Brief Announcement: Coupling for Markov Decision Processes - Application to Self-Stabilization with Arbitrary Schedulers

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In [5], the method of coupling has been applied to the proof of self-stabilization of randomized distributed algorithms: A randomized distributed algorithm $\mathcal{A}$ is seen as a Markov chain $(X_t)_{t\geq 0}$ on a set Ω of configurations; the self-stabilization property of $\mathcal{A}$ (i.e., the convergence of $\mathcal{A}$ towards a closed subset $\mathcal{L}$ of Ω) follows from the existence of a coupling process $(X_t, Y_t)_{t\geq 0}$ where X_t and Y_t are faithful copies of $\mathcal{A}$, and coalesce in finite expected time. Furthermore, the coalescence time (or coupling time) gives an upper bound on the "hitting time", i.e. on the expected time for $\mathcal{A}$ to reach $\mathcal{L}$. The idea has been enhanced by using the path-coupling technique of Bubley-Dyer [2], allowing to focus on a set S of adjacent pairs (for a certain metric δ) instead of considering the whole space $\Omega \times \Omega$. More precisely, suppose that, for all (x, y) of S, the following contraction condition $E[\delta(X_{t+1}, Y_{t+1})|(X_t, Y_t) = (x, y)] \leq \alpha\delta(x, y)$ holds for some $0 < \alpha < 1$, where (X_t, Y_t) is a (partial) coupling for $\mathcal{A}$. Then $\mathcal{A}$ is self-stabilizing w.r.t. $\mathcal{L}$, and the hitting time is bounded by $\delta_{max}/(1 - \alpha)$.

We now want to extend this result to the case where the randomized distributed algorithm $\mathcal{A}$ is given not under the form of a Markov chain, but under the form of a *Markov decision process*. In other terms, at each step t, there is now a certain number of positions of machines that are *enabled*, i.e., subject to possible actions of the algorithm. In this context, a *scheduler* is a mechanism that selects the subset of enabled machines that are activated at step t. Given a scheduler, the algorithm $\mathcal{A}$ can be seen as a Markov chain as before. But we would like to prove now that, under any scheduler, $\mathcal{A}$ self-stabilizes. Furthermore, we would like to find an upper bound on the hitting time over all schedulers (including "malicious" ones, i.e. those selecting, at each step, the positions where actions hinder the progress towards $\mathcal{L}$ as much as possible). The coupling-based result of [5] can be extended as follows. Suppose that, for all scheduler σ, there exists a scheduler τ such that:
$\forall\ (x, y) \in S$: $E[\delta(X^{\sigma}_{t+1}, Y^{\tau}_{t+1})|(X^{\sigma}_t, Y^{\tau}_t) = (x, y)] \leq \alpha\delta(x, y)$
for some $0 < \alpha < 1$, where $(X^{\sigma}_t, Y^{\tau}_t)$ is a (partial) coupling of faithful copies of $\mathcal{A}$ under schedulers σ and τ. Then $\mathcal{A}$ is *self-stabilizing* under arbitrary scheduler. Note that the kind of coupling that we will use involves faithful copies of *different* Markov chains $(X^{\sigma}_t)_{t\geq 0}$ and $(Y^{\tau}_t)_{t\geq 0}$. Note also that the scheduler τ of the right component is determined from the scheduler σ of the left one. A further enhancement of this result will exploit an extension of coupling, *variable-length coupling*: a variable-length coupling, denoted by $(\overline{X}^{\sigma}_t, \overline{Y}^{\tau}_t, T)$, is a random variable taking values in $\Omega^* \times \Omega^* \times \mathbb{N}$, where T is a stopping time (see [6]; cf [3, 4]). By converting the variable-length couplings into fixed-length couplings ($T = constant$), such couplings can be defined for all $(x, y) \in \Omega^2$. Theorem 3 of [6] becomes in our context:
Suppose that there exists $0 < \alpha < 1$ such that, for all scheduler σ, there exist a scheduler τ and a variable-length partial coupling $(\overline{X}^{\sigma}_t, \overline{Y}^{\tau}_t, T)$ satisfying:
$\forall (x, y) \in S\ E[\delta(X^{\sigma}_{t+T}, Y^{\tau}_{t+T})|(X^{\sigma}_t, Y^{\tau}_t) = (x, y)] \leq \alpha\delta(x, y)$
and let $M := max_{(x,y)\in S, \sigma\in\Sigma}T$. Then there exists a fixed-length full coupling $(\overline{X}^{\sigma}, \overline{Y}^{\tau}, M)$ satisfying:
$\forall (x, y) \in \Omega^2\ E[\delta(X^{\sigma}_{t+M}, Y^{\tau}_{t+M})|(X^{\sigma}_t, Y^{\tau}_t) = (x, y)] \leq \alpha\delta(x, y)$
It follows that, if M is bounded, $\mathcal{A}$ is self-stabilizing w.r.t. $\mathcal{L}$ (for arbitrary scheduler σ), and the hitting time $\mathbf{H}_{\mathcal{L}}$ satisfies: $\mathbf{H}_{\mathcal{L}} \leq M\delta_{max}/(1 - \alpha)$.
Even in the case where there is no good natural upper bound M on the stopping time T, Hayes and Vigoda [6] explain how to define a truncated version of the (partial) coupling, which allows us to extend the above result for unbounded stopping times. Using this technique, we are able to prove the self-stabilization of the algorithm of [1] under arbitrary scheduler.

References

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