The Notion of model

I. What we have seen last time

A theory = a set of axioms + a decidable and non confusing congruence (often defined with a reduction system)

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \wedge\text{-intro}$$

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash C} \wedge\text{-intro if } C \equiv A \wedge B$$

Proof reduction does not always terminate

But all (1) purely computational theories where (2) proof reduction terminates have the witness property

Next

Examples of theories

Prove termination of proof-reduction for some of these theories
For this: the notion of model

II. Models valued in $\{0, 1\}$

The algebra $\{0,1\}$

$\mathcal{B} = \{0,1\}$
$\leq$: natural order on this set
$\tilde{\top} = 1$, $\tilde{\perp} = 0$
$\tilde{\wedge}$, function from $\{0,1\} \times \{0,1\}$ to $\{0,1\}$

$\tilde{\wedge}$	0	1
0	0	0
1	0	1

$\tilde{\vee}$ and $\tilde{\Rightarrow}$ similar
$\tilde{\forall}$ and $\tilde{\exists}$ functions from $\mathcal{P}^{+}(\{0,1\})$ to $\{0,1\}$

$\tilde{\forall}$	$\{0\}$	$\{0,1\}$	$\{1\}$
	0	0	1

$\tilde{\exists}$	$\{0\}$	$\{0,1\}$	$\{1\}$
	0	1	1

Models valued in $\{0,1\}$

A model for a language $\mathcal{L}$ is formed with

- for each sort s, a non empty set $\mathcal{M}_s$
- for each function symbol f of arity $\langle s_1, ..., s_n, s' \rangle$, a function $\hat{f}$ from $\mathcal{M}_{s_1} \times ... \times \mathcal{M}_{s_n}$ to $\mathcal{M}_{s'}$
- for each predicate symbol P of arity $\langle s_1, ..., s_n \rangle$, a function $\hat{P}$ from $\mathcal{M}_{s_1} \times ... \times \mathcal{M}_{s_n}$ to $\mathcal{B}$

Interpretation in a model

$[\![\ ]\!]$ maps every term t of sort s, to an element $[\![t]\!]$ of $\mathcal{M}_s$
every proposition A to an element $[\![A]\!]$ of $\mathcal{B}$
Morphism
$[\![f(t_1, ..., t_n)]\!] = \hat{f}([\![t_1]\!], ..., [\![t_n]\!])$
$[\![P(t_1, ..., t_n)]\!] = \hat{P}([\![t_1]\!], ..., [\![t_n]\!])$
$[\![A \wedge B]\!] = [\![A]\!] \tilde{\wedge} [\![B]\!]$, etc.
Completely defined by its image on the variables

Valuations

Function ϕ of finite domain associating to the variables $x_1, \ldots, x_n$ of sorts $s_1, \ldots, s_n$ elements $a_1, \ldots, a_n$ of $\mathcal{M}_{s_1}$, ..., $\mathcal{M}_{s_n}$

Any valuation ϕ extends to a morphism $[\![\ ]\!]_\phi$ between

- the terms and the propositions whose free variables are in the domain of ϕ
- and the model $\mathcal{M}$

- $[\![x]\!]_\phi = \phi(x)$
- $[\![f(t_1, ..., t_n)]\!]_\phi = \hat{f}([\![t_1]\!]_\phi, ..., [\![t_n]\!]_\phi)$
- $[\![P(t_1, ..., t_n)]\!]_\phi = \hat{P}([\![t_1]\!]_\phi, ..., [\![t_n]\!]_\phi)$
- $[\![\top]\!]_\phi = \tilde{\top}$, $[\![\bot]\!]_\phi = \tilde{\bot}$
- $[\![A \wedge B]\!]_\phi = [\![A]\!]_\phi \mathbin{\tilde{\wedge}} [\![B]\!]_\phi$, $[\![A \vee B]\!]_\phi = [\![A]\!]_\phi \mathbin{\tilde{\vee}} [\![B]\!]_\phi$
- $[\![A \Rightarrow B]\!]_\phi = [\![A]\!]_\phi \mathrel{\tilde{\Rightarrow}} [\![B]\!]_\phi$
- $[\![\forall x\ A]\!]_\phi = \tilde{\forall}\ \{[\![A]\!]_{\phi, x=a} \mid a \in \mathcal{M}_s\}$
- $[\![\exists x\ A]\!]_\phi = \tilde{\exists}\ \{[\![A]\!]_{\phi, x=a} \mid a \in \mathcal{M}_s\}$

Validity

A valid in $\mathcal{M}$ if for all ϕ, $[\![A]\!]_\phi \geq \tilde{\top}$
$A_1, ..., A_n \vdash B$ valid in $\mathcal{M}$ if the proposition $(A_1 \wedge ... \wedge A_n) \Rightarrow B$ is
$\mathcal{T}$ valid in $\mathcal{M}$ if all its axioms are

Soundness: If the proposition A has a classical proof in $\mathcal{T}$, then it is valid in all models of $\mathcal{T}$
Completeness: If the proposition A is valid in all models of $\mathcal{T}$, then it has a classical proof in $\mathcal{T}$

Contrapositive of soundness

If a model $\mathcal{M}$ of $\mathcal{T}$ s.t. A not valid in $\mathcal{M}$, then A not provable in $\mathcal{T}$

Exercise: two proposition symbols P and Q
A single axiom P
Q is not provable
$\neg Q$ is not provable

III. Many valued models

Four problems

Adapt the notion of model to prove indep. of Q with single model
Adapt the notion of model to constructive provability
Adapt the notion of model to Deduction modulo theory
Adapt the notion of model to prove termination of proof-reduction

One solution: many valued models

Algebras

A set $\mathcal{B}$

a binary relation $\leq$ on $\mathcal{B}$

two elements $\tilde{\top}$ and $\tilde{\bot}$ of $\mathcal{B}$
three functions $\tilde{\wedge}$, $\tilde{\vee}$, and $\tilde{\Rightarrow}$ from $\mathcal{B} \times \mathcal{B}$ to $\mathcal{B}$
a subset $\mathcal{A}$ of $\mathcal{P}^+(\mathcal{B})$, a function $\tilde{\forall}$ from $\mathcal{A}$ to $\mathcal{B}$
a subset $\mathcal{E}$ of $\mathcal{P}^+(\mathcal{B})$, a function $\tilde{\exists}$ from $\mathcal{E}$ to $\mathcal{B}$

Models

A model for a language $\mathcal{L}$ is formed with

- for each sort s, a non empty set $\mathcal{M}_s$
- an algebra $\mathcal{B} = \langle \mathcal{B}, \leq, \tilde{\top}, \tilde{\bot}, \tilde{\wedge}, \tilde{\vee}, \mathcal{A}, \tilde{\forall}, \mathcal{E}, \tilde{\exists}, \tilde{\Rightarrow} \rangle$,
- for each function symbol f of arity $\langle s_1, ..., s_n, s' \rangle$, a function $\hat{f}$ from $\mathcal{M}_{s_1} \times ... \times \mathcal{M}_{s_n}$ to $\mathcal{M}_{s'}$
- for each predicate symbol P of arity $\langle s_1, ..., s_n \rangle$, a function $\hat{P}$ from $\mathcal{M}_{s_1} \times ... \times \mathcal{M}_{s_n}$ to $\mathcal{B}$

valued in the algebra $\mathcal{B}$

Valuation (as above): a function ϕ of finite domain associating to the variables $x_1, ..., x_n$ of sorts $s_1, ..., s_n$ elements $a_1, ..., a_n$ of $\mathcal{M}_{s_1}$, ..., $\mathcal{M}_{s_n}$

Interpretation (as above):

- $[\![x]\!]_\phi = \phi(x)$, $[\![f(t_1, ..., t_n)]\!]_\phi = \hat{f}([\![t_1]\!]_\phi, ..., [\![t_n]\!]_\phi)$
- $[\![P(t_1, ..., t_n)]\!]_\phi = \hat{P}([\![t_1]\!]_\phi, ..., [\![t_n]\!]_\phi)$
- $[\![A \wedge B]\!]_\phi = [\![A]\!]_\phi \mathrel{\tilde{\wedge}} [\![B]\!]_\phi$, etc.
- $[\![\forall x\ A]\!]_\phi = \tilde{\forall}\ \{[\![A]\!]_{\phi, x=a} \mid a \in \mathcal{M}_s\}$

Validity (as above): for all ϕ, $[\![A]\!]_\phi \geq \tilde{\top}$

Examples of algebras

$\{0, 1\}$

But also:
$\mathcal{B} = \mathcal{P}(\{3, 4\}) = \{\varnothing, \{3\}, \{4\}, \{3, 4\}\}$, $\leq \ = \ \subseteq$,
$\tilde{\top} = \{3, 4\}$, $\tilde{\bot} = \varnothing$,
$a \ \tilde{\wedge} \ b = a \cap b$, $a \ \tilde{\vee} \ b = a \cup b$, $a \ \tilde{\Rightarrow} \ b = (\{3, 4\} \setminus a) \cup b$,
$\tilde{\forall} \ E = \bigcap_{x \in E} x$, $\tilde{\exists} \ E = \bigcup_{x \in E} x$
Note $\tilde{\neg} \ a = a \ \tilde{\Rightarrow} \ \tilde{\bot} = \{3, 4\} \setminus a$

$\hat{P} = \tilde{\top} = \{3, 4\}$
$\hat{Q} = \{4\}$
Neither Q nor $\neg Q$ is valid

Aggregates two models in one
$\mathcal{M}_3 : \hat{P} = 1, \hat{Q} = 0$
$\mathcal{M}_4 : \hat{P} = 1, \hat{Q} = 1$
$\mathcal{M} : \hat{A} = \{i \mid \text{A valid in } \mathcal{M}_i\}$

From $\mathcal{P}(\{3,4\})$ to pre-Boolean algebras

Models where $\mathcal{B}$ is a powerset

Generalize: models where $\mathcal{B}$ is a Boolean algebra
A set with, an order, greatest lowerbounds ($\tilde{\top}$, $\tilde{\wedge}$, $\tilde{\forall}$) and least upperbounds ($\tilde{\bot}$, $\tilde{\vee}$, $\tilde{\exists}$) and a relative complement $\tilde{\Rightarrow}$

Generalize further: order: reflexive, antisymmetric, transitive
Antisymmetry useless and complicates proofs: drop it
Intuition: $A \leq B$ if $A \Rightarrow B$ provable: reflexive, transitive, not antisymmetric

Pre-Boolean algebras

Set $\mathcal{B}$, binary relation $\leq$, $\tilde{\top}$ and $\tilde{\bot}$ elements of $\mathcal{B}$, $\tilde{\wedge}$, $\tilde{\vee}$, and $\tilde{\Rightarrow}$ binary functions, $\tilde{\forall}$ function from a subset $\mathcal{A}$ of $\mathcal{P}^+(\mathcal{B})$ to $\mathcal{B}$, and $\tilde{\exists}$ function from a subset $\mathcal{E}$ of $\mathcal{P}^+(\mathcal{B})$ to $\mathcal{B}$

$a \leq a$, if $a \leq b$ and $b \leq c$ then $a \leq c$
$a \mathbin{\tilde{\wedge}} b \leq a$, $a \mathbin{\tilde{\wedge}} b \leq b$, if $c \leq a$ and $c \leq b$ then $c \leq a \mathbin{\tilde{\wedge}} b$
etc.
$a \leq b \mathbin{\tilde{\Rightarrow}} c$ if and only if $a \mathbin{\tilde{\wedge}} b \leq c$
$\tilde{\top} \leq (a \mathbin{\tilde{\vee}} (a \mathbin{\tilde{\Rightarrow}} b))$

Examples of pre-Boolean algebras

$\{0, 1\}$
$\mathcal{P}(\{3, 4\})$
but also $\{0\}$
and any set equipped with the full relation and any operations

Soundness and completeness

Soundness: If the proposition A has a classical proof in $\mathcal{T}$, then it is valid in all models of $\mathcal{T}$

Completeness: If the proposition A is valid in all models of $\mathcal{T}$, then it has a classical proof in $\mathcal{T}$

More models: soundness stronger, completeness weaker

IV. Models and constructive proofs

The validity of the excluded-middle

Models valued in $\{0, 1\}$ are all models of the excluded-middle

$$[\![A \vee \neg A]\!]_\phi = [\![A]\!]_\phi \mathrel{\tilde{\vee}} \tilde{\neg}[\![A]\!]_\phi = max([\![A]\!]_\phi, 1 - [\![A]\!]_\phi) = 1$$

Models valued in $\mathcal{P}(E)$ also

$$[\![A \vee \neg A]\!]_\phi = [\![A]\!]_\phi \mathrel{\tilde{\vee}} \tilde{\neg}[\![A]\!]_\phi = [\![A]\!]_\phi \cup (E \setminus [\![A]\!]_\phi) = E$$

Models valued in a (pre-)Boolean algebra also

$$\tilde{\top} \leq (a \mathrel{\tilde{\vee}} (a \mathrel{\tilde{\Rightarrow}} b))$$

Valid in all models but no constructive proof

From pre-Boolean algebra to pre-Heyting algebras

Just drop the condition

$$\tilde{\top} \leq (a \mathbin{\tilde{\vee}} (a \mathbin{\tilde{\Rightarrow}} b))$$

pre-Heyting algebra

A pre-Heyting algebra that is not a pre-Boolean algebra

Instead of $\mathcal{P}(\mathbb{R})$, the open sets only

Pre-order $\subseteq$ (antisymmetric in this case)

Everything works (open sets stable by unions, finite intersections)
except infinite intersections and complement
In this case take the interior
$\hat{P} = (-\infty, 0)$
$\tilde{\neg}\hat{P} = \overset{\circ}{\overline{[0, +\infty)}} = (0, +\infty)$
$\hat{P} \mathbin{\tilde{\vee}} \tilde{\neg}\, \hat{P} = (-\infty, 0) \cup (0, +\infty) = \mathbb{R} \setminus \{0\}$

Another pre-Heyting algebra that is not pre-Boolean

$\{0, 1/2, 1\}$

natural order

$\tilde{\top} = 1$, $\tilde{\bot} = 0$, $a \mathbin{\tilde{\wedge}} b = glb(a, b)$, $a \mathbin{\tilde{\vee}} b = lub(a, b)$,
$\tilde{\forall}\ A = glb_{x \in A} x$, $\tilde{\exists}\ E = lub_{x \in E} x$

$a \mathbin{\tilde{\Rightarrow}} b = b$ if $a > b$, and 1 otherwise

$a \leq (b \mathbin{\tilde{\Rightarrow}} c)$ if and only if $(a \mathbin{\tilde{\wedge}} b) \leq c$ (three cases: $b \leq c$, $b > c$ and $a \leq c$, and $b > c$ and $a > c$)

$1/2 \mathbin{\tilde{\vee}} (1/2 \mathbin{\tilde{\Rightarrow}} 0) = 1/2 \mathbin{\tilde{\vee}} 0 = 1/2$

Soundness

If the proposition A has a constructive proof in $\mathcal{T}$, then it is valid in all models of $\mathcal{T}$

Lemma: If $\Gamma \vdash A$ has a constructive proof, then it is valid in all pre-Heyting models (By induction over proof structure)

Completeness

If the proposition A is valid in all models of $\mathcal{T}$, then it has a constructive proof in $\mathcal{T}$

Weak

A simple proof: build a single model where valid propositions are exactly those that have a constructive proof in $\mathcal{T}$

The Lindenbaum model

Idea: Interpret each term (resp. proposition) by itself
$\mathcal{M}_s$ = set of terms (of sort s), $\mathcal{B}$ = set of propositions

Closed terms and prop. of $\mathcal{L} \cup S$, S infinite set of constants

$A \leq B$ if $A \Rightarrow B$ has a constructive proof in $\mathcal{T}$
The operations $\tilde{\top}$, $\tilde{\bot}$, $\tilde{\wedge}$, $\tilde{\vee}$, and $\tilde{\Rightarrow}$, are $\top$, $\bot$, $\wedge$, $\vee$, and $\Rightarrow$

$\mathcal{A} = \mathcal{E}$ set of subsets of $\mathcal{B}$ of the form $\{(t/x)A \mid t \in \mathcal{M}_s\}$ for some A

A unique

$\tilde{\forall}\ \{(t/x)A \mid t \in \mathcal{M}_s\} = (\forall x\ A)$

$\tilde{\exists}\ \{(t/x)A \mid t \in \mathcal{M}_s\} = (\exists x\ A)$

$\hat{f}$: function mapping $t_1, ..., t_n$ to $f(t_1, ..., t_n)$

$\hat{P}$: function mapping $t_1, ..., t_n$ to $P(t_1, ..., t_n)$

Algebra of Lindenbaum model of $\mathcal{T}$: a pre-Heyting algebra

Lindenbaum model of $\mathcal{T}$: model of $\mathcal{T}$

A valid in the Lindenbaum model of $\mathcal{T}$ then A has a constructive proof in $\mathcal{T}$

V. Models and Deduction modulo theory

Validity of a theory in Deduction modulo theory

$\equiv$ valid in $\mathcal{M}$ if for all A and B such that $A \equiv B$, for all ϕ
$[\![A]\!]_\phi = [\![B]\!]_\phi$

$\mathcal{T}, \equiv$ valid in $\mathcal{M}$ if all axioms of $\mathcal{T}$ and $\equiv$ are valid in $\mathcal{M}$

Soundness

If the proposition A has a constructive proof in $\mathcal{T}$, then it is valid in all models of $\mathcal{T}$

Lemma: If $\Gamma \vdash A$ has a constructive proof, then it is valid in all pre-Heyting models
(By induction over proof structure using the fact that the model is a model of the congruence to justify the replacement of a proposition by a congruent one in each rule)

Completeness

Lindenbaum model:
Replace terms by classes of terms modulo $\equiv$
Replace propositions by classes of propositions modulo $\sim$
Only difficulty $\sim$ not $\equiv$: $\{(t/x)A \mid t \in \mathcal{M}_s\}$ does not uniquely define A

A reason to drop antisymmetry

If $A \Leftrightarrow B$ provable in $\mathcal{T}, \equiv$,
for all ϕ, $[\![A]\!]_\phi \leq [\![B]\!]_\phi$ and $[\![B]\!]_\phi \leq [\![A]\!]_\phi$
If $A \equiv B$, then for all ϕ, $[\![A]\!]_\phi = [\![B]\!]_\phi$

With antisymmetry: same notion
Without
$4 = 4$ and $2 + 2 = 4$ same interpretation
Fermat's little theorem and Fermat's last theorem different $\leq \geq$ interpretations

Consistency

If $\mathcal{T}$ has a model if and only if $\mathcal{T}$ consistent

Here: even non consistent theories have models

But: A pre-Heyting algebra is trivial if $a \leq b$ always
A theory $\mathcal{T}, \equiv$ is consistent if and only if it has a model whose pre-Heyting algebra is non trivial

VI. Super-consistency

An exercise

A model valued in $\{0, 1\}$ of the congruence defined with the reduction rule

$$P \longrightarrow (Q \Rightarrow Q)$$

That is: find $\hat{P}$ and $\hat{Q}$ such that $\hat{P} = (\hat{Q} \mathrel{\tilde{\Rightarrow}} \hat{Q})$
A solution: $\hat{Q} = 1$ and $\hat{P} = (1 \mathrel{\tilde{\Rightarrow}} 1) = 1$

But ...

No property of the algebra $\{0,1\}$ really used

$\hat{Q} = \tilde{\top}$ and $\hat{P} = (\tilde{\top} \mathrel{\tilde{\Rightarrow}} \tilde{\top})$ works in any pre-Heyting algebra $\mathcal{B}$

Thus, the congruence $\equiv$ has a model valued in $\{0,1\}$ and also in any algebra $\mathcal{B}$

Super-consistency

A theory is super-consistent if it has a model valued in any (full, ordered, and complete) pre-Heyting algebra

Why do we care: as we shall see super-consistency implies termination of proof-reduction, hence the witness property

Next time

Arithmetic