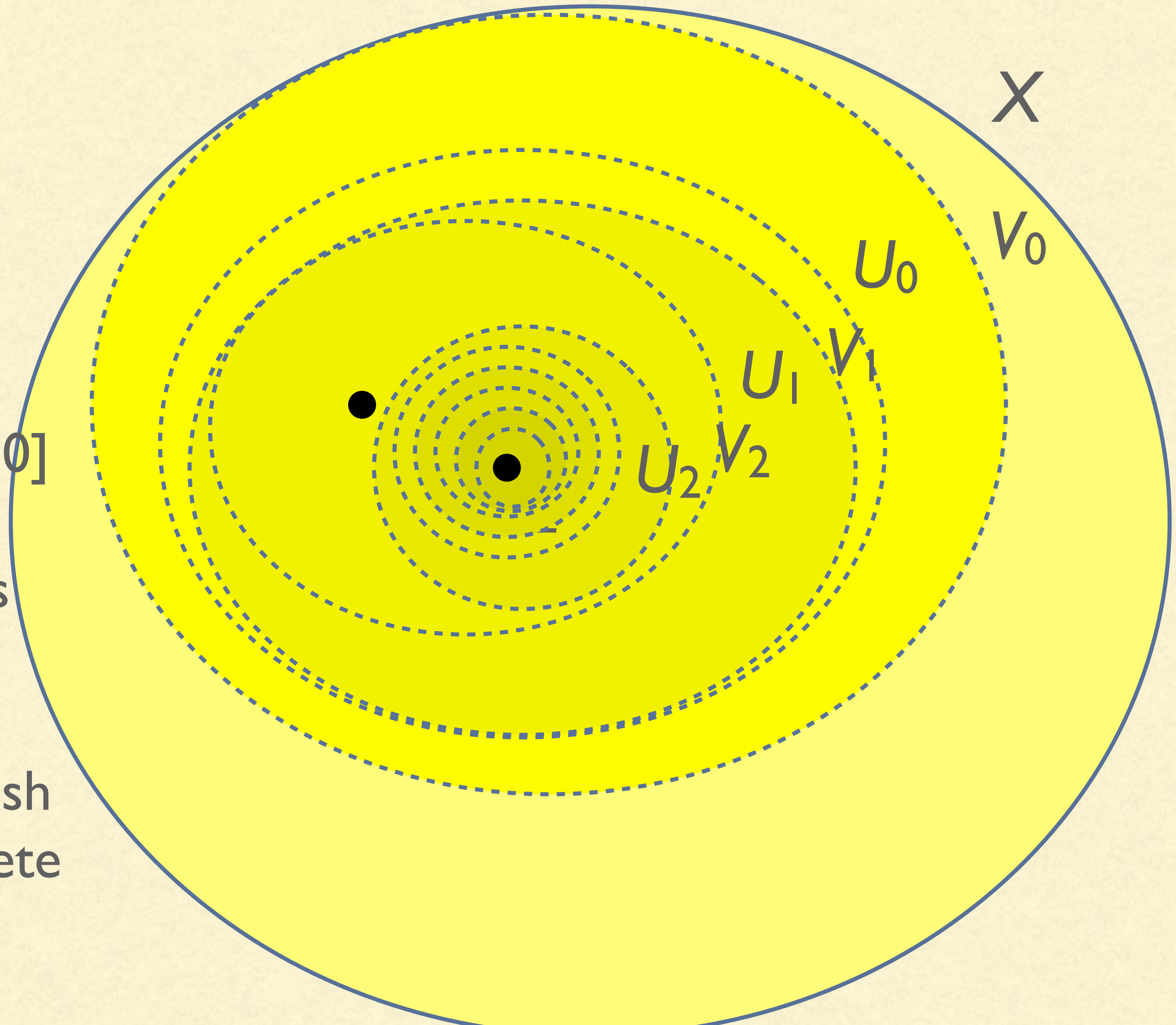
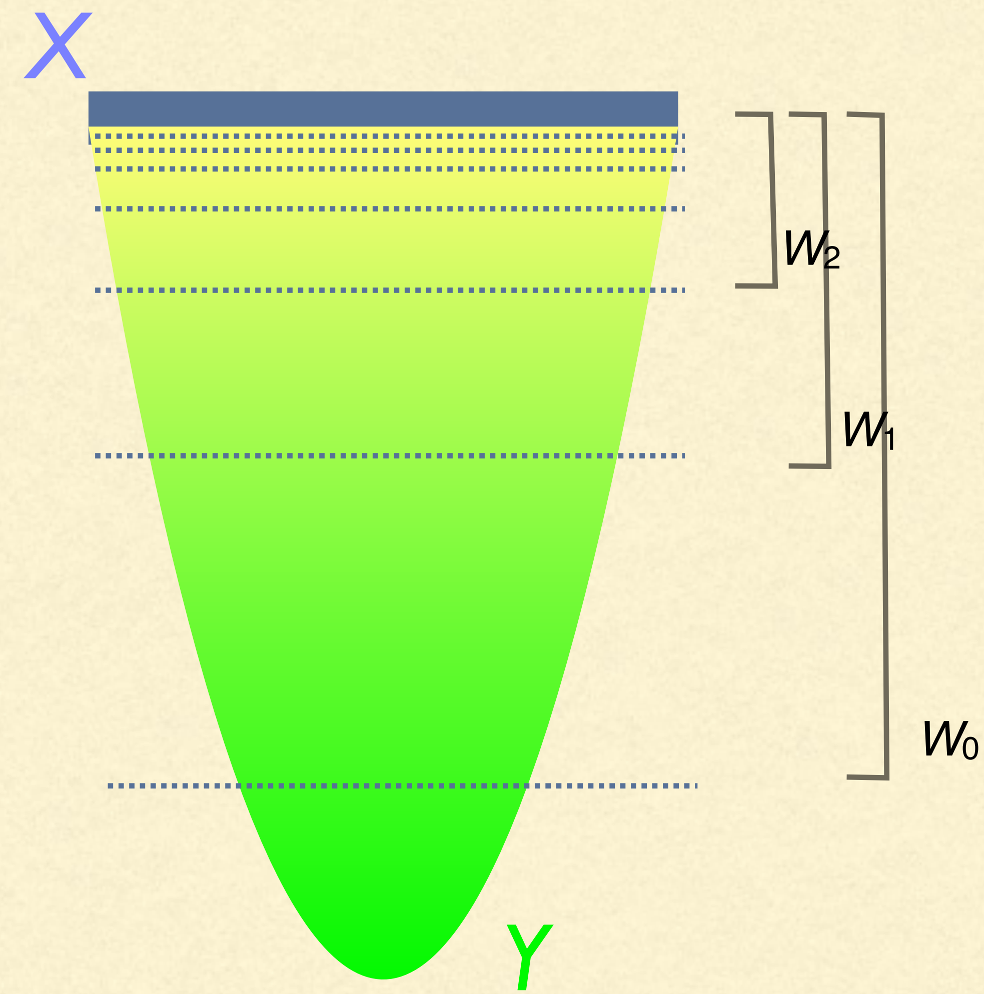
- **Thm** [deBrecht13]. Quasi-Polish = convergence Choquet-complete + countably-based



DOMAIN-COMPLETE ⇒ CONVERGENCE CHOQUET-COMPLETE



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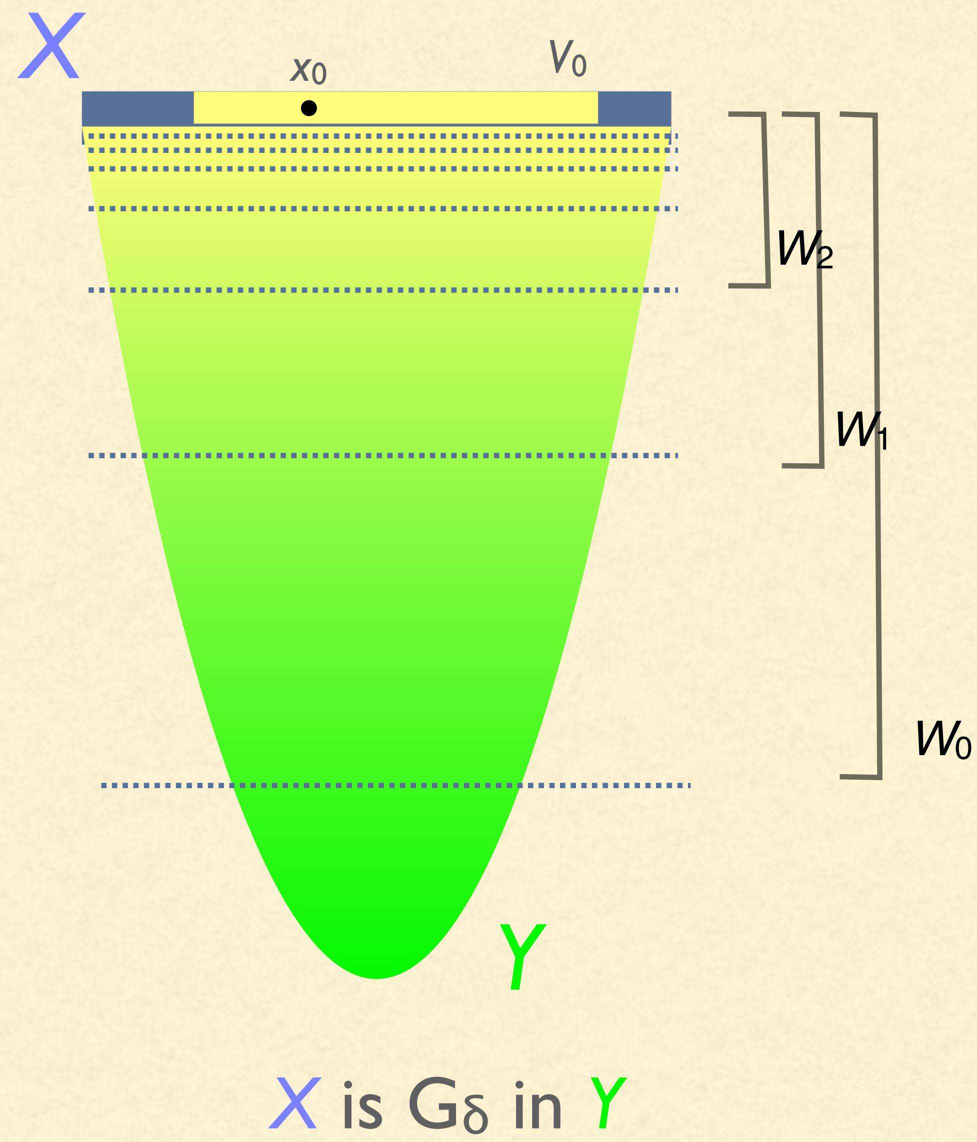

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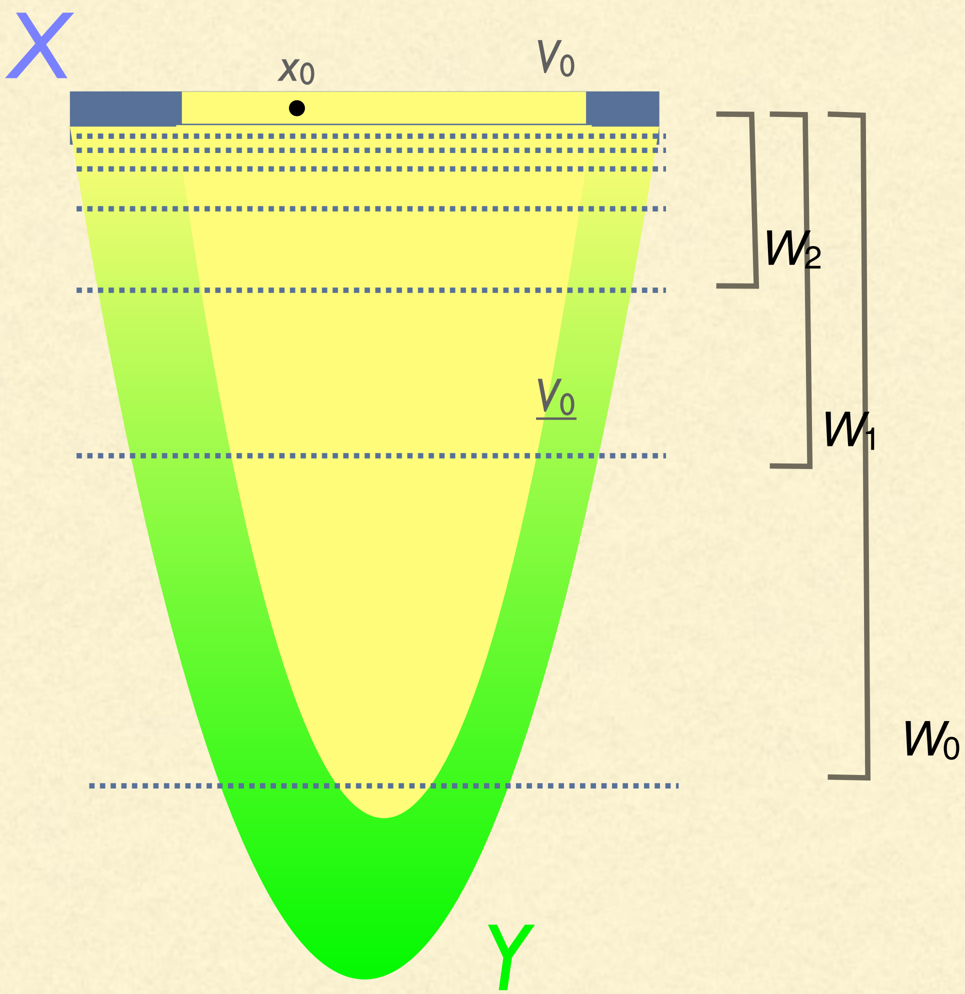

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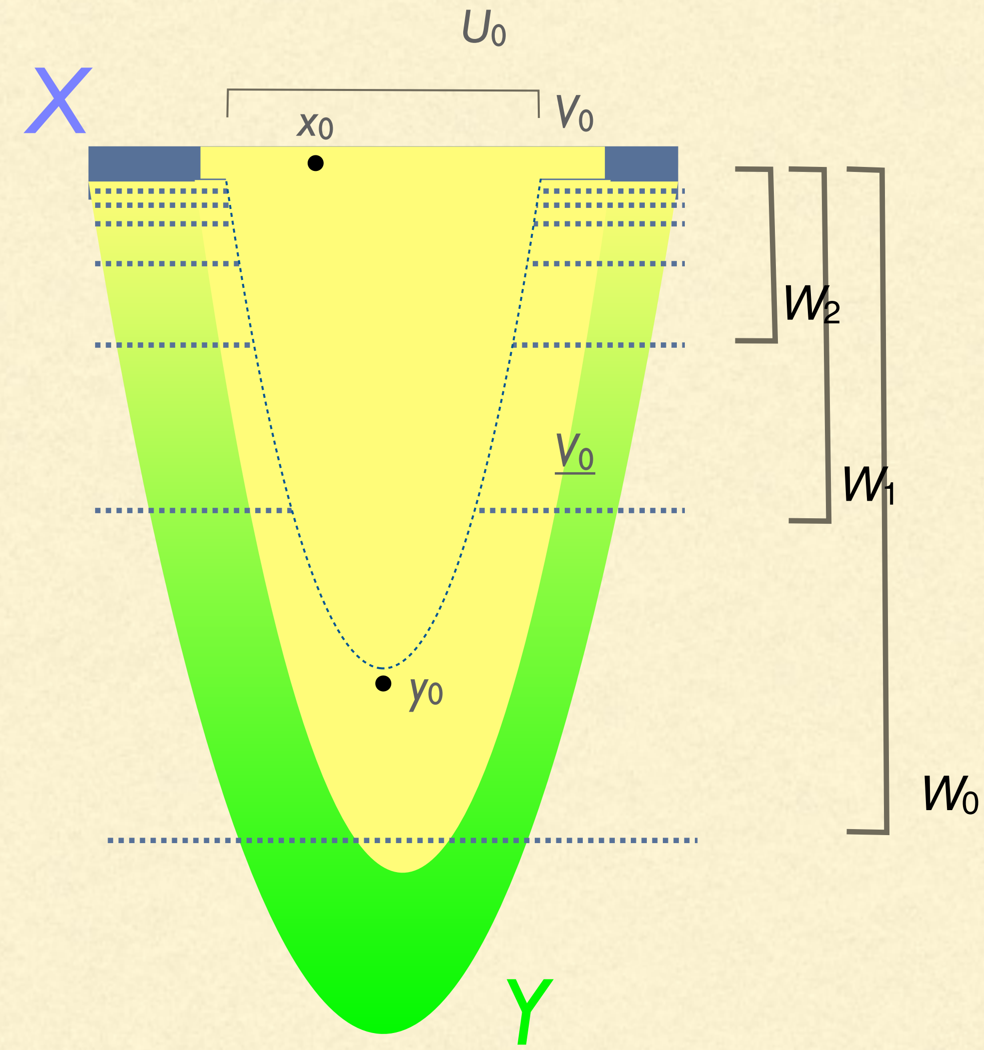
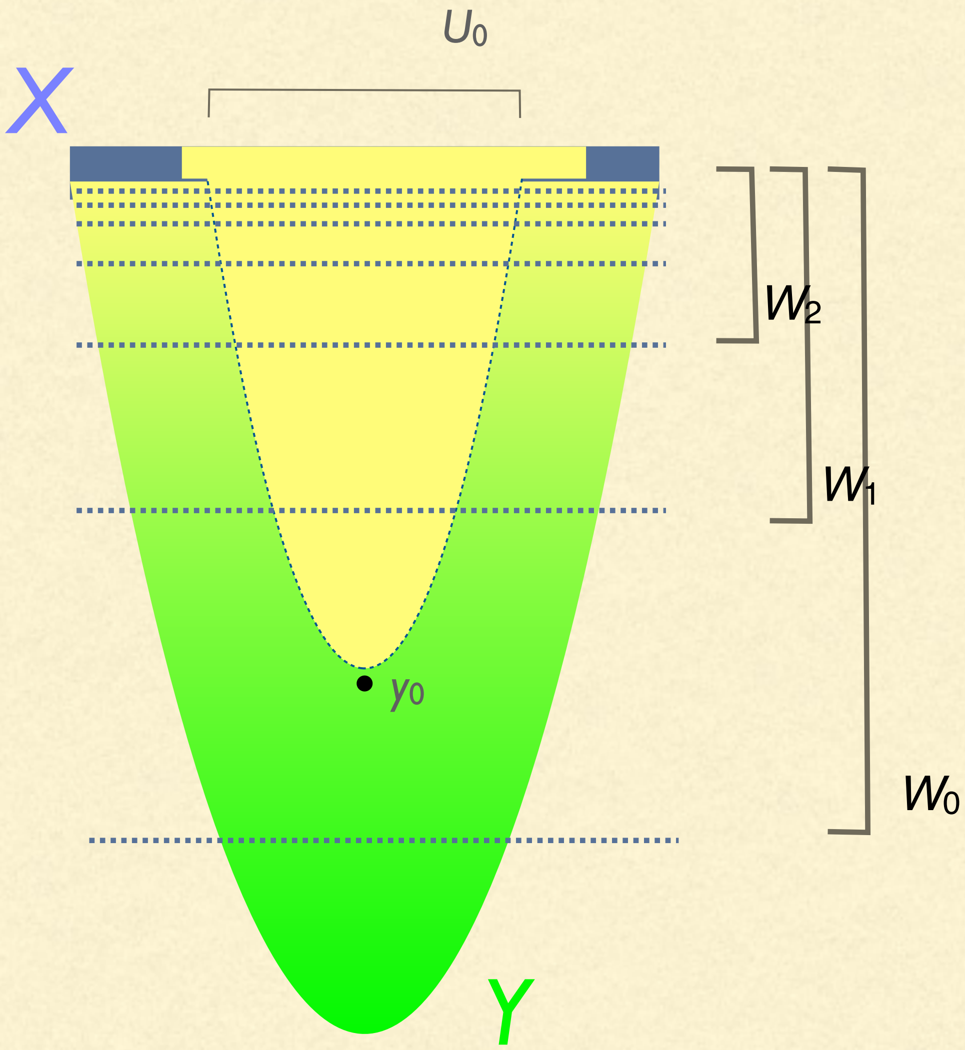
- α finds $y_0 \ll x_0$, in $\underline{V_0} \cap W_0$, and plays $U_0 = \uparrow y_0 \cap X$.



X is G_δ in Y

DOMAIN-COMPLETE ⇒ CONVERGENCE CHOQUET-COMPLETE



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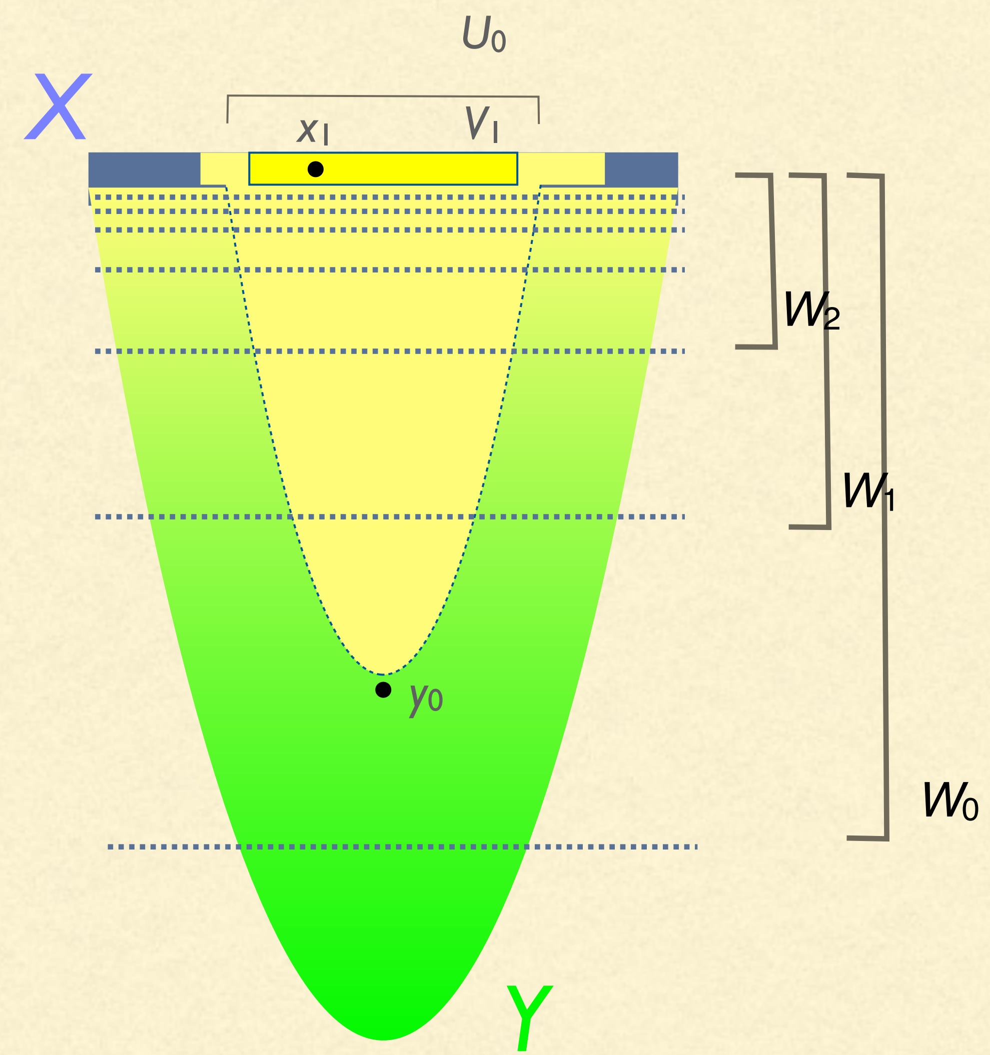

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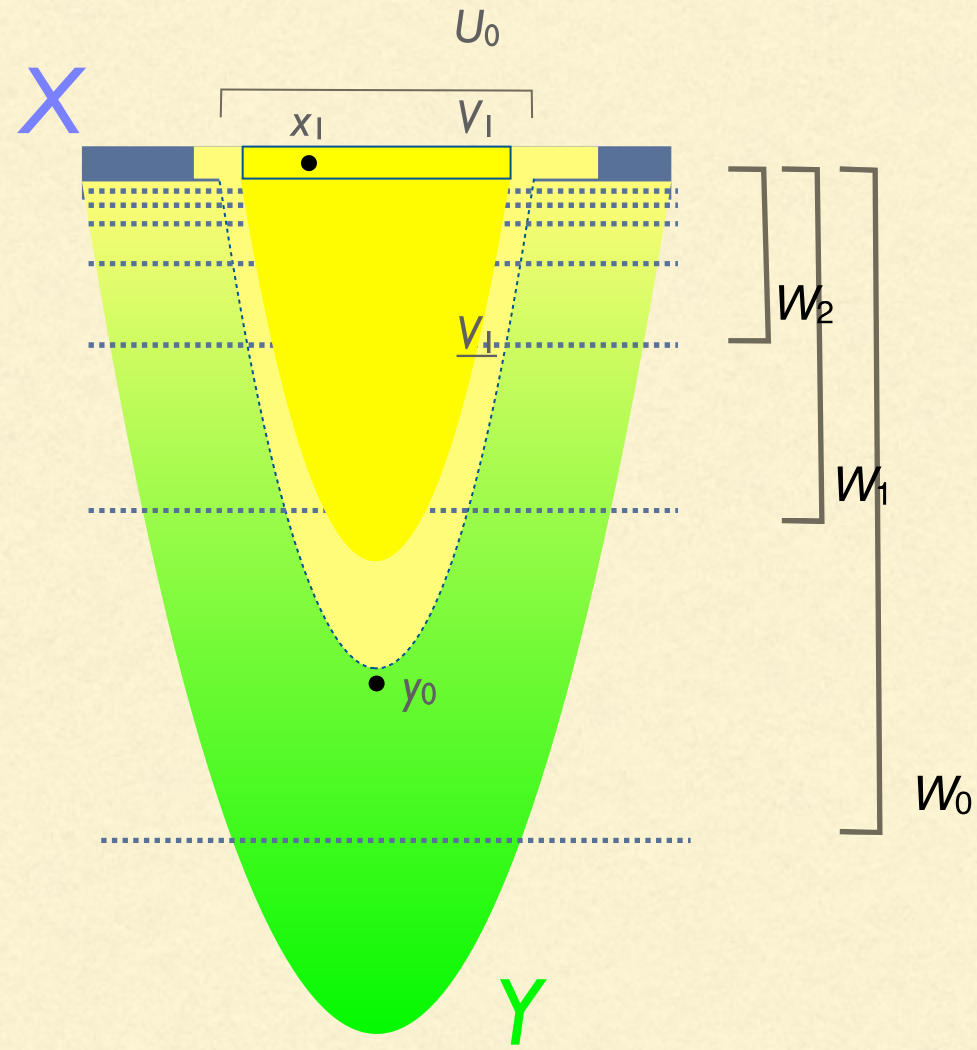

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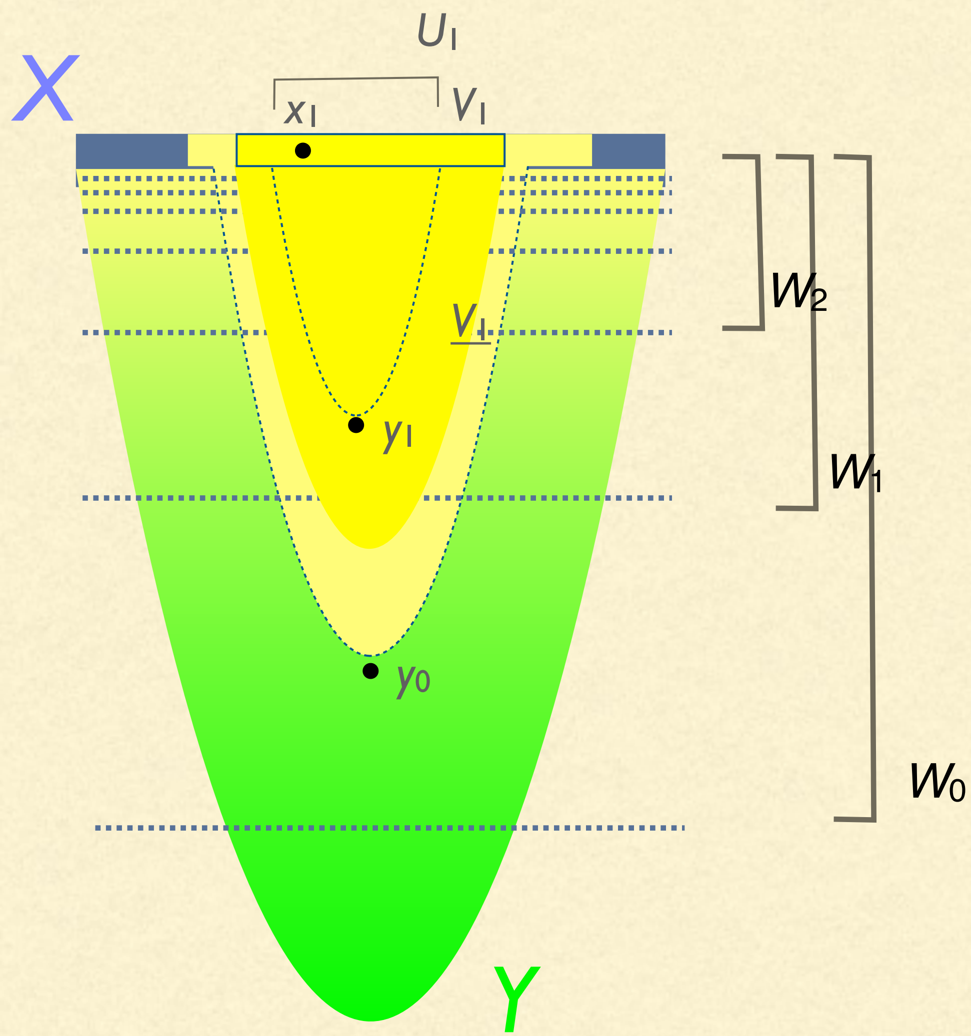

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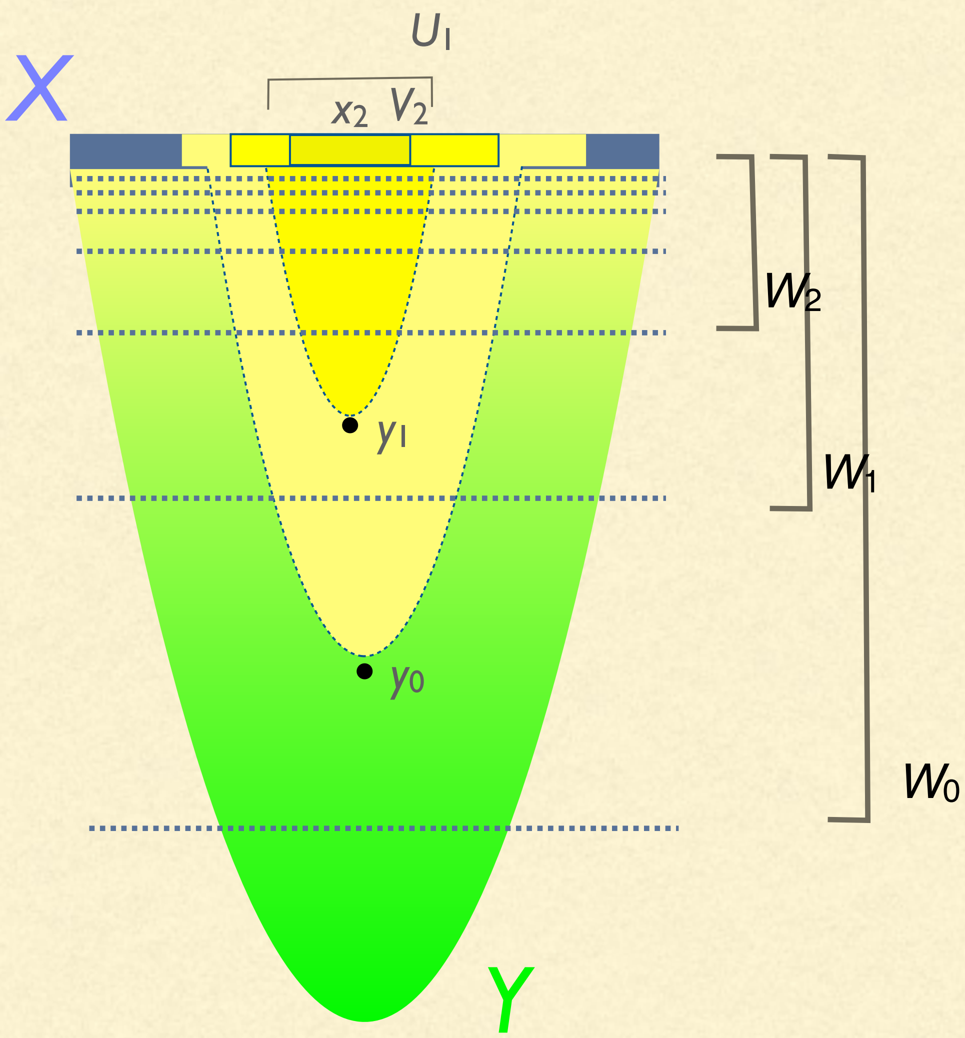

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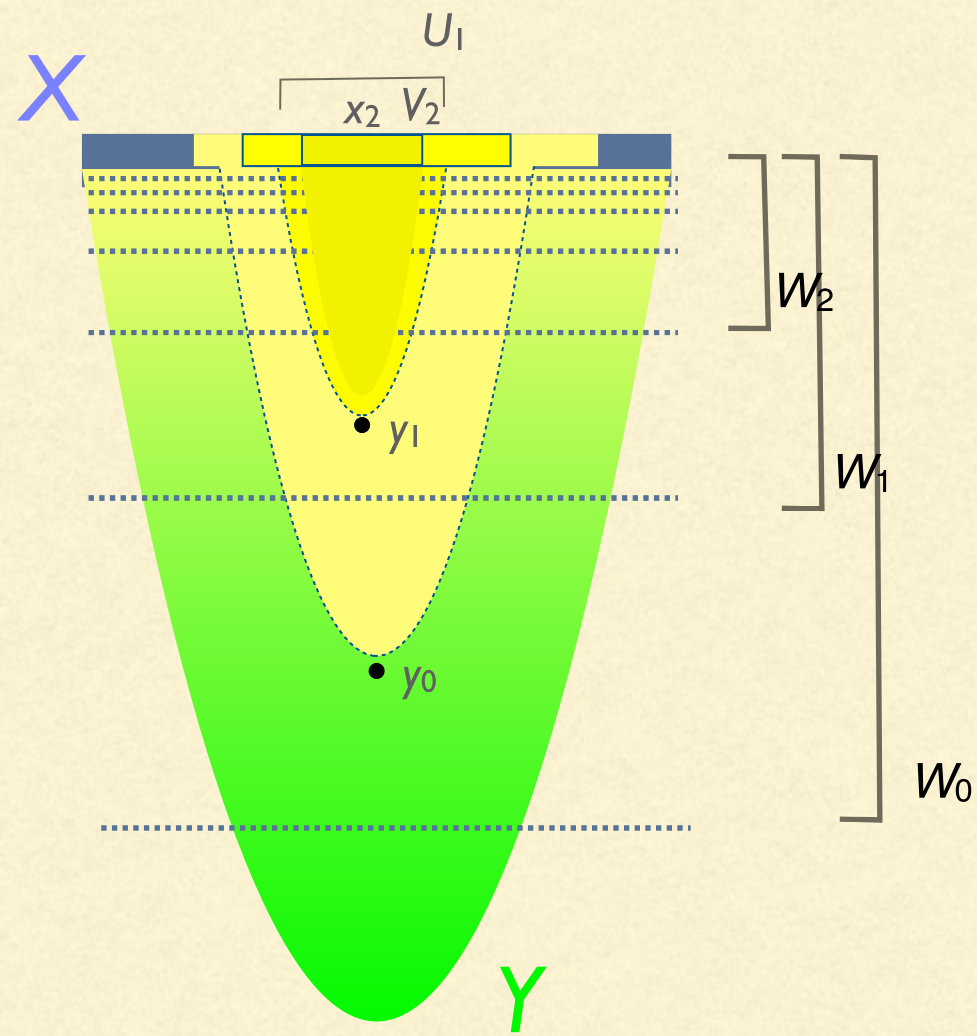

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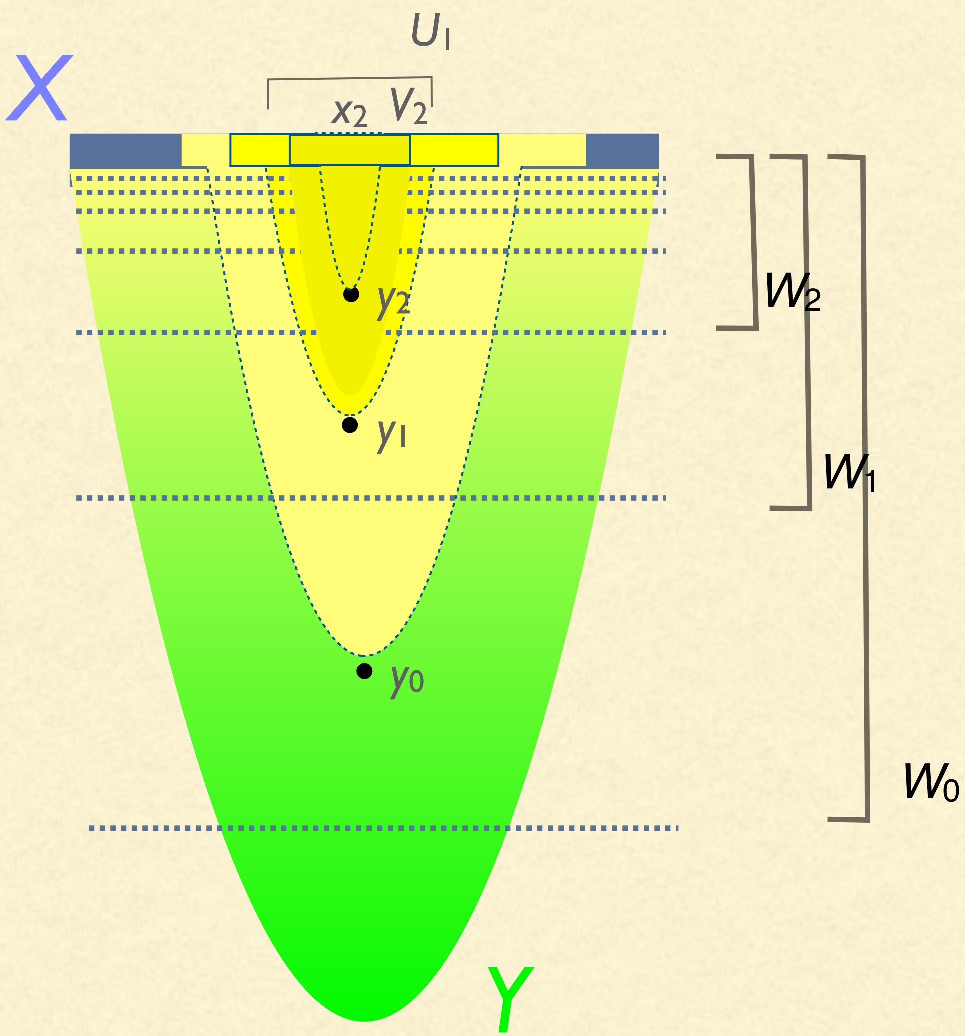

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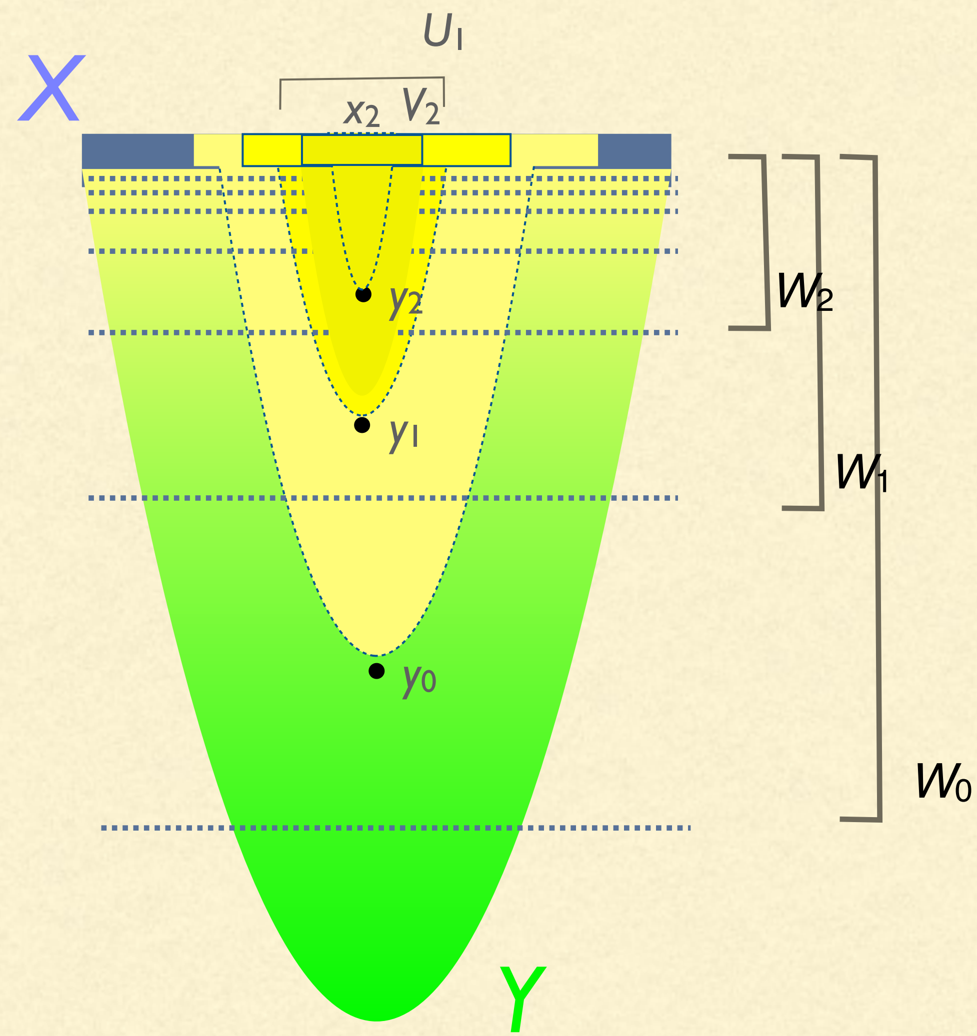

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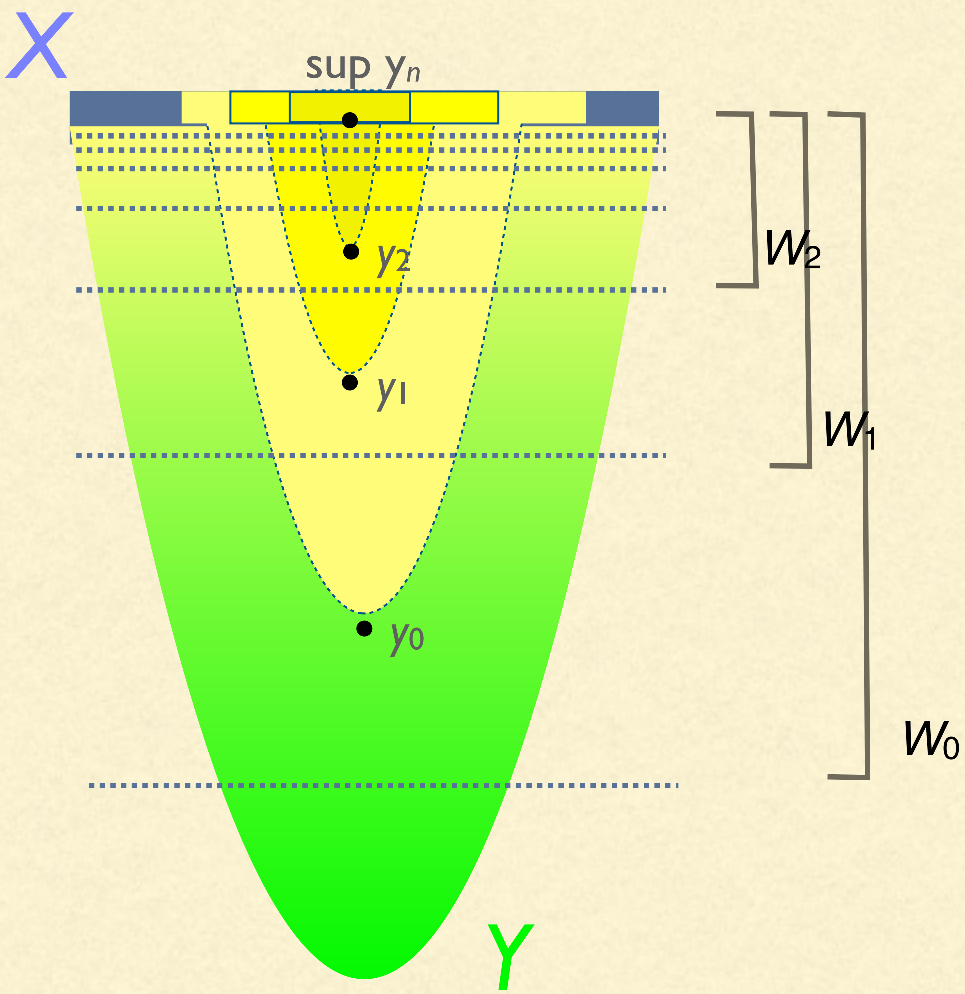
DOMAIN-COMPLETE ⇒ CONVERGENCE CHOQUET-COMPLETE

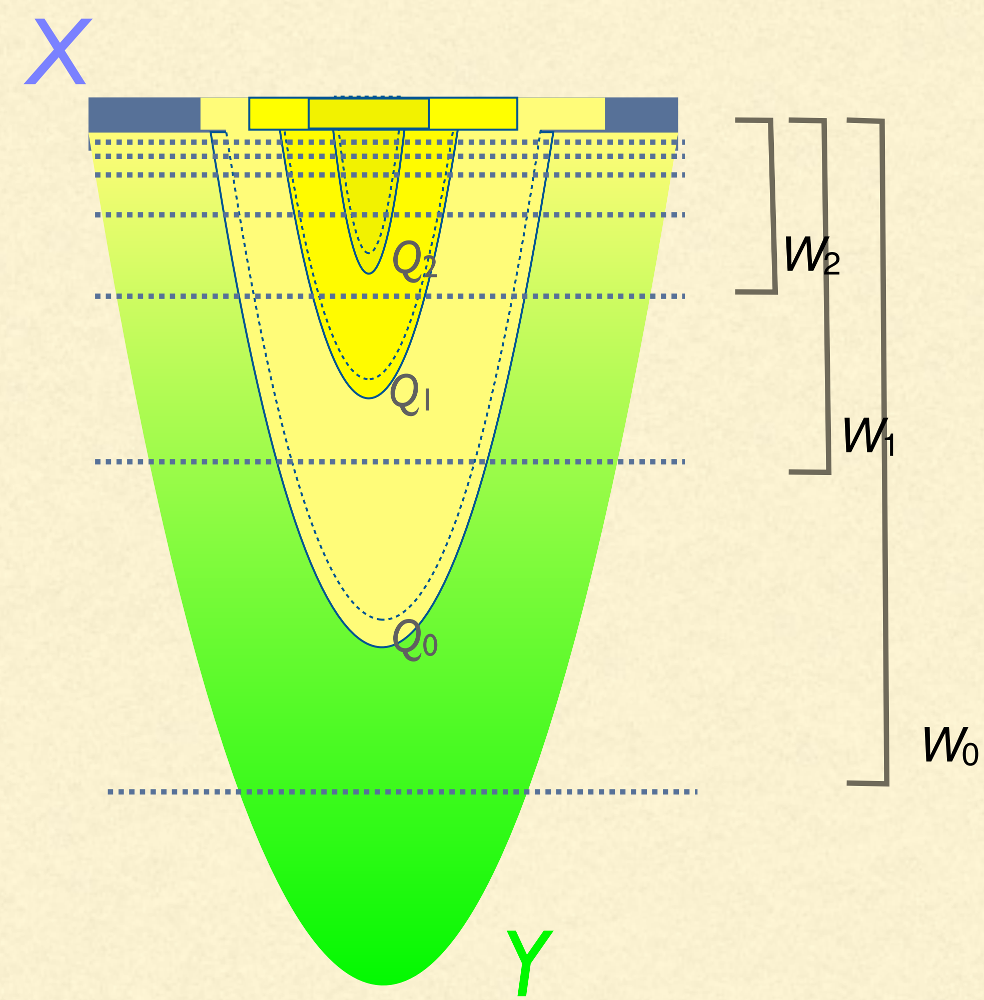
- For every n, $U_n = \uparrow y_n \cap X$
- In the end, $(U_n)_n$ is a base of neighborhoods of sup y_n. □



X is G_δ in Y

LCS-COMPLETE ⇒ COMPACTLY CHOQUET-COMPLETE

- For LCS-complete spaces, replace $\uparrow y_n$ by compact saturated sets Q_n
- $U_n = int(Q_n) \cap X$
- In the end, $(U_n)_n$ is a base of neighborhoods of ~~sup y_n~~ *a non-empty compact saturated set Q.* □



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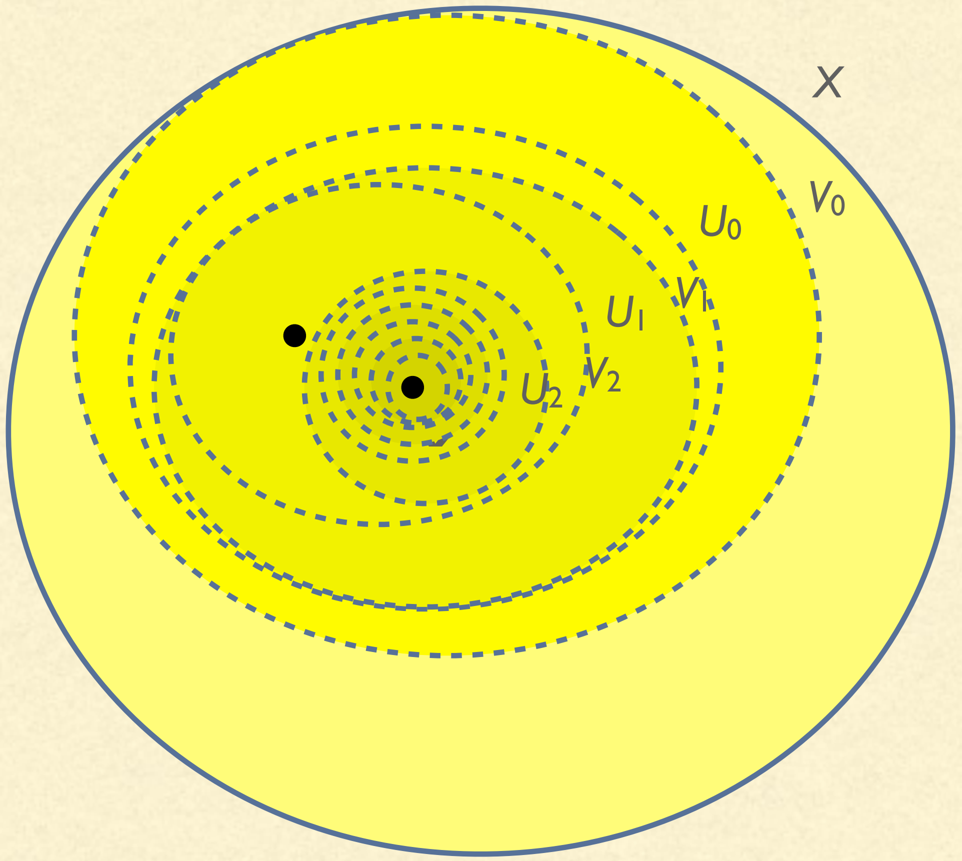
COMPACT CHOQUET-COMPLETENESS

- X is **compactly Choquet-complete** iff α can ensure that $(U_n)_n$ is a base of neighborhoods of some non-empty compact sat. set Q.



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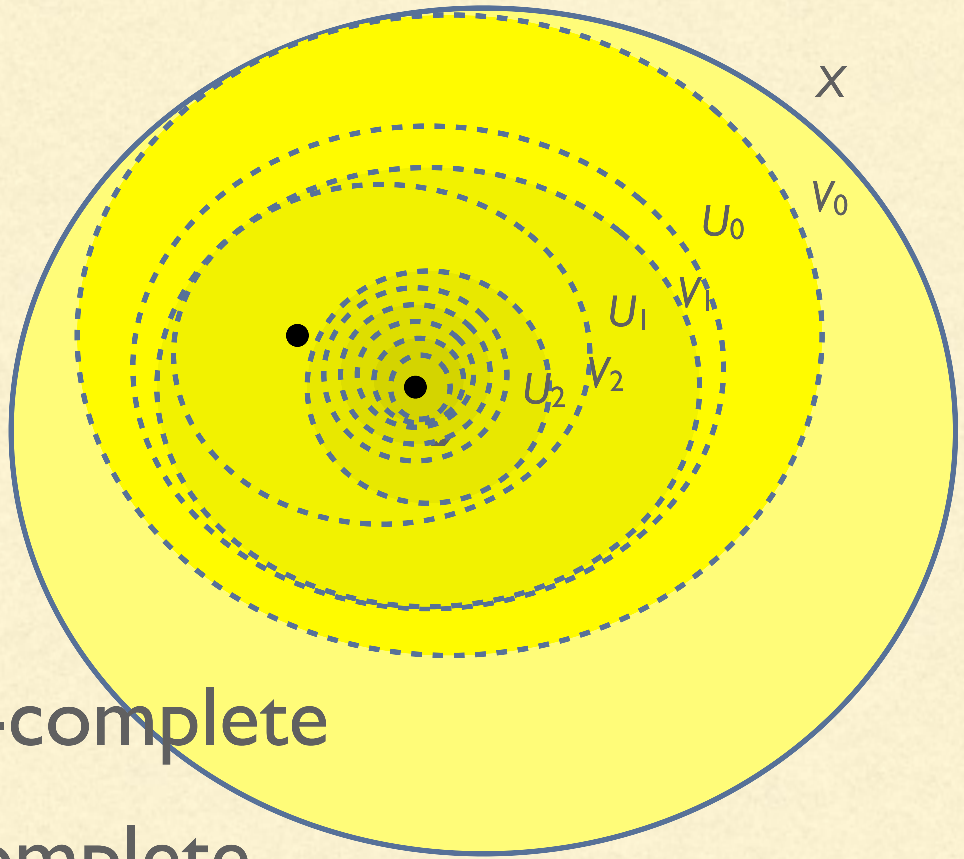

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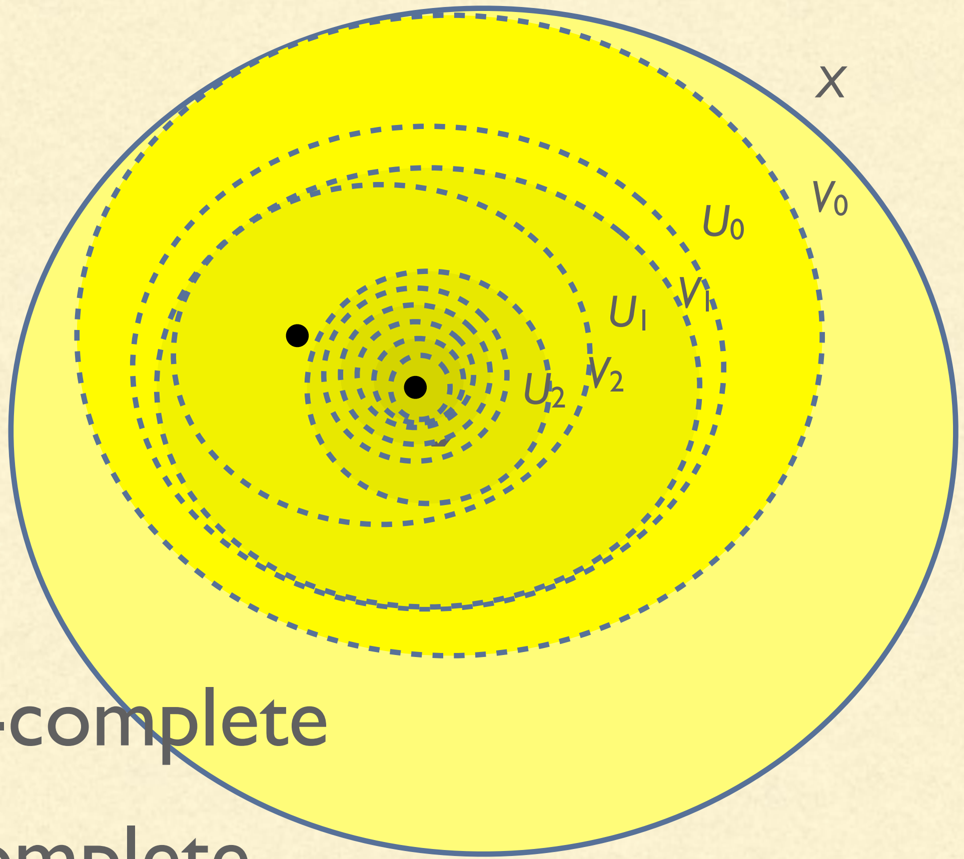

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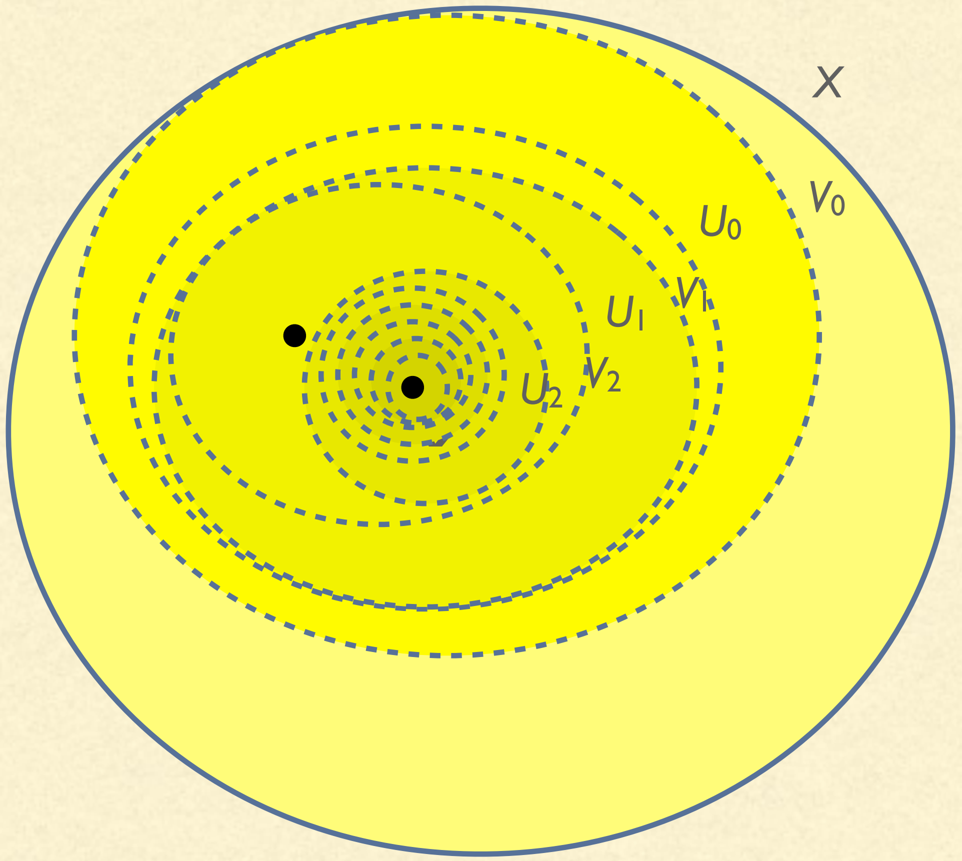
- Used everywhere in the theory.

SOME CONSEQUENCES (1)



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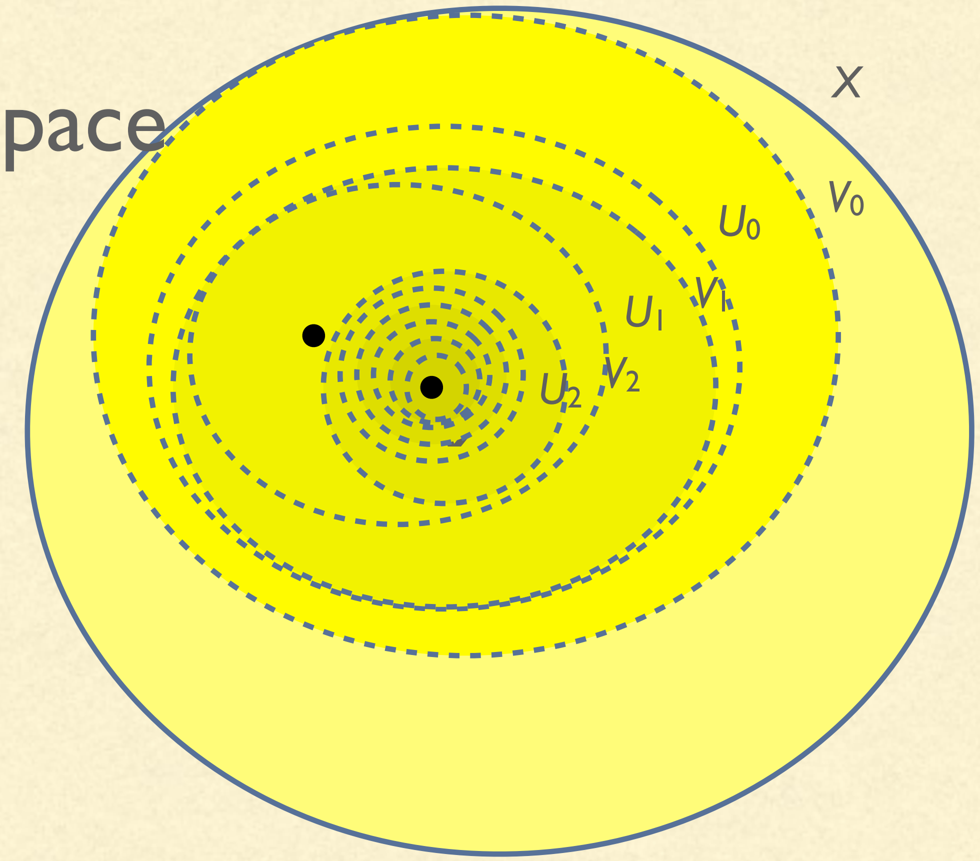

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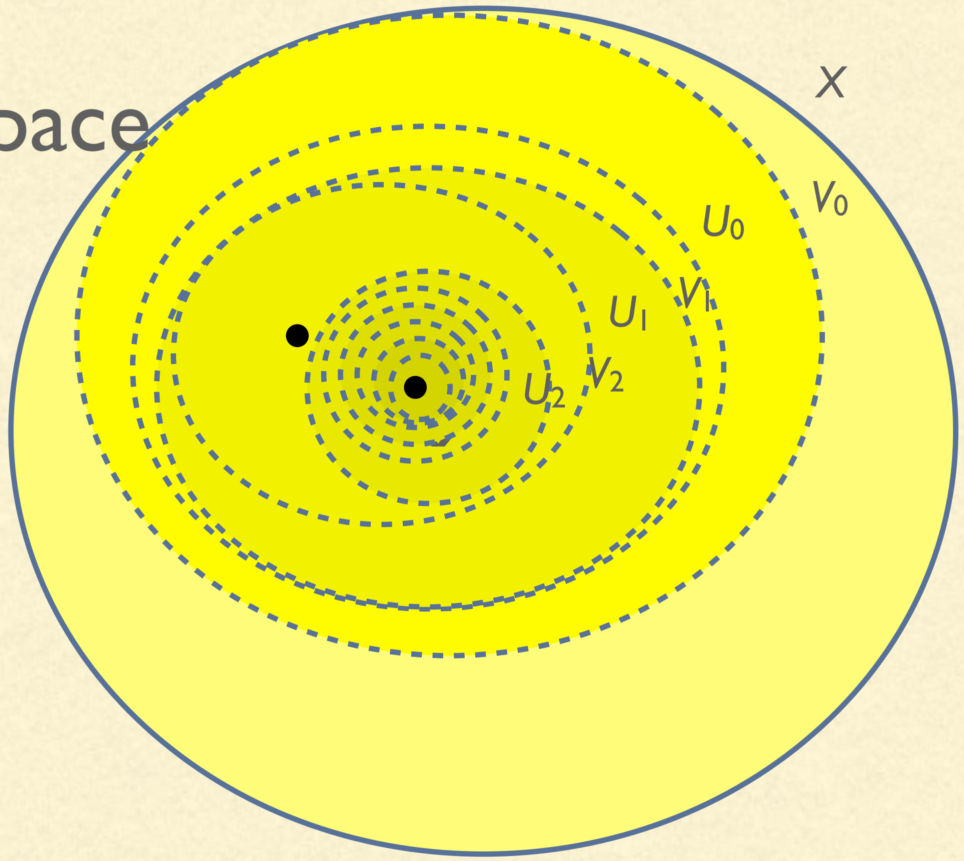

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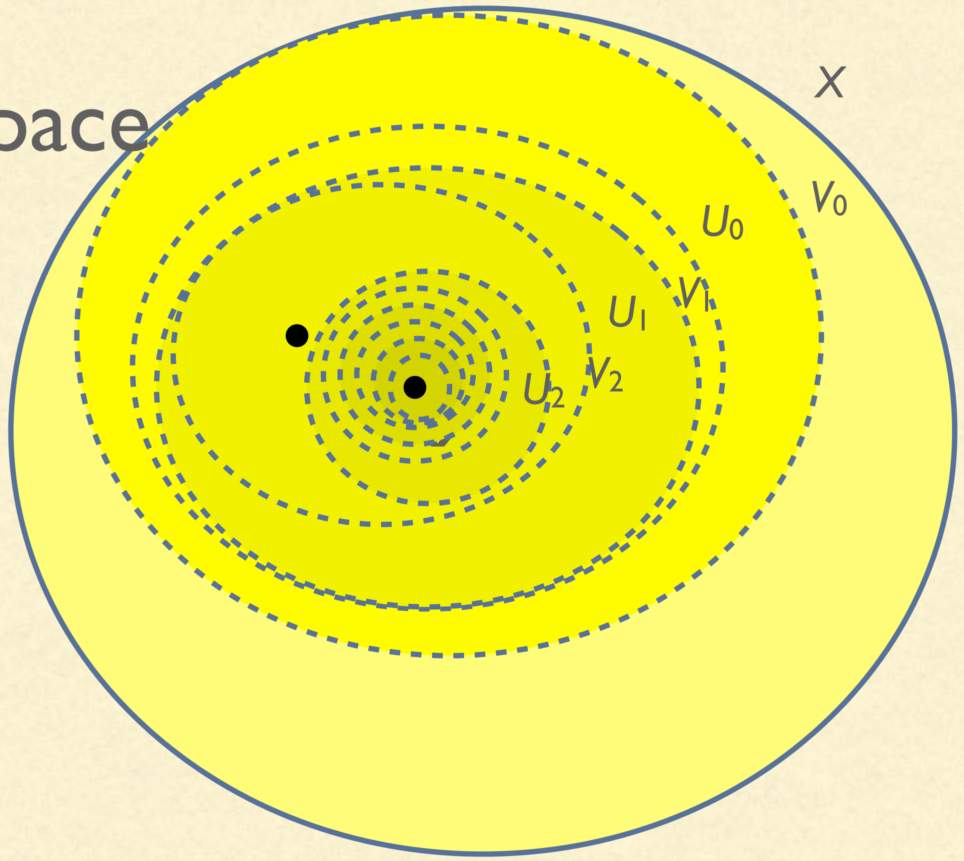

- **Thm.** LCS-complete≠domain-complete
  ($\{0,1\}^I$, with I uncountable, is compact T_2,
  but not convergence Choquet-complete)

- **Prop.** $\mathbb{Q}$ is not LCS-complete
  (not Choquet-complete: let β remove the first point of U_n
  in some fixed enumeration of $\mathbb{Q}$; α cannot win)

SOME CONSEQUENCES

- **Thm.** LCS-complete + countably-based
  = quasi-Polish

- **Proof.** Let B_n form a countable base.
  Instead of playing U_n, α plays the intersection
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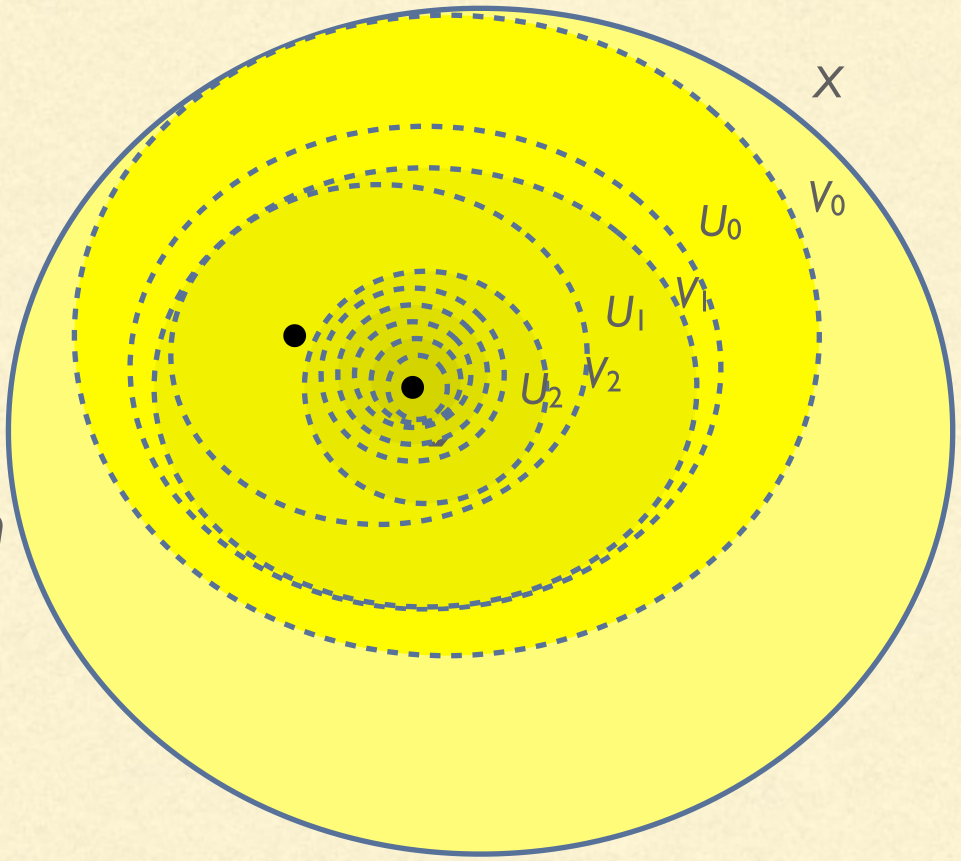

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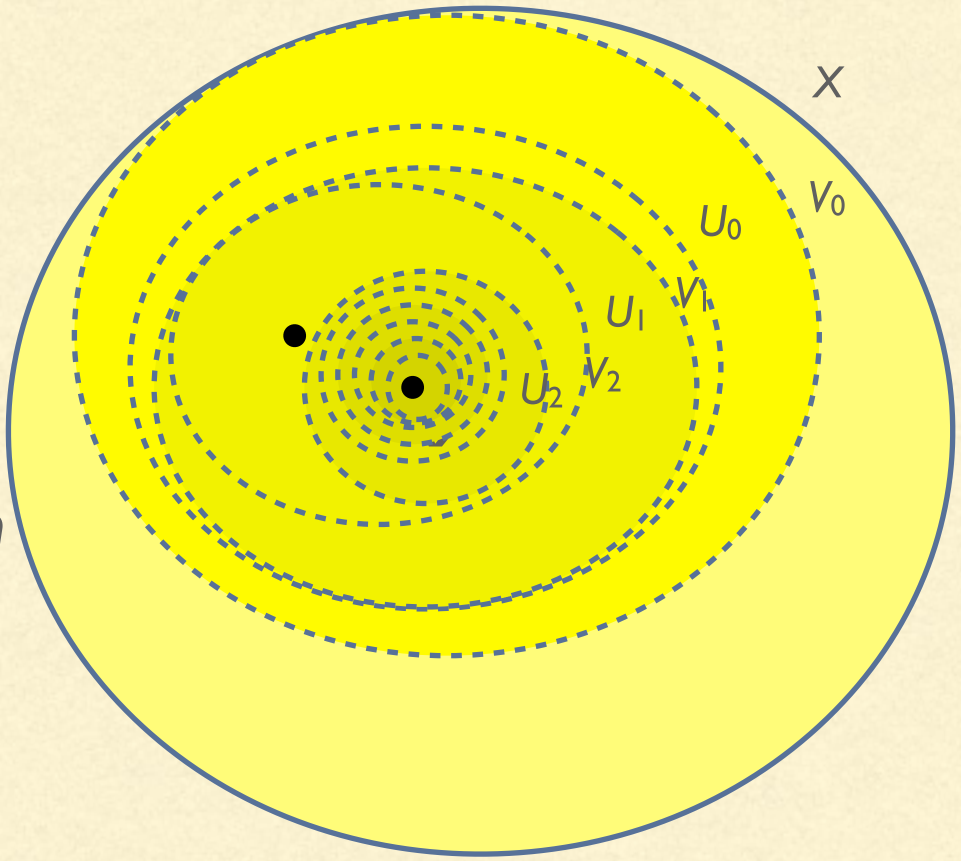

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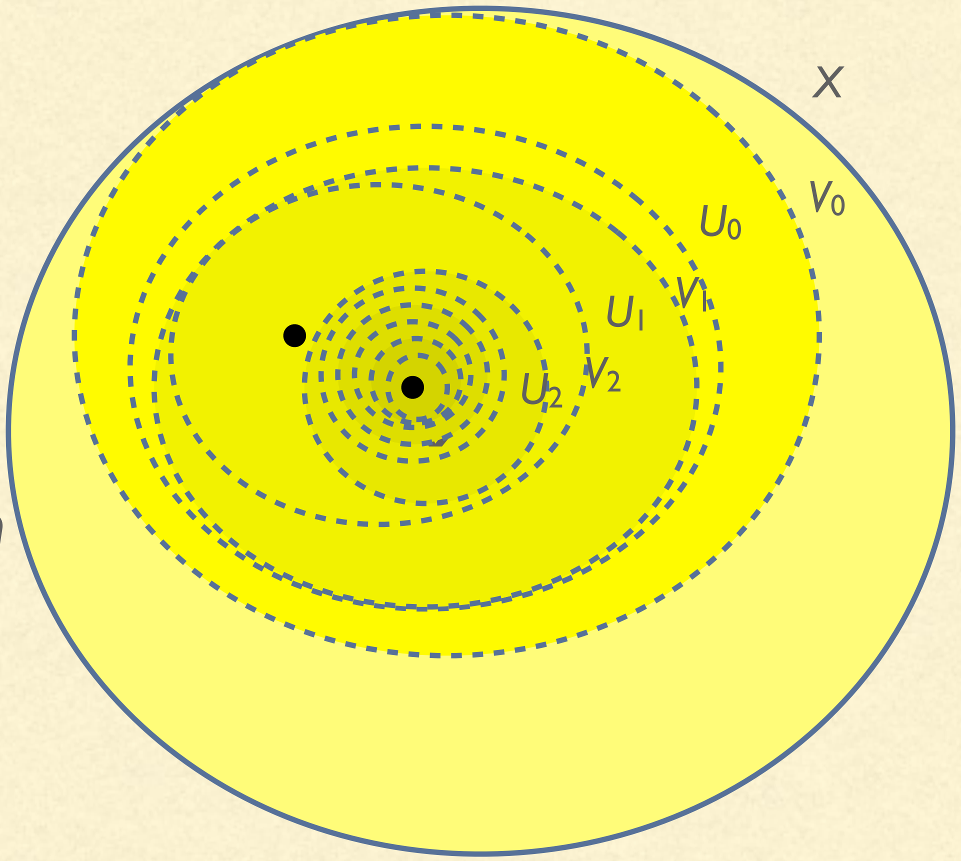

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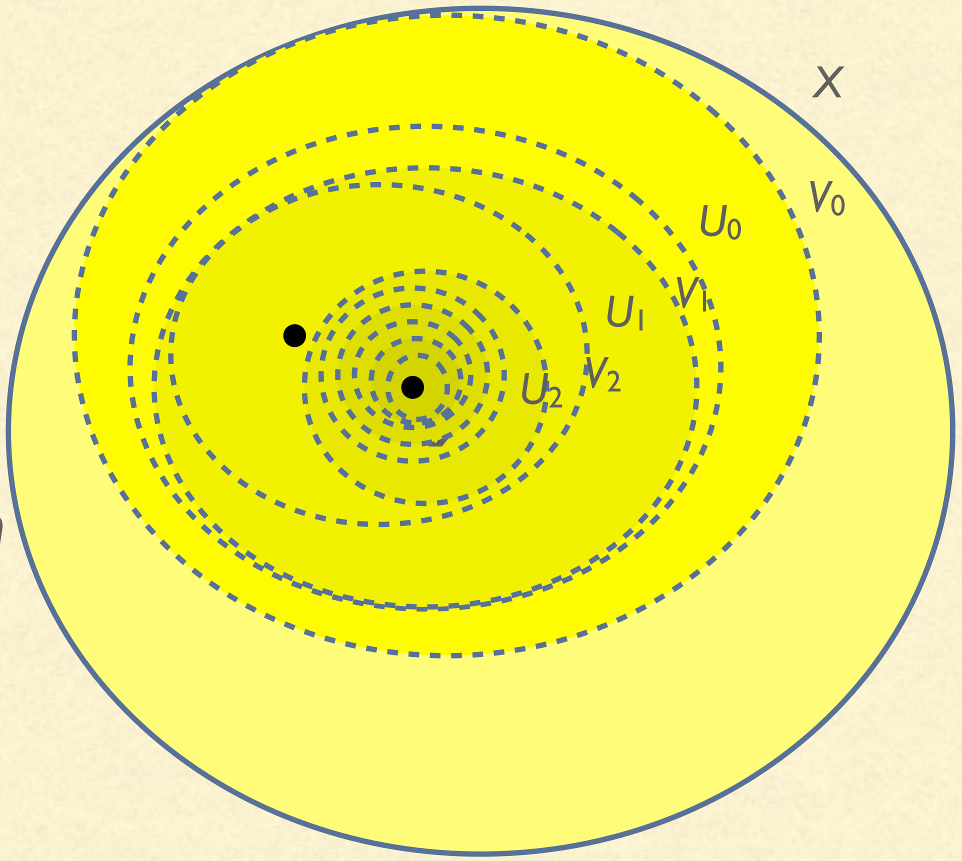

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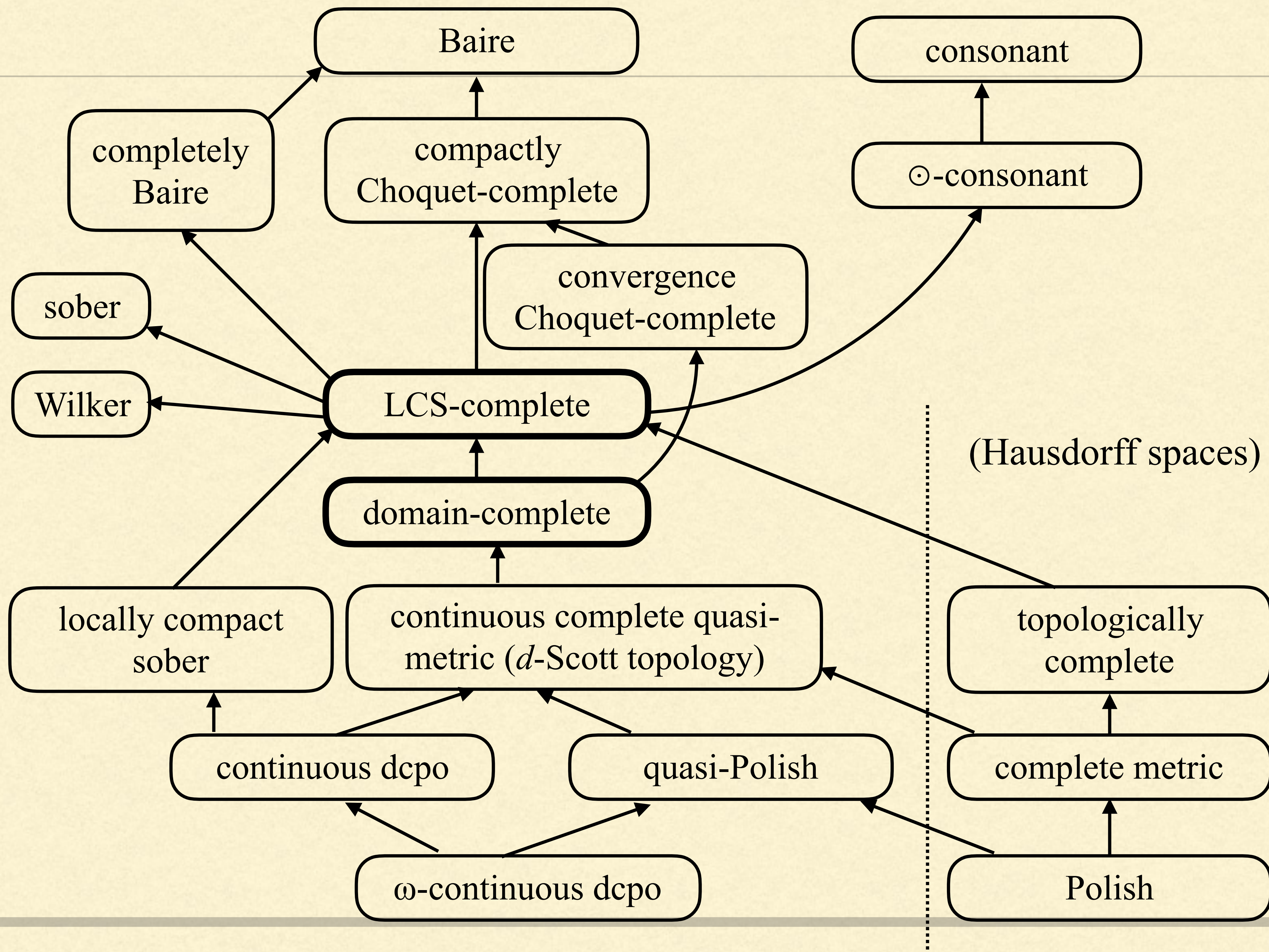
- Recall [deBrecht13]: this + countably-based ⇒ quasi-Polish. □

THE FINAL PICTURE



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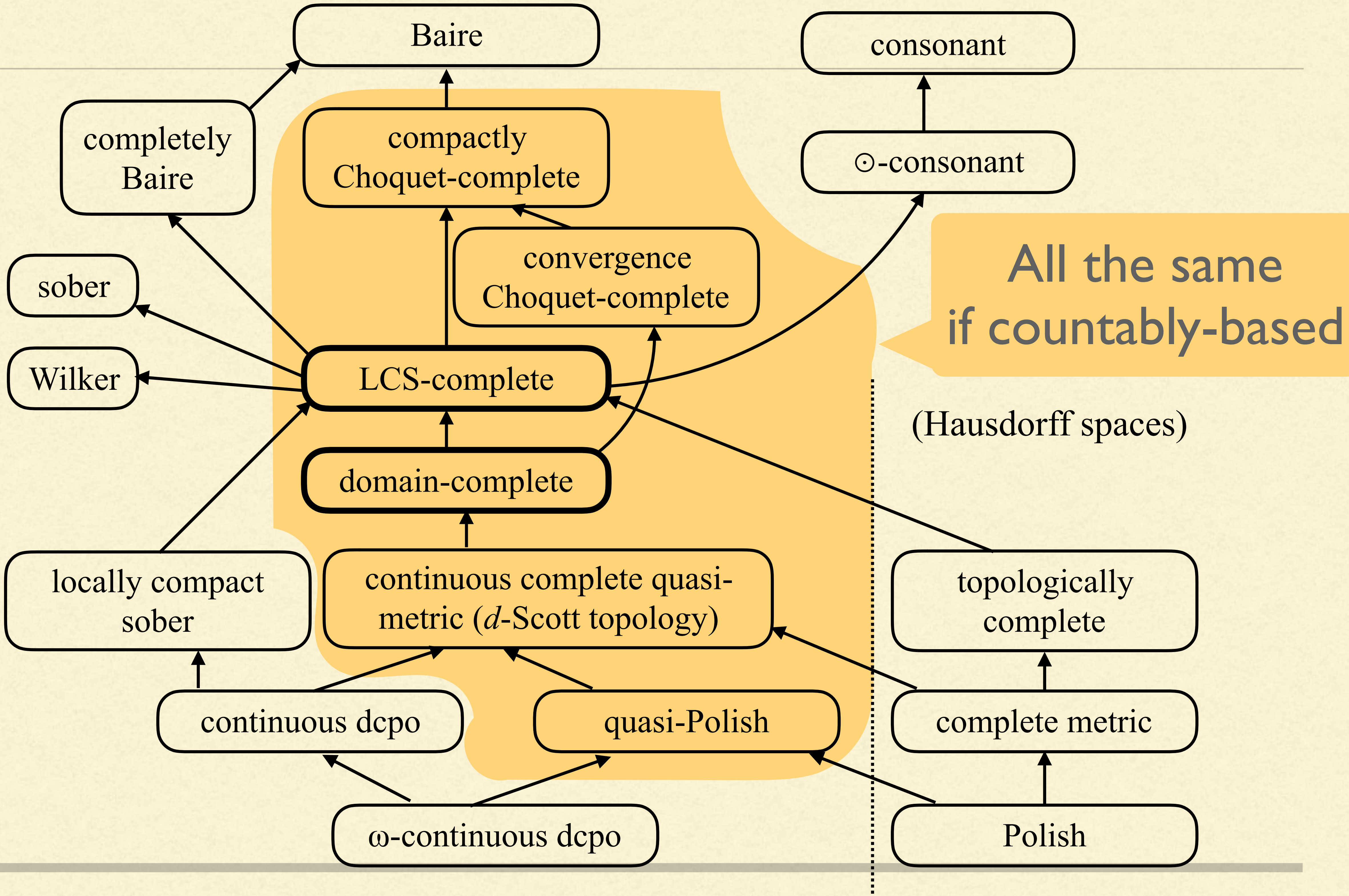
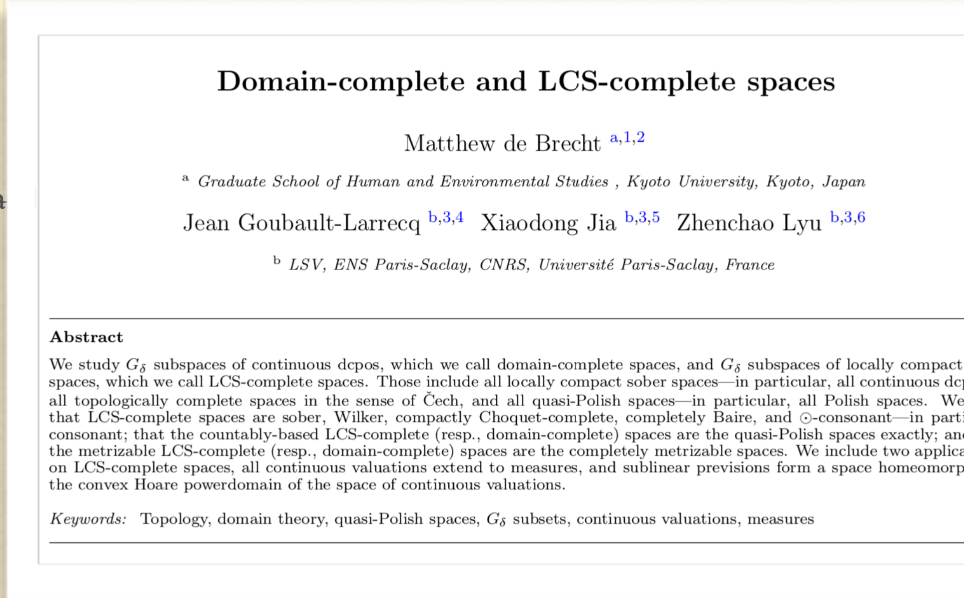


CONCLUSION

- A very rich theory,
  extending both domains and (quasi-)Polish spaces,
  with applications in topological measure theory
- Much more to be read about in the paper!
  (19 sections, 8 theorems, 14 propositions, 10 lemmata~~, and 72 essential vitamins a~~



Domain-complete and LCS-complete spaces

Matthew de Brecht [a,1,2]

[a] *Graduate School of Human and Environmental Studies , Kyoto University, Kyoto, Japan*

Jean Goubault-Larrecq [b,3,4] Xiaodong Jia [b,3,5] Zhenchao Lyu [b,3,6]

[b] *LSV, ENS Paris-Saclay, CNRS, Université Paris-Saclay, France*

Abstract

We study G_δ subspaces of continuous dcpos, which we call domain-complete spaces, and G_δ subspaces of locally compact
spaces, which we call LCS-complete spaces. Those include all locally compact sober spaces—in particular, all continuous dc
all topologically complete spaces in the sense of Čech, and all quasi-Polish spaces—in particular, all Polish spaces. We
that LCS-complete spaces are sober, Wilker, compactly Choquet-complete, completely Baire, and ⊙-consonant—in parti
consonant; that the countably-based LCS-complete (resp., domain-complete) spaces are the quasi-Polish spaces exactly; an
the metrizable LCS-complete (resp., domain-complete) spaces are the completely metrizable spaces. We include two applica
on LCS-complete spaces, all continuous valuations extend to measures, and sublinear previsions form a space homeomorp
the convex Hoare powerdomain of the space of continuous valuations.

Keywords: Topology, domain theory, quasi-Polish spaces, G_δ subsets, continuous valuations, measures

- Questions?

- Beyond domains and quasi-Polish spaces
- Motivating example: measure extension theorems
- Locating LCS-complete spaces
- So time permits after all! Stone duality, consonance, …

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STONE DUALITY

- $\mathbf{O} : \mathbf{Top} \to \mathbf{Frm}^{op}$
  maps X to
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- $\mathrm{pt} : \mathbf{Frm}^{op} \to \mathbf{Top}$
  maps L to space of
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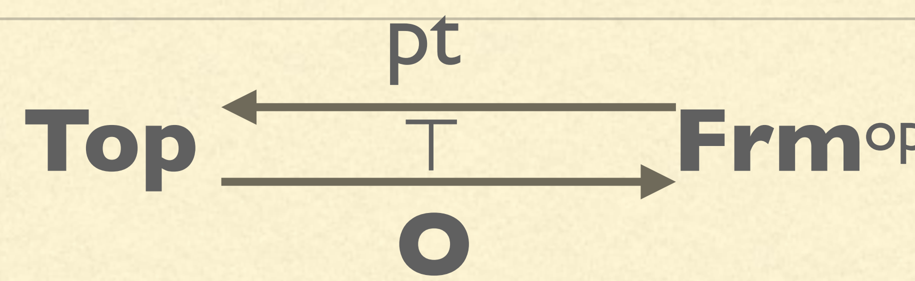

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- Adjunction, which restricts to
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$$\mathbf{Top} \underset{\mathbf{O}}{\overset{\mathrm{pt}}{\rightleftarrows}} \mathbf{Frm}^{op} \quad (\top)$$

Sober spaces $\rightleftarrows$ spatial locales

loc. compact sober $\rightleftarrows$ continuous
distr. complete lattices

continuous dcpos $\rightleftarrows$ completely
distributive lattices

…

quasi-Polish $\rightleftarrows$ countably
presented locales
[Heckmann15]

STONE DUALITY

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$$\mathbf{Top} \underset{\mathbf{O}}{\overset{\mathrm{pt}}{\rightleftarrows}} \mathbf{Frm}^{op} \quad (\mathbf{O} \dashv \mathrm{pt})$$

quasi-Polish ⇄ countably presented locales [Heckmann15]

domain-complete ⇄ quotient of completely distributive lattice

LCS-complete ⇄ quotient of continuous distr. complete lattice

… by **countably many relations** $u = \top$

(this paper)

CATEGORICAL PROPERTIES (1)

- Let **LCS** be the category of LCS-complete spaces
- **Prop. LCS** is closed under:
  — countable topological products
  — arbitrary sums.

CATEGORICAL PROPERTIES (2)

- **Prop. LCS** does not have:

— equalizers

(ℚ would arise as eq. of $f, g : \mathbb{R} \to \mathbb{P}(\mathbb{R})$
with $f(x)=(\mathbb{R}-\{x\}) \cup \mathbb{Q}$, $g(x)=\mathbb{R}$)

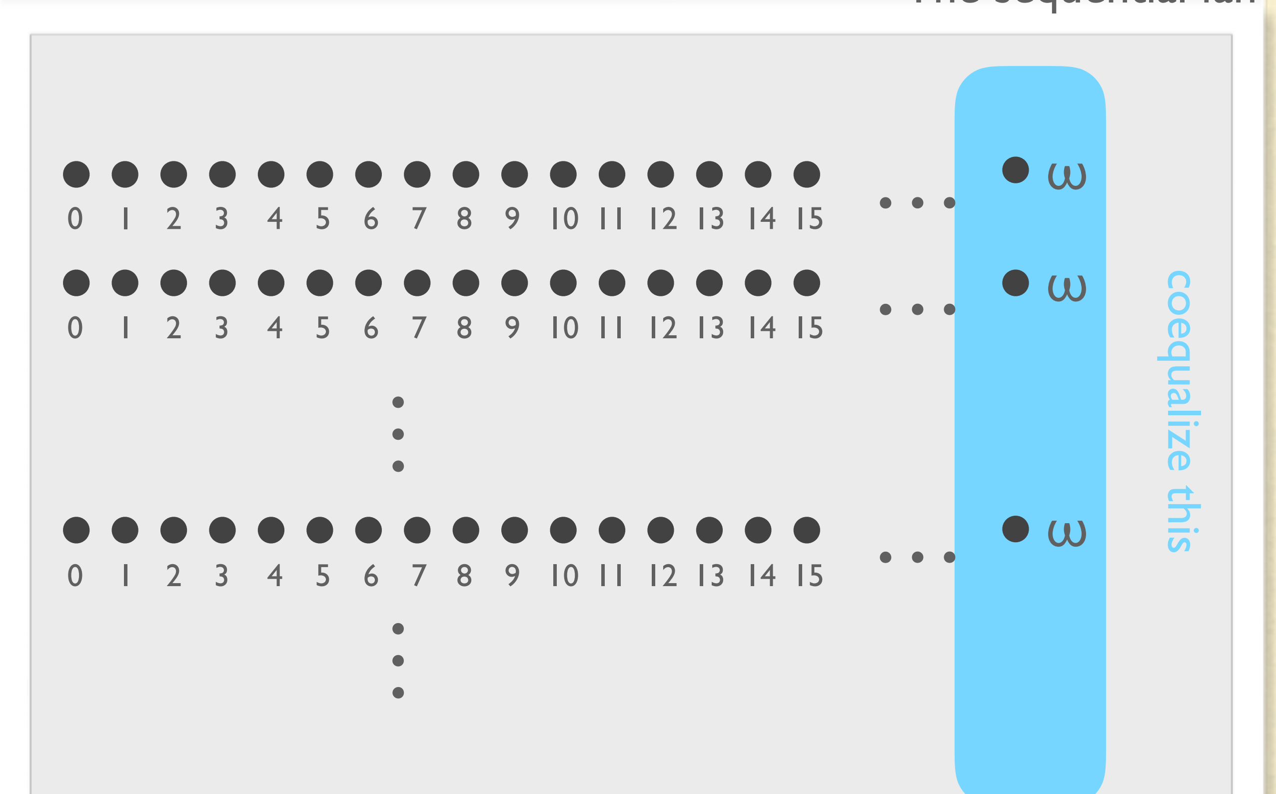
Note that the category of quasi-Polish spaces **has** equalizers.

— coequalizers

(the **sequential fan** would arise
as such a coequalizer
but is not first-countable

however every countable
LCS-complete space is first-countable)

The sequential fan



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- **Corl. LCS** is not Cartesian-closed
- **Thm.** (Bonus.) The exponentiable objects
  in the category of quasi-Polish spaces
  are exactly the countably-based locally compact sober spaces.

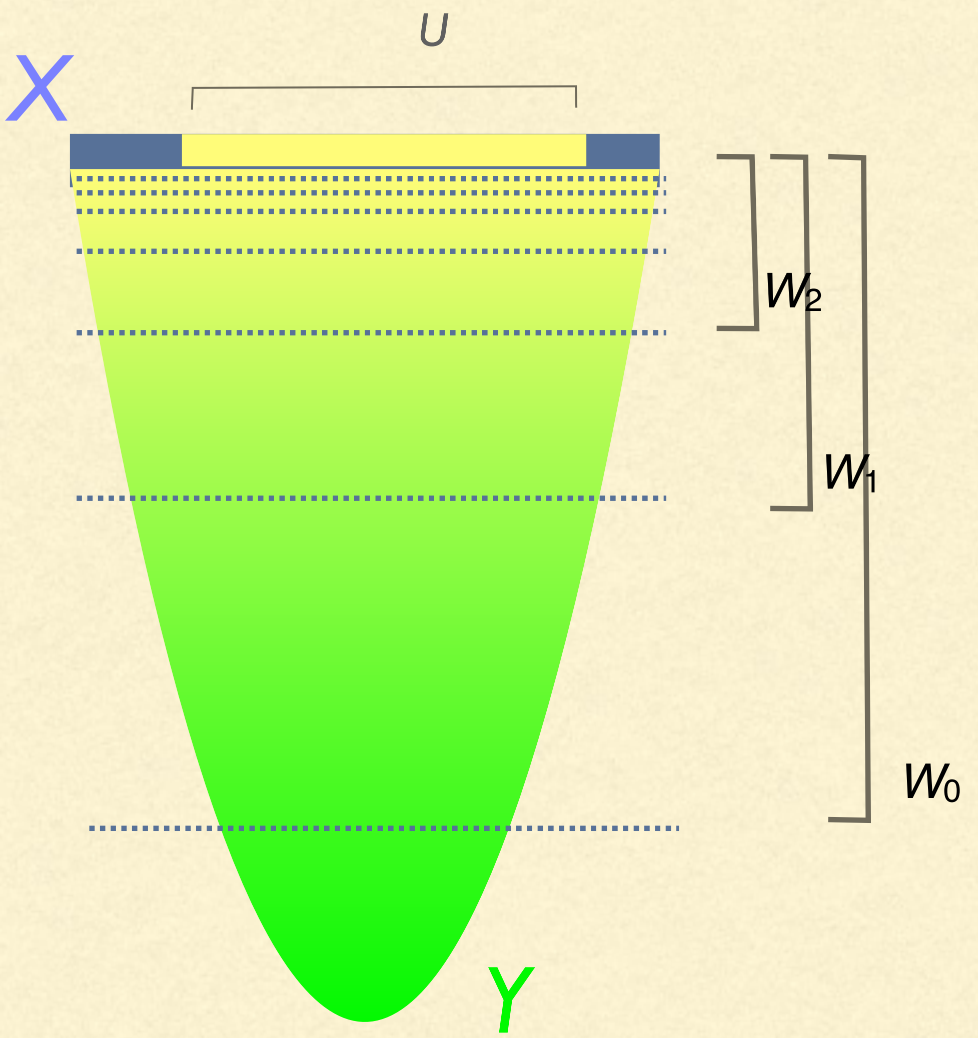
CONSONANCE

- For Q compact saturated, $\blacksquare Q =_{\text{def}}$ collection of opens $U \supseteq Q$
- $\blacksquare Q$ is a Scott-open filter in the complete lattice $\mathbf{O}X$ of opens
- Every union $\bigcup_i \blacksquare Q_i$ is Scott-open in $\mathbf{O}X$.
- **Defn.** X is **consonant** iff those are the only Scott-opens of $\mathbf{O}X$.

LCS-COMPLETE ⇒ CONSONANT

- **Thm.** Every LCS-complete space X is consonant.

- **Proof.**
  Let $\boldsymbol{F}$ be Scott-open in $\mathbf{O}Y$, $U \in \boldsymbol{F}$.
  We must find Q / $U \in$ ■Q $\subseteq \boldsymbol{F}$.



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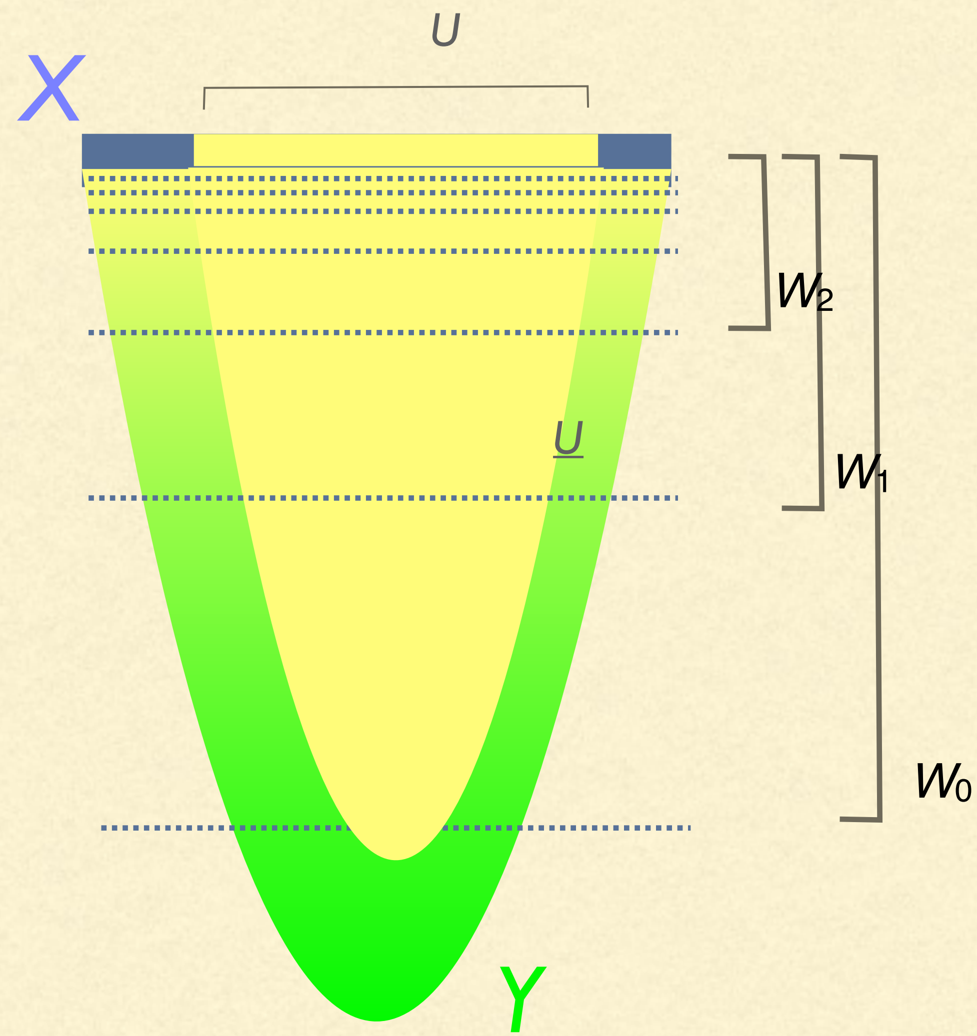

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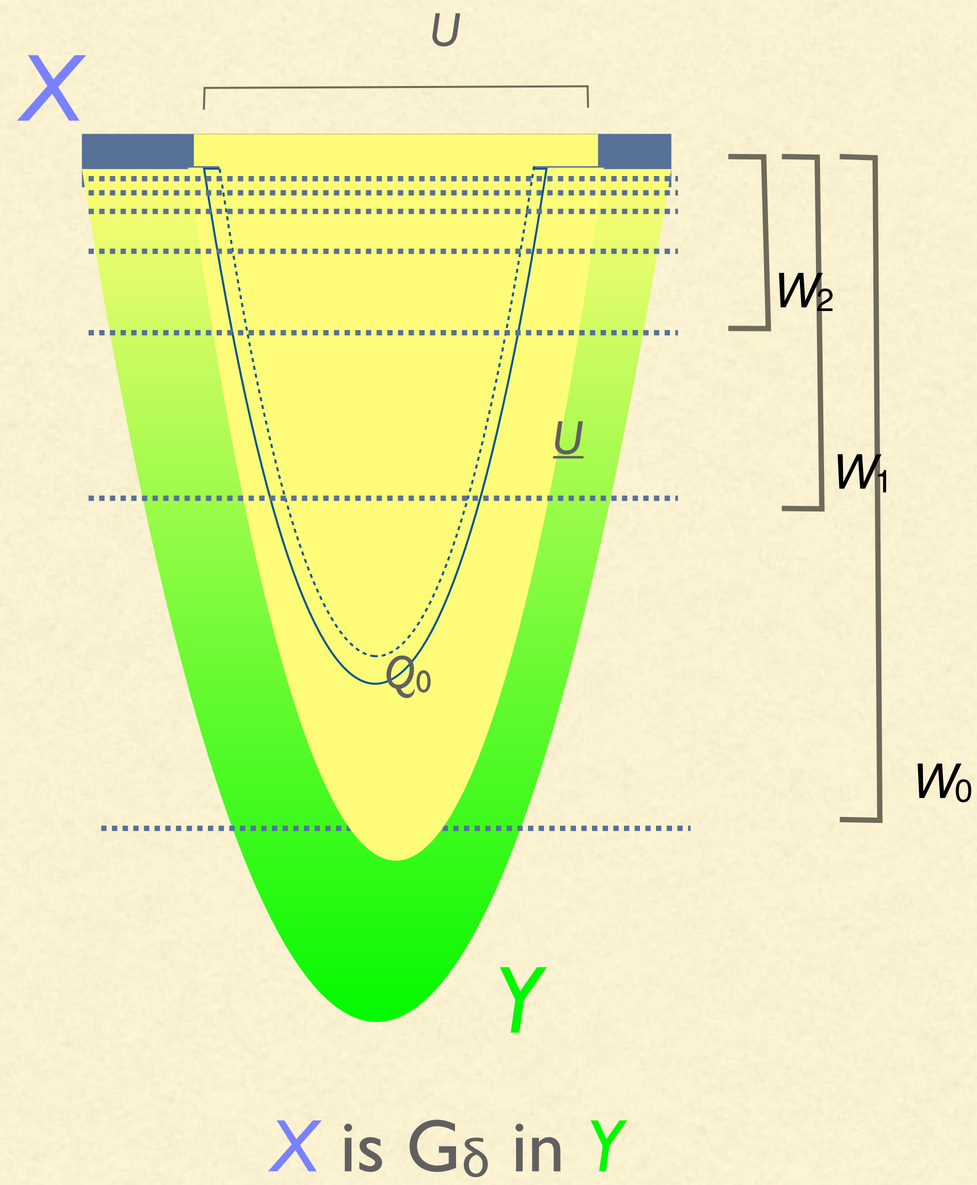
- Y locally compact ⇒ approximate
  $\underline{U} \cap W_0$ by Q_0 with $\mathrm{int}(Q_0) \cap Y \in \boldsymbol{F}$



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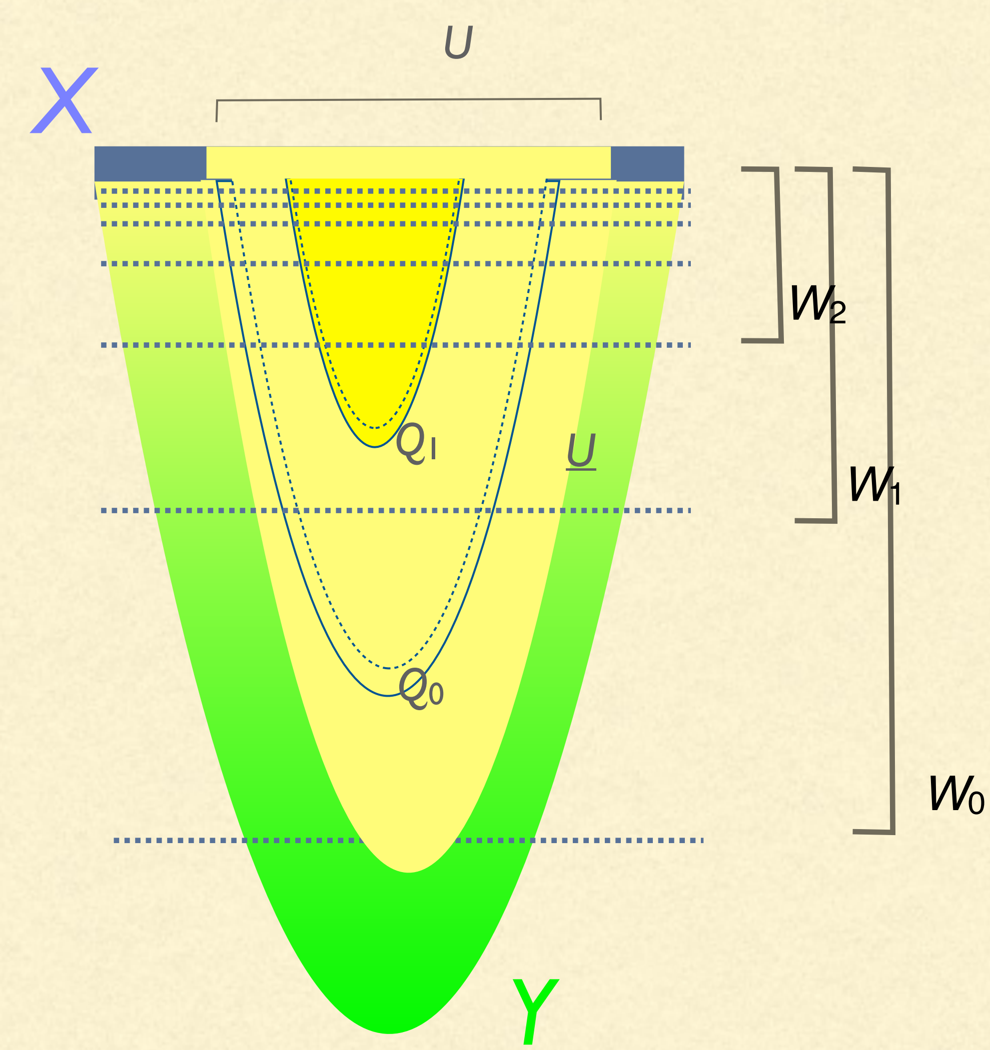

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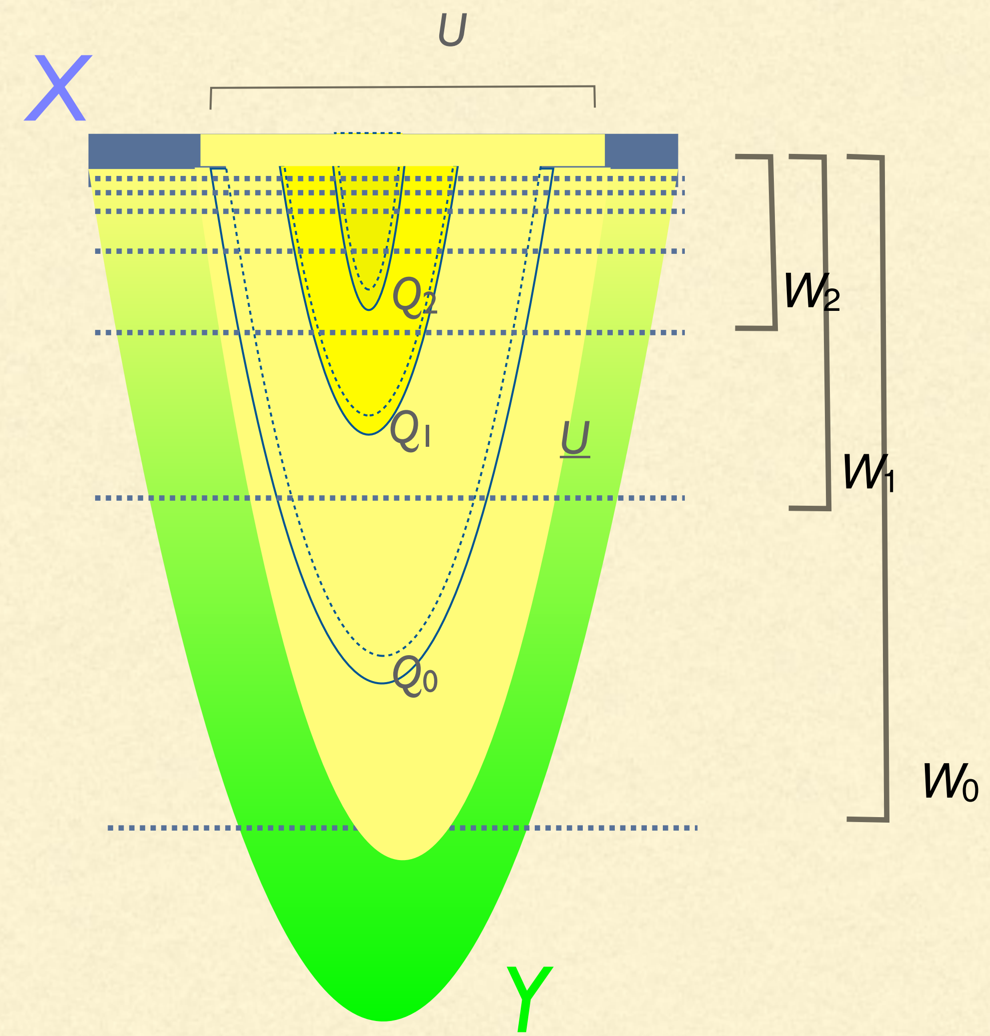

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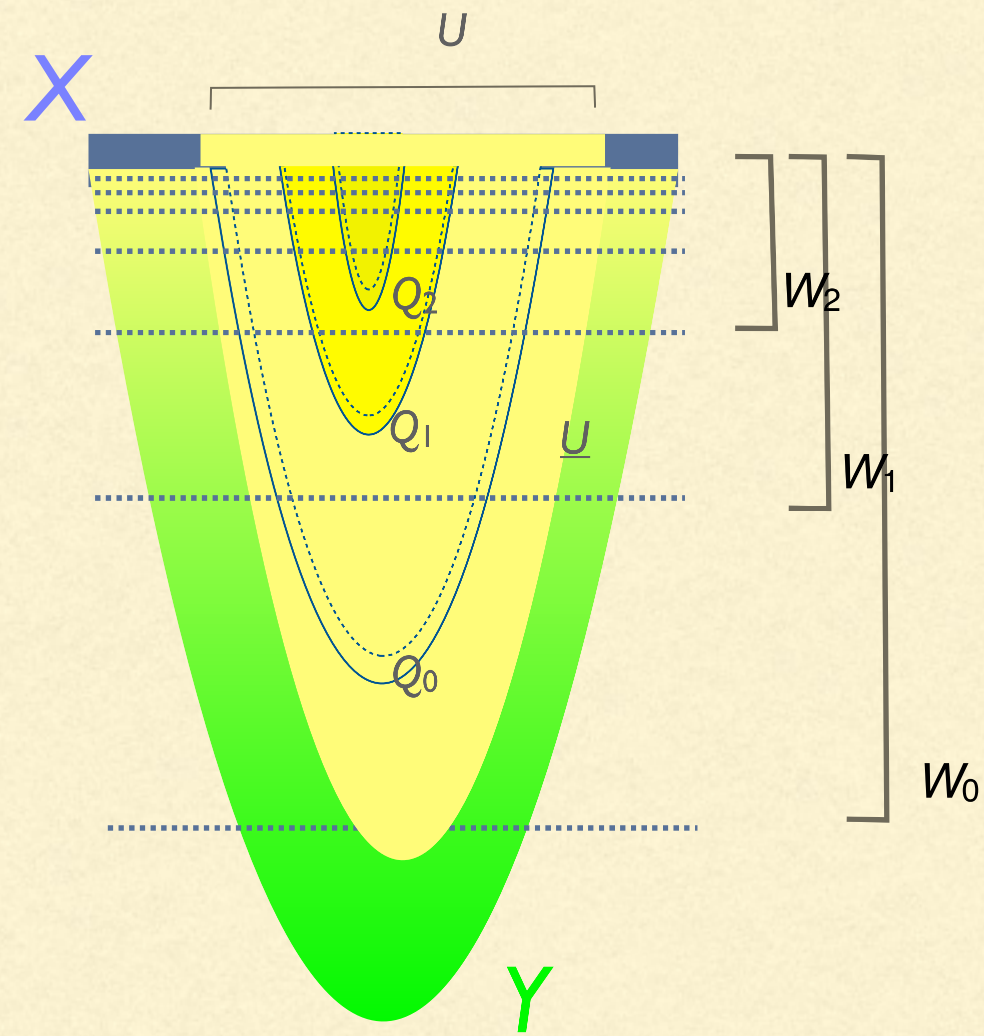
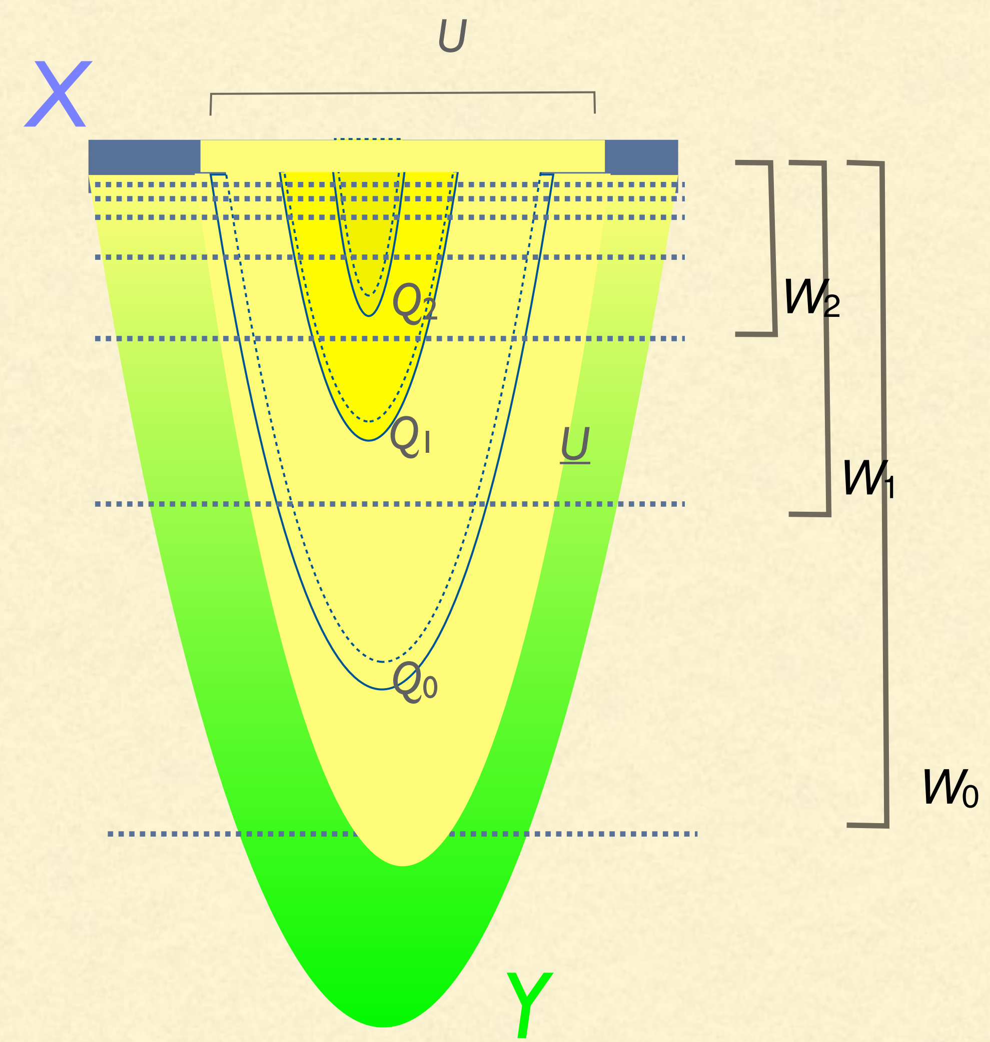


X is Gδ in Y

LCS-COMPLETE ⇒ CONSONANT

- **Thm.** Every LCS-complete space X is consonant.

- **Corl.** … and X+X+…+X is consonant, too, i.e. X is ⊙**-consonant.**



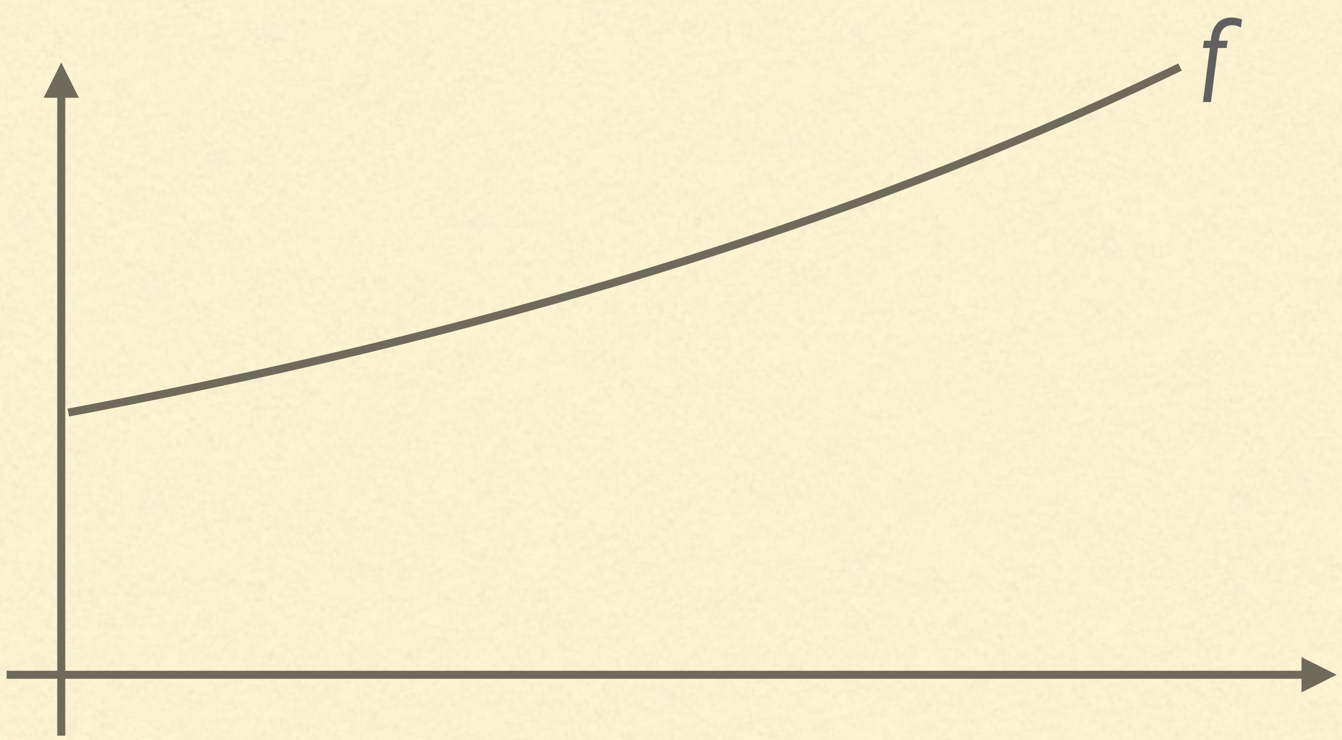
X is G_δ in Y

THE SPACE OF LSC MAPS

- Let $\mathcal{L}X$ = {lower semicontinuous maps : $X \to \mathbb{R}_+ \cup \{\infty\}$ }
  with the Scott topology

- **Thm.** If X is LCS-complete, then Scott=compact-open on $\mathcal{L}X$.

- **Proof.** Let f in Scott-open $\mathcal{U}$.
  Find a step function $\sup_i a_i \chi_{U_i} \leq f$ in $\mathcal{U}$.
  By ⊙**-consonance**,
  find Q_i large enough $\subseteq U_i$,
  and b_i large enough $< a_i$
  Then $\cap_i [Q_i > b_i]$ contains f
  and is included in $\mathcal{U}$. □



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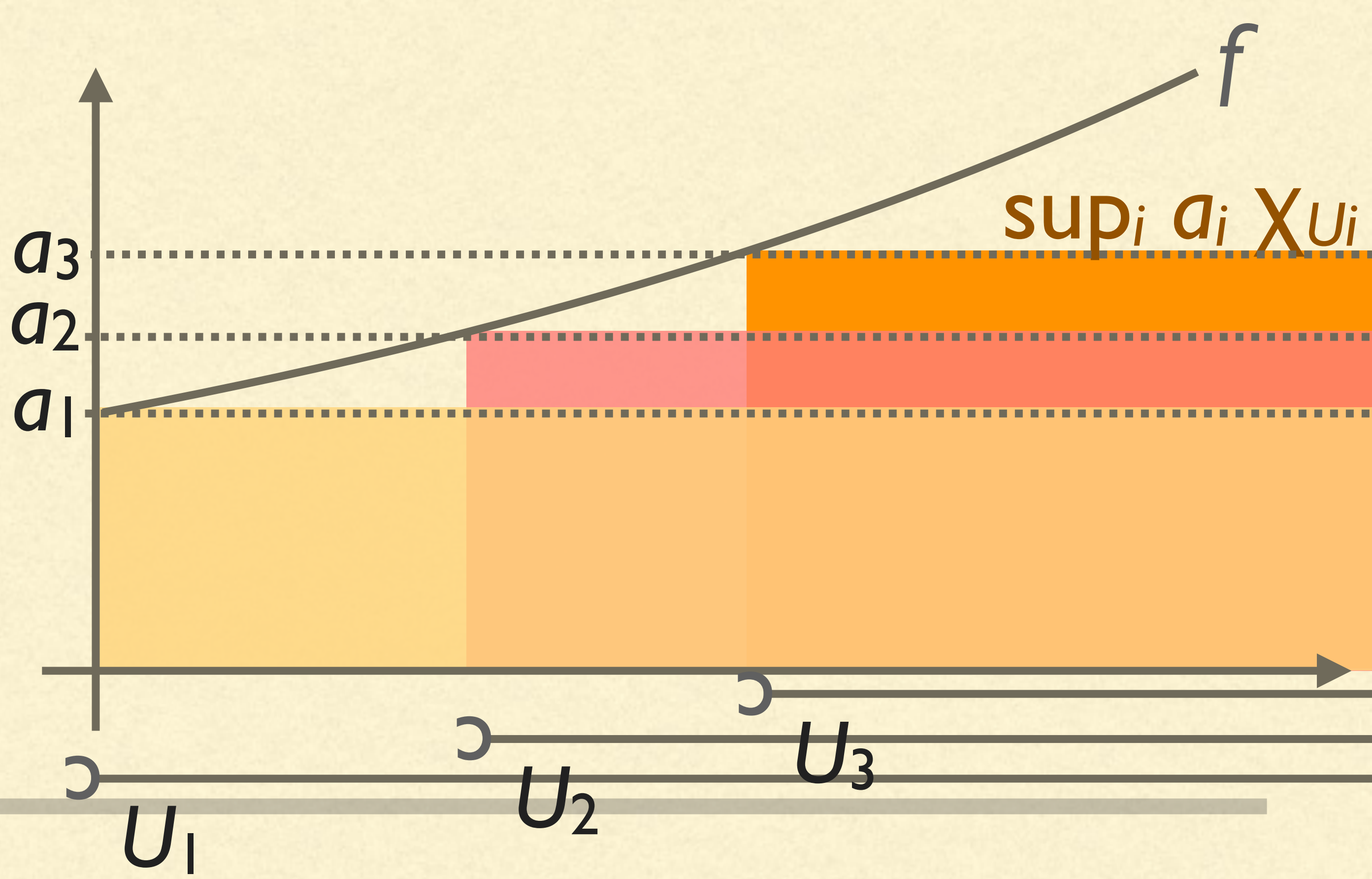

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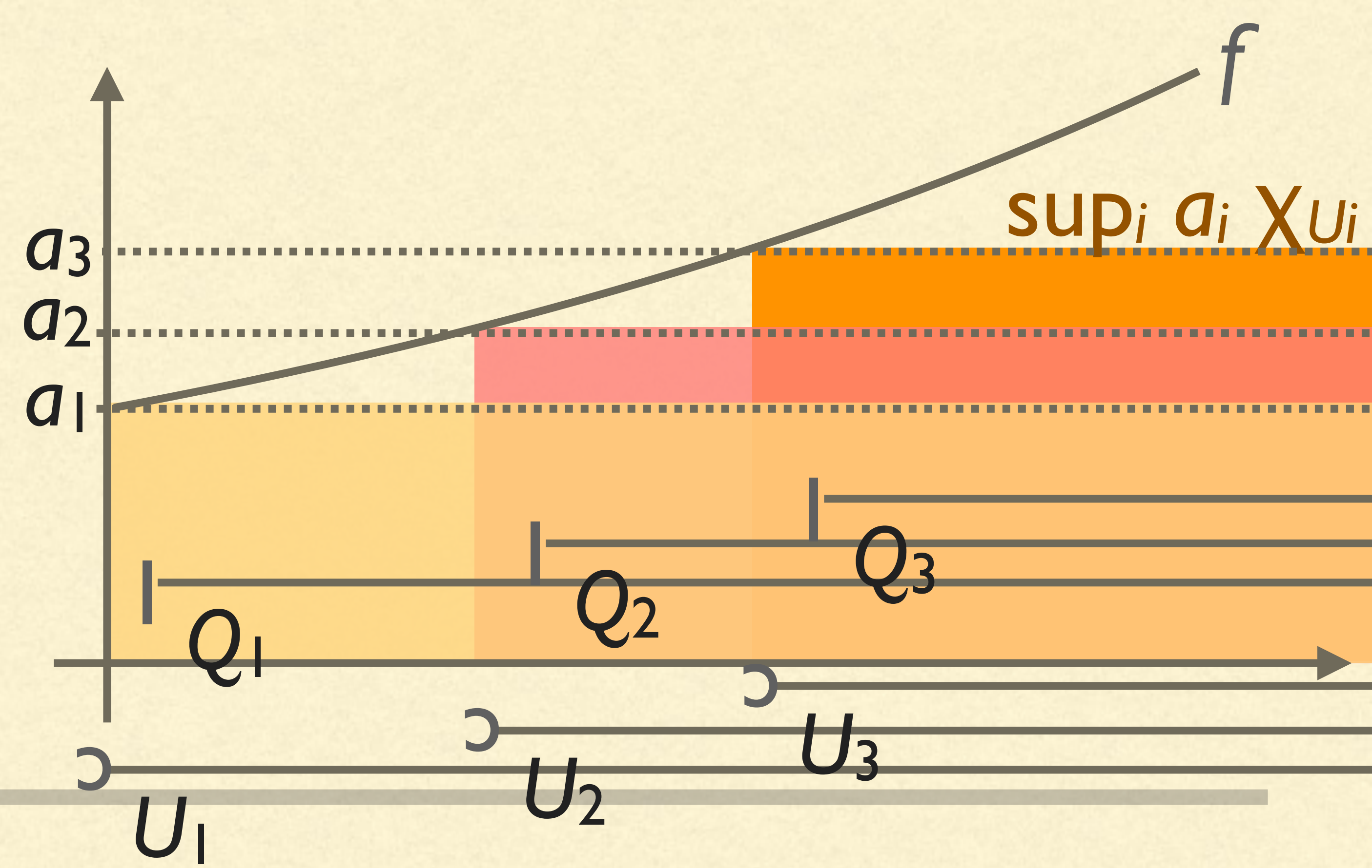

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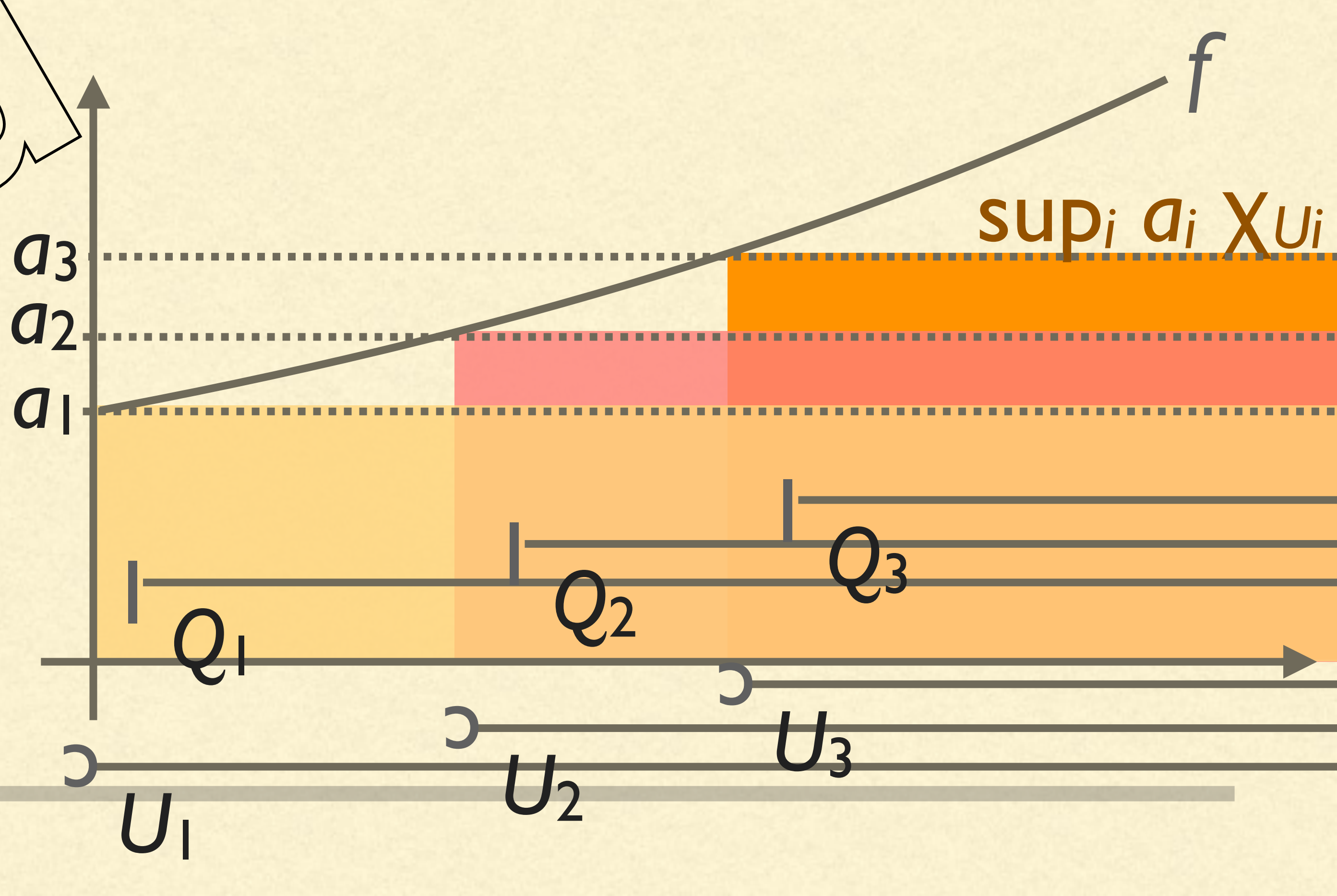

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  … hence the isomorphism theorems of [JGL17] apply, e.g.:

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- **Corl.** If X is LCS-complete, then
  the space of sublinear cont. functionals : $\mathcal{L}X \to \mathbb{R}_+ \cup \{\infty\}$
  ≅ the space of convex closed sets of cont. valuations on X